

DEPTH AND STANLEY DEPTH OF THE EDGE IDEALS OF SOME m -LINE GRAPHS AND m -CYCLIC GRAPHS WITH A COMMON VERTEX

GUANGJUN ZHU

ABSTRACT. We give some precise formulas for the depth of the quotient rings of the edge ideals associated to a graph consisting, either of the union of some line graphs $L_{3r_1}, \dots, L_{3r_{k_1}}, L_{3s_1+1}, \dots, L_{3s_{k_2}+1}$ and $L_{3t_1+2}, \dots, L_{3t_{k_3}+2}$ or of the union of cycle graphs $C_{3r_1}, \dots, C_{3r_{k_1}}, C_{3s_1+1}, \dots, C_{3s_{k_2}+1}$ and $C_{3t_1+2}, \dots, C_{3t_{k_3}+2}$, with a common vertex. We also give some tight bounds for their Stanley depths.

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1. INTRODUCTION

Let $S = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field K and M a finitely generated \mathbb{Z}^n -graded S -module. For a homogeneous element $u \in M$ and a subset $Z \subseteq \{x_1, \dots, x_n\}$, $uK[Z]$ denotes the K -subspace of M generated by all the homogeneous elements of the form uv , where v is a monomial in $K[Z]$. The \mathbb{Z}^n -graded K -subspace $uK[Z]$ is said to be a Stanley space of dimension $|Z|$ if it is a free $K[Z]$ -module, where, as usual, $|Z|$ denotes the number of elements of Z . A Stanley decomposition of M is a decomposition of M as a finite direct sum of \mathbb{Z}^n -graded K -vector spaces

$$\mathcal{D} : M = \bigoplus_{i=1}^r u_i K[Z_i]$$

where each $u_i K[Z_i]$ is a Stanley space of M . The number

$$\text{sdepth}(\mathcal{D}) = \min\{|Z_i| : i = 1, \dots, r\}$$

is called the Stanley depth of decomposition \mathcal{D} and the quantity

$$\text{sdepth}(M) := \max\{\text{sdepth}(\mathcal{D}) \mid \mathcal{D} \text{ is a Stanley decomposition of } M\}$$

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is called the Stanley depth of M . Stanley [10] conjectured that

$$\text{sdepth}(M) \geq \text{depth}(M)$$

for all \mathbb{Z}^n -graded S -modules M . This conjecture proves to be false, in general, for $M = S/I$ and $M = J/I$, where $I \subset J \subset S$ are monomial ideals, see [4].

Herzog, Vlădoiu and Zheng [6] introduced a method to compute the Stanley depth of a factor of two monomial ideals which was later developed into an effective algorithm by Rinaldo [9] implemented in CoCoA [3]. However, it is difficult to compute this invariant, even in some very particular cases. For instance in [1] Biró et al. proved that $\text{sdepth}(\mathfrak{m}) = \lceil \frac{n}{2} \rceil$ where $\mathfrak{m} = (x_1, \dots, x_n)$ is the graded maximal ideal of S and $\lceil \frac{n}{2} \rceil$ denote the smallest integer $\geq \frac{n}{2}$. For a friendly introduction on Stanley depth we refer the reader to [5].

Let I_n and J_n be the edge ideals associated to the line, respectively, cycle graph of length n . Morey [7] proved that $\text{depth}(S/I_n) = \lceil \frac{n}{3} \rceil$. Replacing depth by stanley depth, Ştefan [11] showed that $\text{sdepth}(S/I_n) = \lceil \frac{n}{3} \rceil$. In [2], Cimpoeaş proved that $\text{depth}(S/J_n) = \lceil \frac{n-1}{3} \rceil$ and $\text{sdepth}(S/J_n) = \lceil \frac{n-1}{3} \rceil$ for $n \equiv 0 \pmod{3}$ and $n \equiv 2 \pmod{3}$. He also proved that $\lceil \frac{n-1}{3} \rceil \leq \text{sdepth}(S/J_n) \leq \lceil \frac{n}{3} \rceil$ for $n \equiv 1 \pmod{3}$. Let I and J be the edge ideals associated to the graph consisting of the union of line graphs $L_{3r_1}, \dots, L_{3r_{k_1}}, L_{3s_1+1}, \dots, L_{3s_{k_2}+1}$ and $L_{3t_1+2}, \dots, L_{3t_{k_3}+2}$ with a common vertex, respectively, the graph consisting of the union of cycle graphs $C_{3r_1}, \dots, C_{3r_{k_1}}, C_{3s_1+1}, \dots, C_{3s_{k_2}+1}$ and $C_{3t_1+2}, \dots, C_{3t_{k_3}+2}$ with a common vertex, then using similar techniques, we prove that

$$(1) \quad \text{sdepth}\left(\frac{S}{I}\right) \geq \text{depth}\left(\frac{S}{I}\right) = \begin{cases} \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i, & \text{if } k_3 = 0, \\ \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i + 1, & \text{otherwise;} \end{cases}$$

$$(2) \quad \text{sdepth}\left(\frac{S}{I}\right) \leq \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i + 1.$$

In the fourth section, we prove that

$$(1) \quad \text{sdepth}\left(\frac{S}{J}\right) \geq \text{depth}\left(\frac{S}{J}\right) = \begin{cases} \sum_{i=1}^{k_2} s_i, & \text{if } k_1 = k_3 = 0, \\ \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i - k_1 + 1, & \text{otherwise;} \end{cases}$$

$$(2) \quad \text{sdepth}\left(\frac{S}{J}\right) \leq \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i - k_1 + 1;$$

$$(3) \quad \text{sdepth}\left(\frac{J}{I}\right) \geq \text{depth}\left(\frac{J}{I}\right) = \begin{cases} \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_3} t_i - k_1 + 2, & \text{if } k_2 = 0, \\ \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i - k_1 + 1, & \text{otherwise.} \end{cases}$$

2. PRELIMINARIES

We first recall some definitions about graphs and their edge ideals in order to make this paper self-contained. However, for more details on the notions, we refer the reader to [13, 14].

Definition 2.1. Let $G_i = (V(G_i), E(G_i))$ be some graphs with vertex set $V(G_i)$ and edge set $E(G_i)$ for $1 \leq i \leq k$. The union of graphs G_1, G_2, \dots, G_k , written as $G_1 \cup G_2 \cup \dots \cup G_k$, is the graph with vertex set $\bigcup_{i=1}^k V(G_i)$ and edge set $\bigcup_{i=1}^k E(G_i)$.

Definition 2.2. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{x_1, \dots, x_n\}$ and edge set $E(G)$. Suppose that x_1, \dots, x_n are variables over the field K . The edge ideal of graph G in the polynomial ring $S = K[x_1, \dots, x_n]$ is the squarefree monomial ideal

$$I(G) = (\{x_i x_j \mid \{x_i, x_j\} \in E(G)\}).$$

For the sake of simplicity, we will use the same notation $x_i x_j$ for the monomial and for the corresponding edge of graph G .

Definition 2.3. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{x_1, \dots, x_m\}$ and edge set $E(G)$. Then G is called a line graph of length m , denoted by L_m , if its edge set $E = \{x_i x_{i+1} \mid 1 \leq i \leq m-1\}$. Similarly, if $m \geq 3$, then G is called a cycle graph of length m , denoted by C_m , if its edge set $E = \{x_i x_{i+1} \mid 1 \leq i \leq m-1\} \cup \{x_m x_1\}$.

We recall the well known Depth Lemma, see for instance [13, Lemma 1.3.9] or [12, Lemma 3.1.4].

Lemma 2.4. (*Depth Lemma*) Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be a short exact sequence of modules over a local ring S , or a Noetherian graded ring with S_0 local, then

- (i) $\text{depth}(M) \geq \min\{\text{depth}(L), \text{depth}(N)\}$;
- (ii) $\text{depth}(L) \geq \min\{\text{depth}(M), \text{depth}(N) + 1\}$;
- (iii) $\text{depth}(N) \geq \min\{\text{depth}(L) - 1, \text{depth}(M)\}$.

The most of the statements of the Depth Lemma are wrong if we replace depth by Stanley depth. Rauf [8] proved the analog of Lemma 2.4 (i) for Stanley depth.

Lemma 2.5. Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be a short exact sequence of finitely generated \mathbb{Z}^n -graded S -modules. Then

$$s\text{depth}(M) \geq \min\{s\text{depth}(L), s\text{depth}(N)\}.$$

Using Depth Lemma, Morey in [7] proved the following result.

Lemma 2.6. Let L_m be a line graph of length m and $I(L_m)$ its edge ideal. Then $\text{depth}(S/I(L_m)) = \lceil \frac{m}{3} \rceil$.

Replacing depth by Stanley depth, Ştefan in [11] showed the following result.

Lemma 2.7. Let L_m be a line graph of length m and $I(L_m)$ its edge ideal. Then $s\text{depth}(S/I(L_m)) = \lceil \frac{m}{3} \rceil$.

3. DEPTH AND STANLEY DEPTH OF THE EDGE IDEALS OF SOME m -LINE GRAPHS WITH A COMMON VERTEX

In this section, we will give some formulas for depth and Stanley depth of the quotient rings of the edge ideals of some m -line graphs with a common vertex. We assume that G is the m -line graph formed by joining m line graphs $L_{3r_1}, \dots, L_{3r_{k_1}}, L_{3s_1+1}, \dots, L_{3s_{k_2}+1}, L_{3t_1+2}, \dots, L_{3t_{k_3}+2}$ at a common vertex, where $k_1 + k_2 + k_3 = m$ and $k_i \geq 0$ for $i = 1, 2, 3$. We adopt the following notation to edges of graph G :

$$E(L_{3r_i}) = \{x_1 x_{2,i}, x_{2,i} x_{3,i}, \dots, x_{3r_i-1,i} x_{3r_i,i}\} \text{ for any } 1 \leq i \leq k_1 \text{ and } r_1 \leq \dots \leq r_{k_1},$$

$$E(L_{3s_i+1,i}) = \{x_1 y_{2,i}, y_{2,i} y_{3,i}, \dots, y_{3s_i,i} y_{3s_i+1,i}\} \text{ for any } 1 \leq i \leq k_2 \text{ and } s_1 \leq \dots \leq s_{k_2},$$

$$E(L_{3t_i+2,i}) = \{x_1 z_{2,i}, z_{2,i} z_{3,i}, \dots, z_{3t_i+1,i} z_{3t_i+2,i}\} \text{ for all } 1 \leq i \leq k_3 \text{ and } t_1 \leq \dots \leq t_{k_3}.$$

Set K be any field, $S = K[x_1, x_{2,1}, \dots, x_{3r_1-1,1}, \dots, x_{2,k_1}, \dots, x_{3r_{k_1},k_1}, y_{2,1}, \dots, y_{3s_1+1,1}, \dots, y_{2,k_2}, \dots, y_{3s_{k_2}+1,k_2}, z_{2,1}, \dots, z_{3t_1+2,1}, \dots, z_{2,k_3}, \dots, z_{3t_{k_3}+2,k_3}]$ the polynomial ring. The edge ideal of graph G is $I = (x_1 x_{2,1}, x_{2,1} x_{3,1}, \dots, x_{3r_1-1,1} x_{3r_1,1}, \dots, x_1 x_{2,k_1}, x_{2,k_1} x_{3,k_1}, \dots, x_{3r_{k_1}-1,k_1} x_{3r_{k_1},k_1}, x_1 y_{2,1}, y_{2,1} y_{3,1}, \dots, y_{3s_1,1} y_{3s_1+1,1}, \dots, x_1 y_{2,k_2}, y_{2,k_2} y_{3,k_2}, \dots, y_{3s_{k_2},k_2} y_{3s_{k_2}+1,k_2}, x_1 z_{2,1}, z_{2,1} z_{3,1}, \dots, z_{3t_1+1,1} z_{3t_1+2,1}, \dots, x_1 z_{2,k_3}, \dots, z_{3t_{k_3}+1,k_3} z_{3t_{k_3}+2,k_3})$.

Example 3.1. The following graph G is the union of 5 line graphs L_3, L_4, L_5, L_6 and L_7 with a common vertex x_1 .

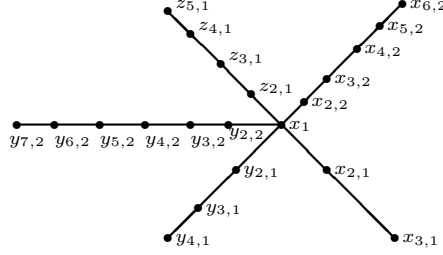


Figure 1

The edge ideal of graph G is $I = (x_1x_{2,1}, x_{2,1}x_{3,1}, x_1x_{2,2}, x_{2,2}x_{3,2}, x_{3,2}x_{4,2}, x_{4,2}x_{5,2}, x_{5,2}x_{6,2}, x_1y_{2,1}, y_{2,1}y_{3,1}, y_{3,1}y_{4,1}, x_1y_{2,2}, y_{2,2}y_{3,2}, y_{3,2}y_{4,2}, y_{4,2}y_{5,2}, y_{5,2}y_{6,2}, y_{6,2}y_{7,2}, x_1z_{2,1}, z_{2,1}z_{3,1}, z_{3,1}z_{4,1}, z_{4,1}z_{5,1})$.

We need the following lemma (See [8, Theorem 3.1]).

Lemma 3.2. Let $I \subset S_1 = K[x_1, \dots, x_m]$, $J \subset S_2 = K[y_1, \dots, y_n]$ be monomial ideals and $S = K[x_1, \dots, x_m, y_1, \dots, y_n]$. Then

$$sdepth(S/(IS, JS)) \geq sdepth(S_1/IS_1) + sdepth(S_2/JS_2).$$

Now, we prove the main results of this section. We adopt the following convention: whenever, in a sum, the index runs from 1 to 0, the sum has to be taken equal to zero.

Theorem 3.3. Let G be a graph consisting of the union of line graphs $L_{3r_1}, \dots, L_{3r_{k_1}}, L_{3s_1+1}, \dots, L_{3s_{k_2}+1}$ and $L_{3t_1+2}, \dots, L_{3t_{k_3}+2}$ with a common vertex x_1 , where $k_i \geq 0$ for $i = 1, 2, 3$. Let I be its edge ideal. Then

$$sdepth\left(\frac{S}{I}\right) \geq depth\left(\frac{S}{I}\right) = \begin{cases} \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i, & \text{if } k_3 = 0, \\ \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i + 1, & \text{otherwise.} \end{cases}$$

In particular, S/I satisfies the Stanley conjecture.

Proof. Note that $(I : x_1) = (x_{2,1}, \dots, x_{2,k_1}, y_{2,1}, \dots, y_{2,k_2}, z_{2,1}, \dots, z_{2,k_3}, x_{3,1}x_{4,1}, \dots, x_{3r_1-1,1}x_{3r_1,1}, \dots, x_{3,k_1}x_{4,k_1}, \dots, x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1}, y_{3,1}y_{4,1}, \dots, y_{3s_1,1}y_{3s_1+1,1}, \dots, y_{3,k_2}y_{4,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2}, z_{3,1}z_{4,1}, \dots, z_{3t_1+1,1}z_{3t_1+2,1}, \dots, z_{3,k_3}z_{4,k_3}, \dots, z_{3t_{k_3}+1,k_3}z_{3t_{k_3}+2,k_3})$ and $(I, x_1) = (x_{2,1}x_{3,1}, \dots, x_{3r_1-1,1}x_{3r_1,1}, \dots, x_{2,k_1}x_{3,k_1}, \dots, x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1}, y_{2,1}y_{3,1}, \dots, y_{3s_1,1}y_{3s_1+1,1}, \dots, y_{2,k_2}y_{3,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2}, z_{2,1}z_{3,1}, \dots, z_{3t_1+1,1}z_{3t_1+2,1}, \dots, z_{2,k_3}z_{3,k_3}, \dots, z_{3t_{k_3}+1,k_3}z_{3t_{k_3}+2,k_3}, x_1)$, thus we get that

$$\begin{aligned} \frac{S}{(I : x_1)} &\cong \frac{K[x_{3,1}, \dots, x_{3r_1,1}]}{(x_{3,1}x_{4,1}, \dots, x_{3r_1-1,1}x_{3r_1,1})} \otimes_K \cdots \otimes_K \frac{K[x_{3,k_1}, \dots, x_{3r_{k_1},k_1}]}{(x_{3,k_1}x_{4,k_1}, \dots, x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1})} \\ &\otimes_K \frac{K[y_{3,1}, \dots, y_{3s_1+1,1}]}{(y_{3,1}y_{4,1}, \dots, y_{3s_1,1}y_{3s_1+1,1})} \otimes_K \cdots \otimes_K \frac{K[y_{3,k_2}, \dots, y_{3s_{k_2}+1,k_2}]}{(y_{3,k_2}y_{4,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})} \\ &\otimes_K \frac{K[z_{3,1}, \dots, z_{3t_1+2,1}]}{(z_{3,1}z_{4,1}, \dots, z_{3t_1+1,1}z_{3t_1+2,1})} \otimes_K \cdots \otimes_K \frac{K[z_{3,k_3}, \dots, z_{3t_{k_3}+2,k_3}]}{(z_{3,k_3}z_{4,k_3}, \dots, z_{3t_{k_3}+1,k_3}z_{3t_{k_3}+2,k_3})} \otimes_K K[x_1], \end{aligned}$$

and

$$\begin{aligned} \frac{S}{(I, x_1)} &\cong \frac{K[x_{2,1}, \dots, x_{3r_1,1}]}{(x_{2,1}x_{3,1}, \dots, x_{3r_1-1,1}x_{3r_1,1})} \otimes_K \cdots \otimes_K \frac{K[x_{2,k_1}, \dots, x_{3r_{k_1},k_1}]}{(x_{2,k_1}x_{3,k_1}, \dots, x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1})} \\ &\otimes_K \frac{K[y_{2,1}, \dots, y_{3s_1+1,1}]}{(y_{2,1}y_{3,1}, \dots, y_{3s_1,1}y_{3s_1+1,1})} \otimes_K \cdots \otimes_K \frac{K[y_{2,k_2}, \dots, y_{3s_{k_2}+1,k_2}]}{(y_{2,k_2}y_{3,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})} \\ &\otimes_K \frac{K[z_{2,1}, \dots, z_{3t_1+2,1}]}{(z_{2,1}z_{3,1}, \dots, z_{3t_1+1,1}z_{3t_1+2,1})} \otimes_K \cdots \otimes_K \frac{K[z_{2,k_3}, \dots, z_{3t_{k_3}+2,k_3}]}{(z_{2,k_3}z_{3,k_3}, \dots, z_{3t_{k_3}+1,k_3}z_{3t_{k_3}+2,k_3})}. \end{aligned}$$

Therefore, by Lemmas 2.6, 2.7, 3.2 and [13, Proposition 2.2.20, Theorem 2.2.21], we obtain that $\text{sdepth}(\frac{S}{(I, x_1)}) \geq \text{depth}(\frac{S}{(I, x_1)}) = \sum_{i=1}^{k_1} \lceil \frac{3r_i-2}{3} \rceil + \sum_{i=1}^{k_2} \lceil \frac{3s_i-1}{3} \rceil + \sum_{i=1}^{k_3} \lceil \frac{3t_i}{3} \rceil + 1 = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i + 1$, and $\text{sdepth}(\frac{S}{(I, x_1)}) \geq \text{depth}(\frac{S}{(I, x_1)}) = \sum_{i=1}^{k_1} \lceil \frac{3r_i-1}{3} \rceil + \sum_{i=1}^{k_2} \lceil \frac{3s_i}{3} \rceil + \sum_{i=1}^{k_3} \lceil \frac{3t_i+1}{3} \rceil = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i + k_3$.

Using Lemma 2.5 on the short exact sequence

$$(1) \quad 0 \longrightarrow S/(I : x_1) \xrightarrow{x_1} S/I \longrightarrow S/(I, x_1) \longrightarrow 0,$$

we conclude that $\text{sdepth}(\frac{S}{I}) \geq \begin{cases} \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i, & \text{if } k_3 = 0; \\ \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i + 1, & \text{otherwise.} \end{cases}$

If $k_3 \neq 0$, then $\text{depth}(\frac{S}{(I, x_1)}) \geq \text{depth}(\frac{S}{(I, x_1)})$. Using Lemma 2.4 on the short exact sequence (1), it follows that $\text{depth}(\frac{S}{(I, x_1)}) = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i + 1$.

Assume that $k_3 = 0$. We claim that we have the S -module isomorphism

$$\begin{aligned} \frac{(I : x_1)}{I} &\cong \bigoplus_{i=1}^{k_1} x_{2,i} \left(\frac{K[x_{3,1}, \dots, x_{3r_1,1}]}{(x_{3,1}x_{4,1}, \dots, x_{3r_1-1,1}x_{3r_1,1})} \otimes_K \cdots \otimes_K \frac{K[x_{3,i-1}, \dots, x_{3r_{i-1},i-1}]}{(x_{3,i-1}x_{4,i-1}, \dots, x_{3r_{i-1}-1,i-1}x_{3r_{i-1},i-1})} \right. \\ &\otimes_K \frac{K[x_{4,i}, \dots, x_{3r_i,i}]}{(x_{4,i}x_{5,i}, \dots, x_{3r_i-1,i}x_{3r_i,i})} \otimes_K \frac{K[x_{2,i+1}, \dots, x_{3r_{i+1},i+1}]}{(x_{2,i+1}x_{3,i+1}, \dots, x_{3r_{i+1}-1,i+1}x_{3r_{i+1},i+1})} \otimes_K \cdots \\ &\otimes_K \frac{K[x_{2,k_1}, \dots, x_{3r_{k_1},k_1}]}{(x_{2,k_1}x_{3,k_1}, \dots, x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1})} \otimes_K \frac{K[y_{2,1}, \dots, y_{3s_1+1,1}]}{(y_{2,1}y_{3,1}, \dots, y_{3s_1,1}y_{3s_1+1,1})} \otimes_K \cdots \\ &\otimes_K \frac{K[y_{2,k_2}, \dots, y_{3s_{k_2}+1,k_2}]}{(y_{2,k_2}y_{3,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})} \Big) [x_{2,i}] \oplus \left(\bigoplus_{i=1}^{k_2} y_{2,i} \left(\frac{K[x_{3,1}, \dots, x_{3r_1,1}]}{(x_{3,1}x_{4,1}, \dots, x_{3r_1-1,1}x_{3r_1,1})} \otimes_K \cdots \right. \right. \\ &\otimes_K \frac{K[x_{3,k_1}, \dots, x_{3r_{k_1},k_1}]}{(x_{3,k_1}x_{4,k_1}, \dots, x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1})} \otimes_K \frac{K[y_{3,1}, \dots, y_{3s_1+1,1}]}{(y_{3,1}y_{4,1}, \dots, y_{3s_1,1}y_{3s_1+1,1})} \otimes_K \cdots \\ &\otimes_K \frac{K[y_{3,i-1}, \dots, y_{3s_{i-1}+1,i-1}]}{(y_{3,i-1}y_{4,i-1}, \dots, y_{3s_{i-1}-1,i-1}y_{3s_{i-1}+1,i-1})} \otimes_K \frac{K[y_{4,i}, \dots, y_{3s_i+1,i}]}{(y_{4,i}y_{5,i}, \dots, y_{3s_i,i}y_{3s_i+1,i})} \otimes_K \cdots \\ &\left. \left. \otimes_K \frac{K[y_{2,i+1}, \dots, y_{3s_{i+1}+1,i+1}]}{(y_{2,i+1}y_{3,i+1}, \dots, y_{3s_{i+1},i+1}y_{3s_{i+1}+1,i+1})} \otimes_K \cdots \otimes_K \frac{K[y_{2,k_2}, \dots, y_{3s_{k_2}+1,k_2}]}{(y_{2,k_2}y_{3,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})} \right) [y_{2,i}], \end{aligned}$$

where $x_{i,0} = y_{j,0} = 0$ for $3 \leq i \leq k_1$, $3 \leq j \leq k_2$. Indeed, if $u \in (I : x_1)$ is a monomial such that $u \notin I$, then $x_{2,i}|u$ or $y_{2,j}|u$ for some $1 \leq i \leq k_1$ or $1 \leq j \leq k_2$.

If $x_{2,1}|u$, then we can write u as $u = x_{2,1}^\alpha v$ with $\alpha \geq 1$ and $x_{2,1} \nmid v$. Since $u \notin I$, we have that $v \in K[x_{4,1}, \dots, x_{3r_1,1}, x_{2,2}, \dots, x_{3r_2,2}, \dots, x_{2,k_1}, \dots, x_{3r_{k_1},k_1}, y_{2,1}, \dots, y_{3s_1+1,1}, \dots, y_{2,k_2}, \dots, y_{3s_{k_2}+1,k_2}]$ and

$v \notin (x_{4,1}x_{5,1}, \dots, x_{3r_1-1,1}x_{3r_1,1}, x_{2,2}x_{3,2}, \dots, x_{3r_2-1,2}x_{3r_2,2}, \dots, x_{2,k_1}x_{3,k_1}, \dots, x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1}, y_{2,1}y_{3,1}, \dots, y_{3s_1,1}y_{3s_1+1,1}, \dots, y_{2,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})$. Similarly, if $x_{2,2}|u$ and $x_{2,1} \nmid u$, then $u = x_{2,2}^\alpha v$ with $\alpha \geq 1$ and $v \in K[x_{3,1}, \dots, x_{3r_1,1}, x_{4,2}, \dots, x_{3r_2,2}, x_{2,3}, \dots, x_{3r_3,3}, \dots, x_{2,k_1}, \dots, x_{3r_{k_1},k_1}, y_{2,1}, \dots, y_{3s_1+1,1}, \dots, y_{2,k_2}, \dots, y_{3s_{k_2}+1,k_2}]$ and $v \notin (x_{3,1}x_{4,1}, \dots, x_{3r_1-1,1}x_{3r_1,1}, x_{4,2}x_{5,2}, \dots, x_{3r_2-1,2}x_{3r_2,2}, x_{2,3}x_{3,3}, \dots, x_{3r_3-1,3}x_{3r_3,3}, \dots, x_{2,k_1}x_{3,k_1}, \dots, x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1}, y_{2,1}y_{3,1}, \dots, y_{3s_1,1}y_{3s_1+1,1}, \dots, y_{2,k_2}y_{3,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})$. Other cases can be shown in a similar way as the above.

Therefore, by [13, Proposition 2.2.20, Theorem 2.2.21] and Lemma 2.6, it follows that $\text{depth}(\frac{I:x_1}{I}) = \sum_{i=1}^{k_1-1} \lceil \frac{3r_i-2}{3} \rceil + \lceil \frac{3r_{k_1}-3}{3} \rceil + \sum_{i=1}^{k_2-1} \lceil \frac{3s_i-1}{3} \rceil + \lceil \frac{3s_{k_2}-2}{3} \rceil + 1 = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i$.

Now, using Lemma 2.4 on the short exact sequence

$$(2) \quad 0 \longrightarrow (I : x_1)/I \longrightarrow S/I \longrightarrow S/(I : x_1) \longrightarrow 0,$$

this completes the proof. \square

Assume $n = \sum_{i=1}^{k_1} 3r_i + \sum_{i=1}^{k_2} (3s_i + 1) + \sum_{i=1}^{k_3} (3t_i + 2) - (k_1 + k_2 + k_3) + 1$. We identify S/I with the \mathbb{Z}^n -graded K -subvector space I^c of S which is generated by all monomials $u \in S \setminus I$. Set the set

$$P = \{a \in \mathbb{N}^n : x^a \in I^c \text{ and } x^a | x_1 \prod_{\substack{2 \leq i \leq 3r_j, \\ 1 \leq j \leq k_1}} x_{i,j} \prod_{\substack{2 \leq i \leq 3s_j + 1, \\ 1 \leq j \leq k_2}} y_{i,j} \prod_{\substack{1, 2 \leq i \leq 3t_j + 2, \\ 1 \leq j \leq k_3}} z_{i,j}\},$$

where $a = (a(1), a(2, 1), \dots, a(3r_1, 1), \dots, a(2, k_1), \dots, a(3r_{k_1}, k_1), b(2, 1), \dots, b(3s_1 + 1, 1), \dots, b(2, k_2), \dots, b(3s_{k_2} + 1, k_2), c(2, 1), \dots, c(3t_1 + 2, 1), \dots, c(2, k_3), \dots, c(3t_{k_3} + 2, k_3)) \in \mathbb{N}^n$ and

$$x^a = x_1^{a(1)} \prod_{\substack{2 \leq i \leq 3r_j, \\ 1 \leq j \leq k_1}} x_{i,j}^{a(i,j)} \prod_{\substack{2 \leq i \leq 3s_j + 1, \\ 1 \leq j \leq k_2}} y_{i,j}^{b(i,j)} \prod_{\substack{1, 2 \leq i \leq 3t_j + 2, \\ 1 \leq j \leq k_3}} z_{i,j}^{c(i,j)}.$$

Consider the natural partial order on \mathbb{N}^n which is given by componentwise comparison, i.e. if $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$, then $a \geq b$ if and only if $a_i \geq b_i$ for all $i = 1, \dots, n$. With respect to the partial order induced on P , it becomes a poset where $a \geq a'$ if and only if $x^{a'} | x^a$.

Let $\mathcal{P} : P = \bigcup_{i=1}^r [F_i, G_i]$ be a partition of P . We denote $\text{sdepth}(\mathcal{P}) = \min\{|G_i| : 1 \leq i \leq r\}$. Also, we define the Stanley depth of P , to be the number

$$\text{sdepth}(P) = \max\{\text{sdepth}(\mathcal{P}) \mid \mathcal{P} \text{ is a partition of } P\}.$$

Herzog, Vlădoiu and Zheng proved in [6] that $\text{sdepth}(\frac{S}{I}) = \text{sdepth}(P)$. Now, for $d \in \mathbb{N}$ and $\sigma \in P$, we denote

$$\mathcal{P}_d := \{a \in P : |a| = d\} \quad \text{and} \quad \mathcal{P}_{d,\sigma} := \{a \in \mathcal{P}_d : x^\sigma | x^a\},$$

where for $a = (a(1), a(2, 1), \dots, a(3r_1, 1), \dots, a(2, k_1), \dots, a(3r_{k_1}, k_1), b(2, 1), \dots, b(3s_1 + 1, 1), \dots, b(2, k_2), \dots, b(3s_{k_2} + 1, k_2), c(2, 1), \dots, c(3t_1 + 2, 1), \dots, c(2, k_3), \dots, c(3t_{k_3} + 2, k_3)) \in \mathbb{N}^n$, $|a| := a(1) + \sum_{j=1}^{k_1} \sum_{i=2}^{3r_j} a(i, j) +$

$$\sum_{j=1}^{k_2} \sum_{i=2}^{3s_j+1} b(i, j) + \sum_{j=1}^{k_3} \sum_{i=2}^{3t_j+2} c(i, j).$$

With these notations, we are able to prove the following result.

Theorem 3.4. *Let G be a graph as in Theorem 3.3 and I be its edge ideal. Then*

$$sdepth\left(\frac{S}{I}\right) \leq \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i + 1.$$

Proof. Firstly, we claim: if $\sigma \in \mathcal{P}$ such that $\mathcal{P}_{d,\sigma} = \emptyset$, then $sdepth(\mathcal{P}) < d$.

Indeed, let $\mathcal{P} : P = \bigcup_{i=1}^r [F_i, G_i]$ be a partition of P with $sdepth(\mathcal{P}) = sdepth(P)$. Since $\sigma \in P$, it follows that $\sigma \in [F_i, G_i]$ for some i . If $|G_i| \geq d$, then it follows that $\mathcal{P}_{d,\sigma} \neq \emptyset$, since there exist some subsets in the interval $[F_i, G_i]$ of cardinality d which contain σ , a contradiction! Thus, $|G_i| < d$ and therefore $sdepth(\mathcal{P}) < d$.

We set $d = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i + 1$ and $\sigma = (a(1), a(2, 1), \dots, a(3r_1, 1), \dots, a(2, k_1), \dots, a(3r_{k_1}, k_1), b(2, 1), \dots, b(3s_1+1, 1), \dots, b(2, k_2), \dots, b(3s_{k_2}+1, k_2), c(2, 1), \dots, c(3t_1+2, 1), \dots, c(2, k_3), \dots, c(3t_{k_3}+2, k_3)) \in \mathbb{N}^n$, where $a(1) = 1$, for any $1 \leq i \leq k_1$, $1 \leq j \leq r_i - 1$, $a(l, i) = \begin{cases} 1 : l = 3j + 1 \text{ or } l = 3r_i \\ 0 : \text{otherwise} \end{cases}$, for any $1 \leq i \leq k_2$, $1 \leq j \leq s_i$, $b(l, i) = \begin{cases} 1 : l = 3j + 1 \\ 0 : \text{otherwise} \end{cases}$, and for any $1 \leq i \leq k_3$, $1 \leq j \leq t_i$, $c(l, i) = \begin{cases} 1 : l = 3j + 1 \\ 0 : \text{otherwise} \end{cases}$. We obtain that $\mathcal{P}_{d+1,\sigma} = \emptyset$. Indeed, if monomial

$$u = x_1 \prod_{i=1}^{k_1} (x_{3r_i, i} \prod_{j=1}^{r_i-1} x_{3j+1, i}) \prod_{i=1}^{k_2} (\prod_{j=1}^{s_i} y_{3j+1, i}) \prod_{i=1}^{k_3} (\prod_{j=1}^{t_i} z_{3j+1, i}),$$

one can easily see that if $a = (a(1), a(2, 1), \dots, a(3r_1, 1), \dots, a(2, k_1), \dots, a(3r_{k_1}, k_1), b(2, 1), \dots, b(3s_1+1, 1), \dots, b(2, k_2), \dots, b(3s_{k_2}+1, k_2), c(2, 1), \dots, c(3t_1+2, 1), \dots, c(2, k_3), \dots, c(3t_{k_3}+2, k_3)) \in \mathbb{N}^n$ such that $a(l, i) \neq 0$ for some $l \notin \{3j+1, 3r_i \mid 1 \leq j \leq r_i - 1, 1 \leq i \leq k_1\}$ or $b(l, i) \neq 0$ for some $l \notin \{3j+1 \mid 1 \leq j \leq s_i, 1 \leq i \leq k_2\}$ or $c(l, i) \neq 0$ for some $l \notin \{3j+1 \mid 1 \leq j \leq t_i, 1 \leq i \leq k_3\}$, then $u \cdot x^a \in I$. Therefore, by previous remark, $sdepth\left(\frac{S}{I}\right) = sdepth(\mathcal{P}) \leq d = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i + 1$, as required. \square

4. DEPTH AND STANLEY DEPTH OF THE EDGE IDEALS OF SOME m -CYCLIC GRAPHS WITH A COMMON VERTEX

In this section, we will give some formulas for depth and Stanley depth of the quotient rings of the edge ideals of some m -cyclic graphs with a common vertex. We assume that G is the m -cyclic graph formed by joining m cycles $C_{3r_1}, \dots, C_{3r_{k_1}}, C_{3s_1+1}, \dots, C_{3s_{k_2}+1}, C_{3t_1+2}, \dots, C_{3t_{k_3}+2}$ at a common vertex, where $k_1 + k_2 + k_3 = m$ and $k_i \geq 0$ for $i = 1, 2, 3$. We adopt the following notation to edges of graph G : $E(C_{3r_i, i}) = \{x_1 x_{2, i}, x_{2, i} x_{3, i}, \dots, x_{3r_i, i} x_1\}$ for any $1 \leq i \leq k_1$ and $r_1 \leq r_2 \leq \dots \leq r_{k_1}$, $E(C_{3s_i+1, i}) = \{x_1 y_{2, i}, y_{2, i} y_{3, i}, \dots, y_{3s_i+1, i} x_1\}$ for any $1 \leq i \leq k_2$ and $s_1 \leq \dots \leq s_{k_2}$, $E(C_{3t_i+2, i}) = \{x_1 z_{2, i}, z_{2, i} z_{3, i}, \dots, z_{3t_i+2, i} x_1\}$ for all $1 \leq i \leq k_3$ and $t_1 \leq t_2 \leq \dots \leq t_{k_3}$. Let K be any field, $S = K[x_1, x_{2, 1}, \dots, x_{3r_1, 1}, \dots, x_{2, k_1}, \dots, x_{3r_{k_1}, k_1}, y_{2, 1}, \dots, y_{3s_1+1, 1}, \dots, y_{2, k_2}, \dots, y_{3s_{k_2}+1, k_2}, z_{2, 1}, \dots, z_{3t_1+2, 1}, \dots, z_{2, k_3}, \dots, z_{3t_{k_3}+2, k_3}]$ the polynomial ring. Then the edge ideal of graph G is $J = (x_1 x_{2, 1}, x_{2, 1} x_{3, 1}, \dots, x_{3r_1, 1} x_1, \dots, x_1 x_{2, k_1}, x_{2, k_1} x_{3, k_1}, \dots, x_{3r_{k_1}, k_1} x_1, x_1 y_{2, 1}, y_{2, 1} y_{3, 1}, \dots, y_{3s_1+1, 1} x_1, \dots, x_1 y_{2, k_2}, y_{2, k_2} y_{3, k_2}, \dots, y_{3s_{k_2}+1, k_2} x_1, x_1 z_{2, 1}, z_{2, 1} z_{3, 1}, \dots, z_{3t_1+2, 1} x_1, \dots, x_1 z_{2, k_3}, z_{2, k_3} z_{3, k_3}, \dots, z_{3t_{k_3}+2, k_3} x_1)$.

Example 4.1. The following graph G is the union of 5 circle graphs C_3, C_4, C_5, C_6 and C_7 with a common vertex x_1 .

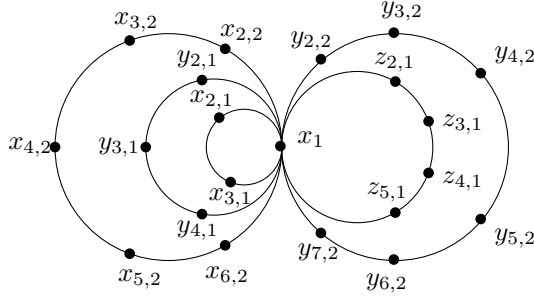


Figure 2

The edge ideal of graph G is $J = (x_1x_{2,1}, x_{2,1}x_{3,1}, x_{3,1}x_1, x_1x_{2,2}, x_{2,2}x_{3,2}, x_{3,2}x_{4,2}, x_{4,2}x_{5,2}, x_{5,2}x_{6,2}, x_{6,2}x_1, x_1y_{2,1}, y_{2,1}y_{3,1}, y_{3,1}y_{4,1}, y_{4,1}x_1, x_1y_{2,2}, y_{2,2}y_{3,2}, y_{3,2}y_{4,2}, y_{4,2}y_{5,2}, y_{5,2}y_{6,2}, y_{6,2}y_{7,2}, y_{7,2}x_1, x_1z_{2,1}, z_{2,1}z_{3,1}, z_{3,1}z_{4,1}, z_{4,1}z_{5,1}, z_{5,1}x_1)$.

Now, we prove the main results of this section.

Theorem 4.2. *Let G be a graph consisting of the union of cycle graphs $C_{3r_1}, \dots, C_{3r_{k_1}}, C_{3s_1+1}, \dots, C_{3s_{k_2}+1}$ and $C_{3t_1+2}, \dots, C_{3t_{k_3}+2}$ with a common vertex x_1 , where $k_i \geq 0$ for $i = 1, 2, 3$. Let J be its edge ideal. Then*

$$sdepth\left(\frac{S}{J}\right) \geq depth\left(\frac{S}{J}\right) = \begin{cases} \sum_{i=1}^{k_2} s_i, & \text{if } k_1 = k_3 = 0, \\ \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i - k_1 + 1, & \text{otherwise.} \end{cases}$$

In particular, S/J satisfies the Stanley conjecture.

Proof. Notice that $(J : x_1) = (x_{2,1}, \dots, x_{2,k_1}, x_{3r_1,1}, \dots, x_{3r_{k_1},k_1}, y_{2,1}, \dots, y_{2,k_2}, y_{3s_1+1,1}, \dots, y_{3s_{k_2}+1,k_2}, z_{2,1}, \dots, z_{2,k_3}, z_{3t_1+2,1}, \dots, z_{3t_{k_3}+2,k_3}, x_{3,1}x_{4,1}, \dots, x_{3r_1-2,1}x_{3r_1-1,1}, \dots, x_{3,k_1}x_{4,k_1}, \dots, x_{3r_{k_1}-2,k_1}x_{3r_{k_1}-1,k_1}, y_{3,1}y_{4,1}, \dots, y_{3s_1-1,1}y_{3s_1,1}, \dots, y_{3,k_2}y_{4,k_2}, \dots, y_{3s_{k_2}-1,k_2}y_{3s_{k_2},k_2}, z_{3,1}z_{4,1}, \dots, z_{3t_1,1}z_{3t_1+1,1}, \dots, z_{3,k_3}z_{4,k_3}, \dots, z_{3t_{k_3},k_3}z_{3t_{k_3}+1,k_3})$ and $(J, x_1) = (x_{2,1}x_{3,1}, \dots, x_{3r_1-1,1}x_{3r_1,1}, \dots, x_{2,k_1}x_{3,k_1}, \dots, x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1}, y_{2,1}y_{3,1}, \dots, y_{3s_1,1}y_{3s_1+1,1}, \dots, y_{2,k_2}y_{3,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2}, z_{2,1}z_{3,1}, \dots, z_{3t_1+1,1}z_{3t_1+2,1}, \dots, z_{2,k_3}z_{3,k_3}, \dots, z_{3t_{k_3}+1,k_3}z_{3t_{k_3}+2,k_3}, x_1)$, thus we get that

$$\begin{aligned} \frac{S}{(J : x_1)} &\cong \frac{K[x_{3,1}, \dots, x_{3r_1-1,1}]}{(x_{3,1}x_{4,1}, \dots, x_{3r_1-2,1}x_{3r_1-1,1})} \otimes_K \cdots \otimes_K \frac{K[x_{3,k_1}, \dots, x_{3r_{k_1}-1,k_1}]}{(x_{3,k_1}x_{4,k_1}, \dots, x_{3r_{k_1}-2,k_1}x_{3r_{k_1}-1,k_1})} \\ &\otimes_K \frac{K[y_{3,1}, \dots, y_{3s_1,1}]}{(y_{3,1}y_{4,1}, \dots, y_{3s_1-1,1}y_{3s_1,1})} \otimes_K \cdots \otimes_K \frac{K[y_{3,k_2}, \dots, y_{3s_{k_2},k_2}]}{(y_{3,k_2}y_{4,k_2}, \dots, y_{3s_{k_2}-1,k_2}y_{3s_{k_2},k_2})} \\ &\otimes_K \frac{K[z_{3,1}, \dots, z_{3t_1+1,1}]}{(z_{3,1}z_{4,1}, \dots, z_{3t_1,1}z_{3t_1+1,1})} \otimes_K \cdots \otimes_K \frac{K[z_{3,k_3}, \dots, z_{3t_{k_3}+1,k_3}]}{(z_{3,k_3}z_{4,k_3}, \dots, z_{3t_{k_3},k_3}z_{3t_{k_3}+1,k_3})} \otimes_K K[x_1], \end{aligned}$$

and

$$\begin{aligned} \frac{S}{(J, x_1)} &\cong \frac{K[x_{2,1}, \dots, x_{3r_1,1}]}{(x_{2,1}x_{3,1}, \dots, x_{3r_1-1,1}x_{3r_1,1})} \otimes_K \cdots \otimes_K \frac{K[x_{2,k_1}, \dots, x_{3r_{k_1},k_1}]}{(x_{2,k_1}x_{3,k_1}, \dots, x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1})} \\ &\otimes_K \frac{K[y_{2,1}, \dots, y_{3s_1+1,1}]}{(y_{2,1}y_{3,1}, \dots, y_{3s_1,1}y_{3s_1+1,1})} \otimes_K \cdots \otimes_K \frac{K[y_{2,k_2}, \dots, y_{3s_{k_2}+1,k_2}]}{(y_{2,k_2}y_{3,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})} \end{aligned}$$

$$\otimes_K \frac{K[z_{2,1}, \dots, z_{3t_1+2,1}]}{(z_{2,1}z_{3,1}, \dots, z_{3t_1+1,1}z_{3t_1+2,1})} \otimes_K \cdots \otimes_K \frac{K[z_{2,k_3}, \dots, z_{3t_{k_3}+2,k_3}]}{(z_{2,k_3}z_{3,k_3}, \dots, z_{3t_{k_3}+1,k_3}z_{3t_{k_3}+2,k_3})}.$$

Therefore, by Lemmas 2.6, 2.7, 3.2 and [13, Proposition 2.2.20, Theorem 2.2.21], we obtain that $\text{sdepth}(\frac{S}{J:x_1}) \geq \text{depth}(\frac{S}{J:x_1}) = \sum_{i=1}^{k_1} \lceil \frac{3r_i-3}{3} \rceil + \sum_{i=1}^{k_2} \lceil \frac{3s_i-2}{3} \rceil + \sum_{i=1}^{k_3} \lceil \frac{3t_i-1}{3} \rceil + 1 = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i - k_1 + 1$,

and $\text{sdepth}(\frac{S}{(J,x_1)}) \geq \text{depth}(\frac{S}{(J,x_1)}) = \sum_{i=1}^{k_1} \lceil \frac{3r_i-1}{3} \rceil + \sum_{i=1}^{k_2} \lceil \frac{3s_i}{3} \rceil + \sum_{i=1}^{k_3} \lceil \frac{3t_i+1}{3} \rceil = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i + k_3$.

Using Lemma 2.5 on the short exact sequence

$$(3) \quad 0 \longrightarrow S/(J : x_1) \xrightarrow{\cdot x_1} S/J \longrightarrow S/(J, x_1) \longrightarrow 0,$$

we conclude that $\text{sdepth}(\frac{S}{J}) \geq \begin{cases} \sum_{i=1}^{k_2} s_i, & \text{if } k_1 = k_3 = 0, \\ \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i - k_1 + 1, & \text{otherwise.} \end{cases}$

If $k_1 \neq 0$ or $k_3 \neq 0$, then $\text{depth}(\frac{S}{(J,x_1)}) \geq \text{depth}(\frac{S}{(J:x_1)})$. Using Lemma 2.4 on the short exact sequence

$$(3), \text{ it follows that } \text{depth}(\frac{S}{J}) = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i - k_1 + 1.$$

Assume that $k_1 = k_3 = 0$. We claim that there exists the S -module isomorphism

$$\begin{aligned} \frac{(J:x_1)}{J} &\cong y_{2,1} \left(\frac{K[y_{4,1}, \dots, y_{3s_1+1,1}, y_{2,2}, \dots, y_{3s_2+1,2}, \dots, y_{2,k_2}, \dots, y_{3s_{k_2}+1,k_2}]}{(y_{4,1}y_{5,1}, \dots, y_{3s_1,1}y_{3s_1+1,1}, y_{2,2}y_{3,2}, \dots, y_{2,k_2}y_{3,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})} \right) [y_{2,1}] \\ &\oplus y_{3s_1+1,1} \left(\frac{K[y_{3,1}, \dots, y_{3s_1-1,1}, y_{2,2}, \dots, y_{3s_2+1,2}, \dots, y_{2,k_2}, \dots, y_{3s_{k_2}+1,k_2}]}{(y_{3,1}y_{4,1}, \dots, y_{3s_1-2,1}y_{3s_1-1,1}, \dots, y_{2,k_2}y_{3,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})} \right) [y_{3s_1+1,1}] \\ &\oplus y_{2,2} \left(\frac{K[y_{3,1}, \dots, y_{3s_1,1}, y_{4,2}, \dots, y_{3s_2+1,2}, \dots, y_{2,k_2}, \dots, y_{3s_{k_2}+1,k_2}]}{(y_{3,1}y_{4,1}, \dots, y_{3s_1-1,1}y_{3s_1,1}, y_{4,2}y_{5,2}, \dots, y_{3s_2,2}y_{3s_2+1,2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})} \right) [y_{2,2}] \\ &\oplus y_{3s_2+1,2} \left(\frac{K[y_{3,1}, \dots, y_{3s_1,1}, y_{3,2}, \dots, y_{3s_2-1,2}, \dots, y_{2,k_2}, \dots, y_{3s_{k_2}+1,k_2}]}{(y_{3,1}y_{4,1}, \dots, y_{3,2}y_{4,2}, \dots, y_{3s_2-2,2}y_{3s_2-1,2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})} \right) [y_{3s_2+1,2}] \\ &\oplus \cdots \\ &\oplus y_{2,k_2} \left(\frac{K[y_{3,1}, \dots, y_{3s_1,1}, \dots, y_{3,k_2-1}, \dots, y_{3s_{k_2-1},k_2-1}, y_{4,k_2}, \dots, y_{3s_{k_2}+1,k_2}]}{(y_{3,1}y_{4,1}, \dots, y_{3s_1-1,1}y_{3s_1,1}, \dots, y_{4,k_2}y_{5,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})} \right) [y_{2,k_2}] \\ &\oplus y_{3s_{k_2}+1,k_2} \left(\frac{K[y_{3,1}, \dots, y_{3s_1,1}, \dots, y_{3,k_2-1}, \dots, y_{3s_{k_2-1},k_2-1}, y_{3,k_2}, \dots, y_{3s_{k_2}-1,k_2}]}{(y_{3,1}y_{4,1}, \dots, y_{3s_1-1,1}y_{3s_1,1}, \dots, y_{3,k_2}y_{4,k_2}, \dots, y_{3s_{k_2}-2,k_2}y_{3s_{k_2}-1,k_2})} \right) [y_{3s_{k_2}+1,k_2}] \end{aligned}$$

Indeed, if $u \in (J : x_1)$ is a monomial such that $u \notin J$, then there exists some $i \in \{1, \dots, k_2\}$ such that $y_{j,i} \mid u$, where $j = 2$ or $3s_i + 1$.

If $y_{2,1} \mid u$, then we can write u as $u = y_{2,1}^\alpha v$ with $\alpha \geq 1$ and $y_{2,1} \nmid v$. Since $u \notin J$, we have that $v \in K[y_{4,1}, \dots, y_{3s_1+1,1}, y_{2,2}, \dots, y_{3s_2+1,2}, \dots, y_{2,k_2}, \dots, y_{3s_{k_2}+1,k_2}]$ and $v \notin (y_{4,1}y_{5,1}, \dots, y_{3s_1,1}y_{3s_1+1,1}, y_{2,2}y_{3,2}, \dots, y_{3s_2,2}y_{3s_2+1,2}, \dots, y_{2,k_2}y_{3,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})$. Similarly, if $y_{3s_1+1,1} \mid u$ and $y_{2,1} \nmid u$, then $u = y_{3s_1+1,1}^\alpha v$ with $\alpha \geq 1$ and $v \in K[y_{3,1}, \dots, y_{3s_1-1,1}, y_{2,2}, \dots, y_{3s_2+1,2}, \dots, y_{2,k_2}, \dots, y_{3s_{k_2}+1,k_2}]$ and $v \notin (y_{3,1}y_{4,1}, \dots, y_{3s_1-2,1}y_{3s_1-1,1}, y_{2,2}y_{3,2}, \dots, y_{3s_2,2}y_{3s_2+1,2}, \dots, y_{2,k_2}y_{3,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})$. If u is a monomial such that $y_{2,2} \mid u$, $y_{2,1} \nmid u$ and $y_{3s_1+1,1} \nmid u$, then we have that $u = y_{2,2}^\alpha v$ with $\alpha \geq 1$ and $v \in K[y_{3,1}, \dots, y_{3s_1,1}, y_{4,2}, \dots, y_{3s_2+1,2}, \dots, y_{2,k_2}, \dots, y_{3s_{k_2}+1,k_2}]$ and $v \notin (y_{3,1}y_{4,1}, \dots, y_{3s_1-1,1}y_{3s_1,1}, y_{4,2}y_{5,2}, \dots, y_{3s_2,2}y_{3s_2+1,2}, \dots, y_{2,k_2}y_{3,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})$. Other cases can be shown in a similar way as the above.

Therefore, by [13, Proposition 2.2.20, Theorem 2.2.21] and Lemma 2.6, it follows that $\text{depth}(\frac{J:x_1}{J}) = \sum_{i=1}^{k_2-1} \lceil \frac{3s_i-2}{3} \rceil + \lceil \frac{3s_{k_2}-3}{3} \rceil + 1 = \sum_{i=1}^{k_2} s_i$.

Now, using Lemma 2.4 on the short exact sequence

$$(4) \quad 0 \longrightarrow (J : x_1)/J \longrightarrow S/J \longrightarrow S/(J : x_1) \longrightarrow 0,$$

this completes the proof. \square

Theorem 4.3. *Let G be a graph as in Theorem 4.2 and J be its edge ideal. Then*

$$s\text{depth}(\frac{S}{J}) \leq \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i - k_1 + 1.$$

Proof. Let $d = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i - k_1 + 1$ and $\sigma = (a(1), a(2, 1), \dots, a(3r_1, 1), \dots, a(2, k_1), \dots, a(3r_{k_1}, k_1), b(2, 1), \dots, b(3s_1+1, 1), \dots, b(2, k_2), \dots, b(3s_{k_2}+1, k_2), c(2, 1), \dots, c(3t_1+2, 1), \dots, c(2, k_3), \dots, c(3t_{k_3}+2, k_3)) \in \mathbb{N}^n$, where $a(1) = 1$, for any $1 \leq i \leq k_1$, $1 \leq j \leq r_i - 1$, $a(l, i) = \begin{cases} 1 : l = 3j + 1 \\ 0 : \text{otherwise} \end{cases}$, for any $1 \leq i \leq k_2$, $1 \leq j \leq s_i - 1$, $b(l, i) = \begin{cases} 1 : l = 3j + 1 \text{ or } l = 3s_i \\ 0 : \text{otherwise} \end{cases}$, and for any $1 \leq i \leq k_3$, $1 \leq j \leq t_i$, $c(l, i) = \begin{cases} 1 : l = 3j + 1 \\ 0 : \text{otherwise} \end{cases}$. From the proof of Theorem 3.4, it is enough to prove that $\mathcal{P}_{d+1, \sigma} = \emptyset$. Indeed, if monomial

$$u = x_1 \prod_{i=1}^{k_1} \left(\prod_{j=1}^{r_i-1} x_{3j+1, i} \right) \prod_{i=1}^{k_2} \left(y_{3s_i, i} \prod_{j=1}^{s_i-1} y_{3j+1, i} \right) \prod_{i=1}^{k_3} \left(\prod_{j=1}^{t_i} z_{3j+1, i} \right),$$

one can easily see that if $a = (a(1), a(2, 1), \dots, a(3r_1, 1), \dots, a(2, k_1), \dots, a(3r_{k_1}, k_1), b(2, 1), \dots, b(3s_1+1, 1), \dots, b(2, k_2), \dots, b(3s_{k_2}+1, k_2), c(2, 1), \dots, c(3t_1+2, 1), \dots, c(2, k_3), \dots, c(3t_{k_3}+2, k_3)) \in \mathbb{N}^n$ such that $a(l, i) \neq 0$ for some $l \notin \{3j + 1 \mid 1 \leq j \leq r_i - 1, 1 \leq i \leq k_1\}$ or $b(l, i) \neq 0$ for some $l \notin \{3j + 1, 3s_i \mid 1 \leq j \leq s_i, 1 \leq i \leq k_2\}$ or $c(l, i) \neq 0$ for some $l \notin \{3j + 1 \mid 1 \leq j \leq t_i, 1 \leq i \leq k_3\}$, then $u \cdot x^a \in I$. Therefore $\mathcal{P}_{d+1, \sigma} = \emptyset$, thus we obtain the required result. \square

Theorem 4.4. *Let G be a graph as in Theorem 4.2. Then*

$$s\text{depth}(\frac{J}{I}) \geq \text{depth}(\frac{J}{I}) = \begin{cases} \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_3} t_i - k_1 + 2, & \text{if } k_2 = 0, \\ \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i - k_1 + 1, & \text{otherwise.} \end{cases}$$

In particular, J/I satisfies the Stanley conjecture.

Proof. We have the S-module isomorphism:

$$\begin{aligned} \frac{J}{I} &\cong \bigoplus_{i=1}^{k_1} x_1 x_{3r_i, i} \left(\frac{K[x_{3,1}, \dots, x_{3r_1-1,1}, \dots, x_{3,i-1}, \dots, x_{3r_{i-1}-1, i-1}, x_{3,i}, \dots, \widehat{x}_{3r_{i-1}, i}, x_{3,i+1}, \dots, x_{3r_{i+1}, i+1}, \dots, x_{3, k_1}, \dots, x_{3r_{k_1}, k_1}]}{(x_{3,1}x_{4,1}, \dots, x_{3r_1-2,1}x_{3r_1-1,1}, \dots, x_{3,i}x_{4,i}, \dots, x_{3r_{i-3}, i}x_{3r_{i-2}, i}, x_{3,i+1}x_{4,i+1}, \dots, x_{3r_{k_1}-1, k_1}x_{3r_{k_1}, k_1})} \right) \\ &\otimes_K \frac{K[y_{3,1}, \dots, y_{3s_1+1,1}, \dots, y_{3,k_2}, \dots, y_{3s_{k_2}+1, k_2}]}{(y_{3,1}y_{4,1}, \dots, y_{3s_1-1,1}y_{3s_1+1,1}, \dots, y_{3s_{k_2}, k_2}y_{3s_{k_2}+1, k_2})} \otimes_K \frac{K[z_{3,1}, \dots, z_{3t_1+2,1}, \dots, z_{3, k_3}, \dots, z_{3t_{k_3}+2, k_3}]}{(z_{3,1}z_{4,1}, \dots, z_{3t_1+1,1}z_{3t_1+2,1}, \dots, z_{3s_{k_3}+1, k_3}z_{3t_{k_3}+2, k_3})} [x_1, x_{3r_i, i}] \\ &\oplus \left(\bigoplus_{i=1}^{k_2} x_1 y_{3s_i+1, i} \left(\frac{K[y_{3,1}, \dots, y_{3s_1,1}, \dots, y_{3,i-1}, \dots, y_{3s_{i-1}, i-1}, y_{3,i}, \dots, \widehat{y}_{3s_i, i}, y_{3,i+1}, \dots, y_{3s_{i+1}, i+1}, \dots, y_{3, k_2}, \dots, y_{3s_{k_2}+1, k_2}]}{(y_{3,1}y_{4,1}, \dots, y_{3s_1-1,1}y_{3s_1,1}, \dots, y_{3,i}y_{4,i}, \dots, y_{3s_{i-2}, i}y_{3s_{i-1}, i}, y_{3,i+1}y_{4,i+1}, \dots, y_{3s_{k_2}, k_2}y_{3s_{k_2}+1, k_2})} \right) \otimes_K \right. \end{aligned}$$

$$\begin{aligned} & \frac{K[x_{3,1}, \dots, x_{3r_1-1,1}, \dots, x_{3,k_1}, \dots, x_{3r_{k_1}-1,k_1}]}{(x_{3,1}x_{4,1}, \dots, x_{3r_1-2,1}x_{3r_1-1,1}, \dots, x_{3r_{k_1}-2,k_1}x_{3r_{k_1}-1,k_1})} \otimes_K \frac{K[z_{3,1}, \dots, z_{3t_1+2,1}, \dots, z_{3,k_3}, \dots, z_{3t_{k_3}+2,k_3}]}{(z_{3,1}z_{4,1}, \dots, z_{3t_1+1,1}z_{3t_1+2,1}, \dots, z_{3s_{k_3}+1,k_3}z_{3t_{k_3}+2,k_3})} (x_1, y_{3s_i+1}, i) \\ & \oplus \left(\bigoplus_{i=1}^{k_3} x_1 z_{3t_i+2}, i \left(\frac{K[z_{3,1}, \dots, z_{3t_1+1,1}, \dots, z_{3,i-1}, \dots, z_{3t_{i-1}+1,i-1}, z_{3,i}, \dots, z_{3t_i+1,i}, z_{3,i+1}, \dots, z_{3t_{i+1}+2,i+1}, \dots, z_{3,k_3}, \dots, z_{3t_{k_3}+2,k_3}]}{(z_{3,1}z_{4,1}, \dots, z_{3t_1+1,1}z_{3t_1+2,1}, \dots, z_{3,i}z_{4,i}, \dots, z_{3t_{i-1}+1,i}z_{3t_i,i}, z_{3,i+1}z_{4,i+1}, \dots, z_{3t_{k_3}+1,k_3}z_{3t_{k_3}+2,k_3})} \otimes_K \right. \right. \\ & \left. \left. \frac{K[x_{3,1}, \dots, x_{3r_1-1,1}, \dots, x_{3,k_1}, \dots, x_{3r_{k_1}-1,k_1}]}{(x_{3,1}x_{4,1}, \dots, x_{3r_1-2,1}x_{3r_1-1,1}, \dots, x_{3r_{k_1}-2,k_1}x_{3r_{k_1}-1,k_1})} \otimes_K \frac{K[y_{3,1}, \dots, y_{3s_1+1,1}, \dots, y_{3,k_2}, \dots, y_{3s_{k_2}+1,k_2}]}{(y_{3,1}y_{4,1}, \dots, y_{3s_1-1,1}y_{3s_1,1}, \dots, y_{3s_{k_2}-1,k_2}y_{3s_{k_2},k_2})} \right) (x_1, z_{3t_i+2}, i), \end{aligned}$$

where $x_{i,0} = y_{j,0} = z_{l,0} = 0$ for $3 \leq i \leq k_1$, $3 \leq j \leq k_2$ and $3 \leq l \leq k_3$.

Indeed, let $u \in J$ be a monomial such that $u \notin I$, then $x_1 x_{3r_i,i} | u$ or $x_1 y_{3s_j+1,j} | u$ or $x_1 z_{3t_l+2,l} | u$ for some $1 \leq i \leq k_1$ or $1 \leq j \leq k_2$ or $1 \leq l \leq k_3$.

If $x_1 x_{3r_1,1} | u$, then we can write u as $u = x_1^\alpha x_{3r_1,1}^\beta v$ with $\alpha, \beta \geq 1$, $x_1 \nmid v$ and $x_{3r_1,1} \nmid v$. Since $u \notin I$, we have that $v \in K[x_{3,1}, \dots, x_{3r_1-2,1}, x_{3,2}, \dots, x_{3r_2,2}, \dots, x_{3,k_1}, \dots, x_{3r_{k_1},k_1}, y_{3,1}, \dots, y_{3s_1+1,1}, \dots, y_{3,k_2}, \dots, y_{3s_{k_2}+1,k_2}, z_{3,1}, \dots, z_{3t_1+2,1}, \dots, z_{3,k_3}, \dots, z_{3t_{k_3}+2,k_3}]$ and $v \notin (x_{3,1}x_{4,1}, \dots, x_{3r_1-3,1}x_{3r_1-2,1}, x_{3,2}x_{4,2}, \dots, x_{3r_2-1,2}x_{3r_2,2}, \dots, x_{3,k_1}x_{4,k_1}, \dots, x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1}, y_{3,1}y_{4,1}, \dots, y_{3s_1,1}y_{3s_1+1,1}, \dots, y_{3,k_2}y_{4,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2}, z_{3,1}z_{4,1}, \dots, z_{3t_1+1,1}z_{3t_1+2,1}, \dots, z_{3,k_3}z_{4,k_3}, \dots, z_{3t_{k_3}+1,k_3}z_{3t_{k_3}+2,k_3})$. Similarly, if $x_1 x_{3r_2,2} | u$ and $x_1 x_{3r_1,1} \nmid u$, then $u = x_1^\alpha x_{3r_2,2}^\beta v$ with $\alpha, \beta \geq 1$ and $v \in K[x_{3,1}, \dots, x_{3r_1-1,1}, x_{3,2}, \dots, x_{3r_2-2,2}, x_{3,3}, \dots, x_{3r_3,3}, \dots, x_{3,k_1}, \dots, x_{3r_{k_1},k_1}, y_{3,1}, \dots, y_{3s_1+1,1}, \dots, y_{3,k_2}, \dots, y_{3s_{k_2}+1,k_2}, z_{3,1}, \dots, z_{3t_1+2,1}, \dots, z_{3,k_3}, \dots, z_{3t_{k_3}+2,k_3}]$ and $v \notin (x_{3,1}x_{4,1}, \dots, x_{3r_1-2,1}x_{3r_1-1,1}, x_{3,2}x_{4,2}, \dots, x_{3r_2-3,2}x_{3r_2-2,2}, x_{3,3}x_{4,3}, \dots, x_{3r_3-1,3}x_{3r_3,3}, \dots, x_{3,k_1}x_{4,k_1}, \dots, x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1}, y_{3,1}y_{4,1}, \dots, y_{3s_1,1}y_{3s_1+1,1}, \dots, y_{3,k_2}y_{4,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2}, z_{3,1}z_{4,1}, \dots, z_{3t_1+1,1}z_{3t_1+2,1}, \dots, z_{3,k_3}z_{4,k_3}, \dots, z_{3t_{k_3}+1,k_3}z_{3t_{k_3}+2,k_3})$. Other cases can be shown in a similar way as the above.

Therefore, by Lemmas 2.4–2.7, 3.2 and [13, Proposition 2.2.20, Theorem 2.2.21], if $k_2 \neq 0$, then $\text{sdepth}(\frac{J}{I}) \geq \text{depth}(\frac{J}{I}) = \min\{\sum_{i=1}^{k_1-1} \lceil \frac{3r_i-3}{3} \rceil + \lceil \frac{3r_{k_1}-4}{3} \rceil + \sum_{i=1}^{k_2} \lceil \frac{3s_i-1}{3} \rceil + \sum_{i=1}^{k_3} \lceil \frac{3t_i}{3} \rceil, \sum_{i=1}^{k_1} \lceil \frac{3r_i-3}{3} \rceil + \sum_{i=1}^{k_2-1} \lceil \frac{3s_i-2}{3} \rceil + \lceil \frac{3s_{k_2}-3}{3} \rceil + \sum_{i=1}^{k_3} \lceil \frac{3t_i}{3} \rceil, \sum_{i=1}^{k_1} \lceil \frac{3r_i-3}{3} \rceil + \sum_{i=1}^{k_2} \lceil \frac{3s_i-2}{3} \rceil + \sum_{i=1}^{k_3-1} \lceil \frac{3t_i-1}{3} \rceil + \lceil \frac{3t_{k_3}-2}{3} \rceil\} + 2 = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i - k_1 + 1.$

If $k_2 = 0$, then $\text{sdepth}(\frac{J}{I}) \geq \text{depth}(\frac{J}{I}) = \min\{\sum_{i=1}^{k_1-1} \lceil \frac{3r_i-3}{3} \rceil + \lceil \frac{3r_{k_1}-4}{3} \rceil + \sum_{i=1}^{k_3} \lceil \frac{3t_i}{3} \rceil, \sum_{i=1}^{k_1} \lceil \frac{3r_i-3}{3} \rceil + \sum_{i=1}^{k_3-1} \lceil \frac{3t_i-1}{3} \rceil + \lceil \frac{3t_{k_3}-2}{3} \rceil\} + 2 = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_3} t_i - k_1 + 2$. This completes the proof. \square

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SCHOOL OF MATHEMATICAL SCIENCES, SOOCHOW UNIVERSITY, SUZHOU 215006, P. R. CHINA
E-mail address: zhuguangjun@suda.edu.cn