

# DEPTH AND STANLEY DEPTH OF THE EDGE IDEALS OF SOME $m$ -LINE GRAPHS AND $m$ -CYCLIC GRAPHS WITH A COMMON VERTEX

GUANGJUN ZHU

ABSTRACT. We give some precise formulas for the depth of the quotient rings of the edge ideals associated to a graph consisting, either of the union of some line graphs  $L_{3r_1}, \dots, L_{3r_{k_1}}, L_{3s_1+1}, \dots, L_{3s_{k_2}+1}$  and  $L_{3t_1+2}, \dots, L_{3t_{k_3}+2}$  or of the union of cycle graphs  $C_{3r_1}, \dots, C_{3r_{k_1}}, C_{3s_1+1}, \dots, C_{3s_{k_2}+1}$  and  $C_{3t_1+2}, \dots, C_{3t_{k_3}+2}$ , with a common vertex. We also give some tight bounds for their Stanley depths.

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## 1. INTRODUCTION

Let  $S = K[x_1, \dots, x_n]$  be the polynomial ring in  $n$  variables over a field  $K$  and  $M$  a finitely generated  $\mathbb{Z}^n$ -graded  $S$ -module. For a homogeneous element  $u \in M$  and a subset  $Z \subseteq \{x_1, \dots, x_n\}$ ,  $uK[Z]$  denotes the  $K$ -subspace of  $M$  generated by all the homogeneous elements of the form  $uv$ , where  $v$  is a monomial in  $K[Z]$ . The  $\mathbb{Z}^n$ -graded  $K$ -subspace  $uK[Z]$  is said to be a Stanley space of dimension  $|Z|$  if it is a free  $K[Z]$ -module, where, as usual,  $|Z|$  denotes the number of elements of  $Z$ . A Stanley decomposition of  $M$  is a decomposition of  $M$  as a finite direct sum of  $\mathbb{Z}^n$ -graded  $K$ -vector spaces

$$\mathcal{D} : M = \bigoplus_{i=1}^r u_i K[Z_i]$$

where each  $u_i K[Z_i]$  is a Stanley space of  $M$ . The number

$$\text{sdepth}(\mathcal{D}) = \min\{|Z_i| : i = 1, \dots, r\}$$

is called the Stanley depth of decomposition  $\mathcal{D}$  and the quantity

$$\text{sdepth}(M) := \max\{\text{sdepth}(\mathcal{D}) \mid \mathcal{D} \text{ is a Stanley decomposition of } M\}$$

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is called the Stanley depth of  $M$ . Stanley [10] conjectured that

$$\text{sdepth}(M) \geq \text{depth}(M)$$

for all  $\mathbb{Z}^n$ -graded  $S$ -modules  $M$ . This conjecture proves to be false, in general, for  $M = S/I$  and  $M = J/I$ , where  $I \subset J \subset S$  are monomial ideals, see [4].

Herzog, Vlădoiu and Zheng [6] introduced a method to compute the Stanley depth of a factor of two monomial ideals which was later developed into an effective algorithm by Rinaldo [9] implemented in CoCoA [3]. However, it is difficult to compute this invariant, even in some very particular cases. For instance in [1] Biró et al. proved that  $\text{sdepth}(\mathfrak{m}) = \lceil \frac{n}{2} \rceil$  where  $\mathfrak{m} = (x_1, \dots, x_n)$  is the graded maximal ideal of  $S$  and  $\lceil \frac{n}{2} \rceil$  denote the smallest integer  $\geq \frac{n}{2}$ . For a friendly introduction on Stanley depth we refer the reader to [5].

Let  $I_n$  and  $J_n$  be the edge ideals associated to the line, respectively, cycle graph of length  $n$ . Morey [7] proved that  $\text{depth}(S/I_n) = \lceil \frac{n}{3} \rceil$ . Replacing depth by stanley depth, Ştefan [11] showed that  $\text{sdepth}(S/I_n) = \lceil \frac{n}{3} \rceil$ . In [2], Cimpoeaş proved that  $\text{depth}(S/J_n) = \lceil \frac{n-1}{3} \rceil$  and  $\text{sdepth}(S/J_n) = \lceil \frac{n-1}{3} \rceil$  for  $n \equiv 0 \pmod{3}$  and  $n \equiv 2 \pmod{3}$ . He also proved that  $\lceil \frac{n-1}{3} \rceil \leq \text{sdepth}(S/J_n) \leq \lceil \frac{n}{3} \rceil$  for  $n \equiv 1 \pmod{3}$ . Let  $I$  and  $J$  be the edge ideals associated to the graph consisting of the union of line graphs  $L_{3r_1}, \dots, L_{3r_{k_1}}, L_{3s_1+1}, \dots, L_{3s_{k_2}+1}$  and  $L_{3t_1+2}, \dots, L_{3t_{k_3}+2}$  with a common vertex, respectively, the graph consisting of the union of cycle graphs  $C_{3r_1}, \dots, C_{3r_{k_1}}, C_{3s_1+1}, \dots, C_{3s_{k_2}+1}$  and  $C_{3t_1+2}, \dots, C_{3t_{k_3}+2}$  with a common vertex, then using similar techniques, we prove that

$$(1) \quad \text{sdepth}\left(\frac{S}{I}\right) \geq \text{depth}\left(\frac{S}{I}\right) = \begin{cases} \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i, & \text{if } k_3 = 0, \\ \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i + 1, & \text{otherwise;} \end{cases}$$

$$(2) \quad \text{sdepth}\left(\frac{S}{I}\right) \leq \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i + 1.$$

In the fourth section, we prove that

$$(1) \quad \text{sdepth}\left(\frac{S}{J}\right) \geq \text{depth}\left(\frac{S}{J}\right) = \begin{cases} \sum_{i=1}^{k_2} s_i, & \text{if } k_1 = k_3 = 0, \\ \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i - k_1 + 1, & \text{otherwise;} \end{cases}$$

$$(2) \quad \text{sdepth}\left(\frac{S}{J}\right) \leq \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i - k_1 + 1;$$

$$(3) \quad \text{sdepth}\left(\frac{J}{I}\right) \geq \text{depth}\left(\frac{J}{I}\right) = \begin{cases} \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_3} t_i - k_1 + 2, & \text{if } k_2 = 0, \\ \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i - k_1 + 1, & \text{otherwise.} \end{cases}$$

## 2. PRELIMINARIES

We first recall some definitions about graphs and their edge ideals in order to make this paper self-contained. However, for more details on the notions, we refer the reader to [13, 14].

**Definition 2.1.** Let  $G_i = (V(G_i), E(G_i))$  be some graphs with vertex set  $V(G_i)$  and edge set  $E(G_i)$  for  $1 \leq i \leq k$ . The union of graphs  $G_1, G_2, \dots, G_k$ , written as  $G_1 \cup G_2 \cup \dots \cup G_k$ , is the graph with vertex set  $\bigcup_{i=1}^k V(G_i)$  and edge set  $\bigcup_{i=1}^k E(G_i)$ .

**Definition 2.2.** Let  $G = (V(G), E(G))$  be a graph with vertex set  $V(G) = \{x_1, \dots, x_n\}$  and edge set  $E(G)$ . Suppose that  $x_1, \dots, x_n$  are variables over the field  $K$ . The edge ideal of graph  $G$  in the polynomial ring  $S = K[x_1, \dots, x_n]$  is the squarefree monomial ideal

$$I(G) = (\{x_i x_j \mid \{x_i, x_j\} \in E(G)\}).$$

For the sake of simplicity, we will use the same notation  $x_i x_j$  for the monomial and for the corresponding edge of graph  $G$ .

**Definition 2.3.** Let  $G = (V(G), E(G))$  be a graph with vertex set  $V(G) = \{x_1, \dots, x_m\}$  and edge set  $E(G)$ . Then  $G$  is called a line graph of length  $m$ , denoted by  $L_m$ , if its edge set  $E = \{x_i x_{i+1} \mid 1 \leq i \leq m-1\}$ . Similarly, if  $m \geq 3$ , then  $G$  is called a cycle graph of length  $m$ , denoted by  $C_m$ , if its edge set  $E = \{x_i x_{i+1} \mid 1 \leq i \leq m-1\} \cup \{x_m x_1\}$ .

We recall the well known Depth Lemma, see for instance [13, Lemma 1.3.9] or [12, Lemma 3.1.4].

**Lemma 2.4.** (*Depth Lemma*) Let  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  be a short exact sequence of modules over a local ring  $S$ , or a Noetherian graded ring with  $S_0$  local, then

- (i)  $\text{depth}(M) \geq \min\{\text{depth}(L), \text{depth}(N)\}$ ;
- (ii)  $\text{depth}(L) \geq \min\{\text{depth}(M), \text{depth}(N) + 1\}$ ;
- (iii)  $\text{depth}(N) \geq \min\{\text{depth}(L) - 1, \text{depth}(M)\}$ .

The most of the statements of the Depth Lemma are wrong if we replace depth by Stanley depth. Rauf [8] proved the analog of Lemma 2.4 (i) for Stanley depth.

**Lemma 2.5.** Let  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  be a short exact sequence of finitely generated  $\mathbb{Z}^n$ -graded  $S$ -modules. Then

$$s\text{depth}(M) \geq \min\{s\text{depth}(L), s\text{depth}(N)\}.$$

Using Depth Lemma, Morey in [7] proved the following result.

**Lemma 2.6.** Let  $L_m$  be a line graph of length  $m$  and  $I(L_m)$  its edge ideal. Then  $\text{depth}(S/I(L_m)) = \lceil \frac{m}{3} \rceil$ .

Replacing depth by Stanley depth, Ştefan in [11] showed the following result.

**Lemma 2.7.** Let  $L_m$  be a line graph of length  $m$  and  $I(L_m)$  its edge ideal. Then  $s\text{depth}(S/I(L_m)) = \lceil \frac{m}{3} \rceil$ .

### 3. DEPTH AND STANLEY DEPTH OF THE EDGE IDEALS OF SOME $m$ -LINE GRAPHS WITH A COMMON VERTEX

In this section, we will give some formulas for depth and Stanley depth of the quotient rings of the edge ideals of some  $m$ -line graphs with a common vertex. We assume that  $G$  is the  $m$ -line graph formed by joining  $m$  line graphs  $L_{3r_1}, \dots, L_{3r_{k_1}}, L_{3s_1+1}, \dots, L_{3s_{k_2}+1}, L_{3t_1+2}, \dots, L_{3t_{k_3}+2}$  at a common vertex, where  $k_1 + k_2 + k_3 = m$  and  $k_i \geq 0$  for  $i = 1, 2, 3$ . We adopt the following notation to edges of graph  $G$ :

$$E(L_{3r_i, i}) = \{x_1 x_{2, i}, x_{2, i} x_{3, i}, \dots, x_{3r_i-1, i} x_{3r_i, i}\} \text{ for any } 1 \leq i \leq k_1 \text{ and } r_1 \leq \dots \leq r_{k_1},$$

$$E(L_{3s_i+1, i}) = \{x_1 y_{2, i}, y_{2, i} y_{3, i}, \dots, y_{3s_i, i} y_{3s_i+1, i}\} \text{ for any } 1 \leq i \leq k_2 \text{ and } s_1 \leq \dots \leq s_{k_2},$$

$$E(L_{3t_i+2, i}) = \{x_1 z_{2, i}, z_{2, i} z_{3, i}, \dots, z_{3t_i+1, i} z_{3t_i+2, i}\} \text{ for all } 1 \leq i \leq k_3 \text{ and } t_1 \leq \dots \leq t_{k_3}.$$

Set  $K$  be any field,  $S = K[x_1, x_{2,1}, \dots, x_{3r_1,1}, \dots, x_{2,k_1}, \dots, x_{3r_{k_1},k_1}, y_{2,1}, \dots, y_{3s_1+1,1}, \dots, y_{2,k_2}, \dots, y_{3s_{k_2}+1,k_2}, z_{2,1}, \dots, z_{3t_1+2,1}, \dots, z_{2,k_3}, \dots, z_{3t_{k_3}+2,k_3}]$  the polynomial ring. The edge ideal of graph  $G$  is  $I = (x_1 x_{2,1}, x_{2,1} x_{3,1}, \dots, x_{3r_1-1,1} x_{3r_1,1}, \dots, x_1 x_{2,k_1}, x_{2,k_1} x_{3,k_1}, \dots, x_{3r_{k_1}-1,k_1} x_{3r_{k_1},k_1}, x_1 y_{2,1}, y_{2,1} y_{3,1}, \dots, y_{3s_1,1} y_{3s_1+1,1}, \dots, x_1 y_{2,k_2}, y_{2,k_2} y_{3,k_2}, \dots, y_{3s_{k_2},k_2} y_{3s_{k_2}+1,k_2}, x_1 z_{2,1}, z_{2,1} z_{3,1}, \dots, z_{3t_1+1,1} z_{3t_1+2,1}, \dots, x_1 z_{2,k_3}, \dots, z_{3t_{k_3}+1,k_3} z_{3t_{k_3}+2,k_3})$ .

**Example 3.1.** The following graph  $G$  is the union of 5 line graphs  $L_3, L_4, L_5, L_6$  and  $L_7$  with a common vertex  $x_1$ .

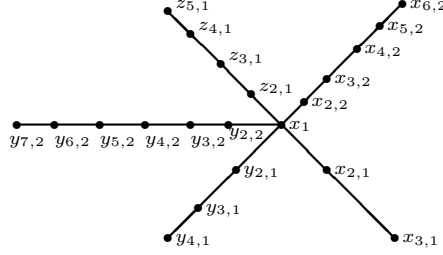


Figure 1

The edge ideal of graph  $G$  is  $I = (x_1x_{2,1}, x_{2,1}x_{3,1}, x_1x_{2,2}, x_{2,2}x_{3,2}, x_{3,2}x_{4,2}, x_{4,2}x_{5,2}, x_{5,2}x_{6,2}, x_1y_{2,1}, y_{2,1}y_{3,1}, y_{3,1}y_{4,1}, x_1y_{2,2}, y_{2,2}y_{3,2}, y_{3,2}y_{4,2}, y_{4,2}y_{5,2}, y_{5,2}y_{6,2}, y_{6,2}y_{7,2}, x_1z_{2,1}, z_{2,1}z_{3,1}, z_{3,1}z_{4,1}, z_{4,1}z_{5,1})$ .

We need the following lemma (See [8, Theorem 3.1]).

**Lemma 3.2.** Let  $I \subset S_1 = K[x_1, \dots, x_m]$ ,  $J \subset S_2 = K[y_1, \dots, y_n]$  be monomial ideals and  $S = K[x_1, \dots, x_m, y_1, \dots, y_n]$ . Then

$$sdepth(S/(IS, JS)) \geq sdepth(S_1/IS_1) + sdepth(S_2/JS_2).$$

Now, we prove the main results of this section. We adopt the following convention: whenever, in a sum, the index runs from 1 to 0, the sum has to be taken equal to zero.

**Theorem 3.3.** Let  $G$  be a graph consisting of the union of line graphs  $L_{3r_1}, \dots, L_{3r_{k_1}}, L_{3s_1+1}, \dots, L_{3s_{k_2}+1}$  and  $L_{3t_1+2}, \dots, L_{3t_{k_3}+2}$  with a common vertex  $x_1$ , where  $k_i \geq 0$  for  $i = 1, 2, 3$ . Let  $I$  be its edge ideal. Then

$$sdepth\left(\frac{S}{I}\right) \geq depth\left(\frac{S}{I}\right) = \begin{cases} \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i, & \text{if } k_3 = 0, \\ \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i + 1, & \text{otherwise.} \end{cases}$$

In particular,  $S/I$  satisfies the Stanley conjecture.

*Proof.* Note that  $(I : x_1) = (x_{2,1}, \dots, x_{2,k_1}, y_{2,1}, \dots, y_{2,k_2}, z_{2,1}, \dots, z_{2,k_3}, x_{3,1}x_{4,1}, \dots, x_{3r_1-1,1}x_{3r_1,1}, \dots, x_{3,k_1}x_{4,k_1}, \dots, x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1}, y_{3,1}y_{4,1}, \dots, y_{3s_1,1}y_{3s_1+1,1}, \dots, y_{3,k_2}y_{4,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2}, z_{3,1}z_{4,1}, \dots, z_{3t_1+1,1}z_{3t_1+2,1}, \dots, z_{3,k_3}z_{4,k_3}, \dots, z_{3t_{k_3}+1,k_3}z_{3t_{k_3}+2,k_3})$  and  $(I, x_1) = (x_{2,1}x_{3,1}, \dots, x_{3r_1-1,1}x_{3r_1,1}, \dots, x_{2,k_1}x_{3,k_1}, \dots, x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1}, y_{2,1}y_{3,1}, \dots, y_{3s_1,1}y_{3s_1+1,1}, \dots, y_{2,k_2}y_{3,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2}, z_{2,1}z_{3,1}, \dots, z_{3t_1+1,1}z_{3t_1+2,1}, \dots, z_{2,k_3}z_{3,k_3}, \dots, z_{3t_{k_3}+1,k_3}z_{3t_{k_3}+2,k_3}, x_1)$ , thus we get that

$$\begin{aligned} \frac{S}{(I : x_1)} &\cong \frac{K[x_{3,1}, \dots, x_{3r_1,1}]}{(x_{3,1}x_{4,1}, \dots, x_{3r_1-1,1}x_{3r_1,1})} \otimes_K \cdots \otimes_K \frac{K[x_{3,k_1}, \dots, x_{3r_{k_1},k_1}]}{(x_{3,k_1}x_{4,k_1}, \dots, x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1})} \\ &\otimes_K \frac{K[y_{3,1}, \dots, y_{3s_1+1,1}]}{(y_{3,1}y_{4,1}, \dots, y_{3s_1,1}y_{3s_1+1,1})} \otimes_K \cdots \otimes_K \frac{K[y_{3,k_2}, \dots, y_{3s_{k_2}+1,k_2}]}{(y_{3,k_2}y_{4,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})} \\ &\otimes_K \frac{K[z_{3,1}, \dots, z_{3t_1+2,1}]}{(z_{3,1}z_{4,1}, \dots, z_{3t_1+1,1}z_{3t_1+2,1})} \otimes_K \cdots \otimes_K \frac{K[z_{3,k_3}, \dots, z_{3t_{k_3}+2,k_3}]}{(z_{3,k_3}z_{4,k_3}, \dots, z_{3t_{k_3}+1,k_3}z_{3t_{k_3}+2,k_3})} \otimes_K K[x_1], \end{aligned}$$

and

$$\begin{aligned} \frac{S}{(I, x_1)} &\cong \frac{K[x_{2,1}, \dots, x_{3r_1,1}]}{(x_{2,1}x_{3,1}, \dots, x_{3r_1-1,1}x_{3r_1,1})} \otimes_K \cdots \otimes_K \frac{K[x_{2,k_1}, \dots, x_{3r_{k_1},k_1}]}{(x_{2,k_1}x_{3,k_1}, \dots, x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1})} \\ &\otimes_K \frac{K[y_{2,1}, \dots, y_{3s_1+1,1}]}{(y_{2,1}y_{3,1}, \dots, y_{3s_1,1}y_{3s_1+1,1})} \otimes_K \cdots \otimes_K \frac{K[y_{2,k_2}, \dots, y_{3s_{k_2}+1,k_2}]}{(y_{2,k_2}y_{3,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})} \\ &\otimes_K \frac{K[z_{2,1}, \dots, z_{3t_1+2,1}]}{(z_{2,1}z_{3,1}, \dots, z_{3t_1+1,1}z_{3t_1+2,1})} \otimes_K \cdots \otimes_K \frac{K[z_{2,k_3}, \dots, z_{3t_{k_3}+2,k_3}]}{(z_{2,k_3}z_{3,k_3}, \dots, z_{3t_{k_3}+1,k_3}z_{3t_{k_3}+2,k_3})}. \end{aligned}$$

Therefore, by Lemmas 2.6, 2.7, 3.2 and [13, Proposition 2.2.20, Theorem 2.2.21], we obtain that  $\text{sdepth}(\frac{S}{(I, x_1)}) \geq \text{depth}(\frac{S}{(I, x_1)}) = \sum_{i=1}^{k_1} \lceil \frac{3r_i-2}{3} \rceil + \sum_{i=1}^{k_2} \lceil \frac{3s_i-1}{3} \rceil + \sum_{i=1}^{k_3} \lceil \frac{3t_i}{3} \rceil + 1 = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i + 1$ , and  $\text{sdepth}(\frac{S}{(I, x_1)}) \geq \text{depth}(\frac{S}{(I, x_1)}) = \sum_{i=1}^{k_1} \lceil \frac{3r_i-1}{3} \rceil + \sum_{i=1}^{k_2} \lceil \frac{3s_i}{3} \rceil + \sum_{i=1}^{k_3} \lceil \frac{3t_i+1}{3} \rceil = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i + k_3$ .

Using Lemma 2.5 on the short exact sequence

$$(1) \quad 0 \longrightarrow S/(I : x_1) \xrightarrow{x_1} S/I \longrightarrow S/(I, x_1) \longrightarrow 0,$$

we conclude that  $\text{sdepth}(\frac{S}{I}) \geq \begin{cases} \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i, & \text{if } k_3 = 0; \\ \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i + 1, & \text{otherwise.} \end{cases}$

If  $k_3 \neq 0$ , then  $\text{depth}(\frac{S}{(I, x_1)}) \geq \text{depth}(\frac{S}{(I, x_1)})$ . Using Lemma 2.4 on the short exact sequence (1), it follows that  $\text{depth}(\frac{S}{(I, x_1)}) = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i + 1$ .

Assume that  $k_3 = 0$ . We claim that we have the  $S$ -module isomorphism

$$\begin{aligned} \frac{(I : x_1)}{I} &\cong \bigoplus_{i=1}^{k_1} x_{2,i} \left( \frac{K[x_{3,1}, \dots, x_{3r_1,1}]}{(x_{3,1}x_{4,1}, \dots, x_{3r_1-1,1}x_{3r_1,1})} \otimes_K \cdots \otimes_K \frac{K[x_{3,i-1}, \dots, x_{3r_{i-1},i-1}]}{(x_{3,i-1}x_{4,i-1}, \dots, x_{3r_{i-1}-1,i-1}x_{3r_{i-1},i-1})} \right. \\ &\otimes_K \frac{K[x_{4,i}, \dots, x_{3r_i,i}]}{(x_{4,i}x_{5,i}, \dots, x_{3r_i-1,i}x_{3r_i,i})} \otimes_K \frac{K[x_{2,i+1}, \dots, x_{3r_{i+1},i+1}]}{(x_{2,i+1}x_{3,i+1}, \dots, x_{3r_{i+1}-1,i+1}x_{3r_{i+1},i+1})} \otimes_K \cdots \\ &\otimes_K \frac{K[x_{2,k_1}, \dots, x_{3r_{k_1},k_1}]}{(x_{2,k_1}x_{3,k_1}, \dots, x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1})} \otimes_K \frac{K[y_{2,1}, \dots, y_{3s_1+1,1}]}{(y_{2,1}y_{3,1}, \dots, y_{3s_1,1}y_{3s_1+1,1})} \otimes_K \cdots \\ &\otimes_K \frac{K[y_{2,k_2}, \dots, y_{3s_{k_2}+1,k_2}]}{(y_{2,k_2}y_{3,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})} \Big) [x_{2,i}] \oplus \left( \bigoplus_{i=1}^{k_2} y_{2,i} \left( \frac{K[x_{3,1}, \dots, x_{3r_1,1}]}{(x_{3,1}x_{4,1}, \dots, x_{3r_1-1,1}x_{3r_1,1})} \otimes_K \cdots \right. \right. \\ &\otimes_K \frac{K[x_{3,k_1}, \dots, x_{3r_{k_1},k_1}]}{(x_{3,k_1}x_{4,k_1}, \dots, x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1})} \otimes_K \frac{K[y_{3,1}, \dots, y_{3s_1+1,1}]}{(y_{3,1}y_{4,1}, \dots, y_{3s_1,1}y_{3s_1+1,1})} \otimes_K \cdots \\ &\otimes_K \frac{K[y_{3,i-1}, \dots, y_{3s_{i-1}+1,i-1}]}{(y_{3,i-1}y_{4,i-1}, \dots, y_{3s_{i-1}-1,i-1}y_{3s_{i-1}+1,i-1})} \otimes_K \frac{K[y_{4,i}, \dots, y_{3s_i+1,i}]}{(y_{4,i}y_{5,i}, \dots, y_{3s_i,i}y_{3s_i+1,i})} \otimes_K \cdots \\ &\left. \left. \otimes_K \frac{K[y_{2,i+1}, \dots, y_{3s_{i+1}+1,i+1}]}{(y_{2,i+1}y_{3,i+1}, \dots, y_{3s_{i+1},i+1}y_{3s_{i+1}+1,i+1})} \otimes_K \cdots \otimes_K \frac{K[y_{2,k_2}, \dots, y_{3s_{k_2}+1,k_2}]}{(y_{2,k_2}y_{3,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})} \right) [y_{2,i}], \end{aligned}$$

where  $x_{i,0} = y_{j,0} = 0$  for  $3 \leq i \leq k_1$ ,  $3 \leq j \leq k_2$ . Indeed, if  $u \in (I : x_1)$  is a monomial such that  $u \notin I$ , then  $x_{2,i}|u$  or  $y_{2,j}|u$  for some  $1 \leq i \leq k_1$  or  $1 \leq j \leq k_2$ .

If  $x_{2,1}|u$ , then we can write  $u$  as  $u = x_{2,1}^\alpha v$  with  $\alpha \geq 1$  and  $x_{2,1} \nmid v$ . Since  $u \notin I$ , we have that  $v \in K[x_{4,1}, \dots, x_{3r_1,1}, x_{2,2}, \dots, x_{3r_2,2}, \dots, x_{2,k_1}, \dots, x_{3r_{k_1},k_1}, y_{2,1}, \dots, y_{3s_1+1,1}, \dots, y_{2,k_2}, \dots, y_{3s_{k_2}+1,k_2}]$  and

$v \notin (x_{4,1}x_{5,1}, \dots, x_{3r_1-1,1}x_{3r_1,1}, x_{2,2}x_{3,2}, \dots, x_{3r_2-1,2}x_{3r_2,2}, \dots, x_{2,k_1}x_{3,k_1}, \dots, x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1}, y_{2,1}y_{3,1}, \dots, y_{3s_1,1}y_{3s_1+1,1}, \dots, y_{2,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})$ . Similarly, if  $x_{2,2}|u$  and  $x_{2,1} \nmid u$ , then  $u = x_{2,2}^\alpha v$  with  $\alpha \geq 1$  and  $v \in K[x_{3,1}, \dots, x_{3r_1,1}, x_{4,2}, \dots, x_{3r_2,2}, x_{2,3}, \dots, x_{3r_3,3}, \dots, x_{2,k_1}, \dots, x_{3r_{k_1},k_1}, y_{2,1}, \dots, y_{3s_1+1,1}, \dots, y_{2,k_2}, \dots, y_{3s_{k_2}+1,k_2}]$  and  $v \notin (x_{3,1}x_{4,1}, \dots, x_{3r_1-1,1}x_{3r_1,1}, x_{4,2}x_{5,2}, \dots, x_{3r_2-1,2}x_{3r_2,2}, x_{2,3}x_{3,3}, \dots, x_{3r_3-1,3}x_{3r_3,3}, \dots, x_{2,k_1}x_{3,k_1}, \dots, x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1}, y_{2,1}y_{3,1}, \dots, y_{3s_1,1}y_{3s_1+1,1}, \dots, y_{2,k_2}y_{3,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})$ . Other cases can be shown in a similar way as the above.

Therefore, by [13, Proposition 2.2.20, Theorem 2.2.21] and Lemma 2.6, it follows that  $\text{depth}(\frac{I:x_1}{I}) = \sum_{i=1}^{k_1-1} \lceil \frac{3r_i-2}{3} \rceil + \lceil \frac{3r_{k_1}-3}{3} \rceil + \sum_{i=1}^{k_2-1} \lceil \frac{3s_i-1}{3} \rceil + \lceil \frac{3s_{k_2}-2}{3} \rceil + 1 = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i$ .

Now, using Lemma 2.4 on the short exact sequence

$$(2) \quad 0 \longrightarrow (I : x_1)/I \longrightarrow S/I \longrightarrow S/(I : x_1) \longrightarrow 0,$$

this completes the proof.  $\square$

Assume  $n = \sum_{i=1}^{k_1} 3r_i + \sum_{i=1}^{k_2} (3s_i + 1) + \sum_{i=1}^{k_3} (3t_i + 2) - (k_1 + k_2 + k_3) + 1$ . We identify  $S/I$  with the  $\mathbb{Z}^n$ -graded  $K$ -subvector space  $I^c$  of  $S$  which is generated by all monomials  $u \in S \setminus I$ . Set the set

$$P = \{a \in \mathbb{N}^n : x^a \in I^c \text{ and } x^a | x_1 \prod_{\substack{2 \leq i \leq 3r_j, \\ 1 \leq j \leq k_1}} x_{i,j} \prod_{\substack{2 \leq i \leq 3s_j + 1, \\ 1 \leq j \leq k_2}} y_{i,j} \prod_{\substack{1, 2 \leq i \leq 3t_j + 2, \\ 1 \leq j \leq k_3}} z_{i,j}\},$$

where  $a = (a(1), a(2, 1), \dots, a(3r_1, 1), \dots, a(2, k_1), \dots, a(3r_{k_1}, k_1), b(2, 1), \dots, b(3s_1 + 1, 1), \dots, b(2, k_2), \dots, b(3s_{k_2} + 1, k_2), c(2, 1), \dots, c(3t_1 + 2, 1), \dots, c(2, k_3), \dots, c(3t_{k_3} + 2, k_3)) \in \mathbb{N}^n$  and

$$x^a = x_1^{a(1)} \prod_{\substack{2 \leq i \leq 3r_j, \\ 1 \leq j \leq k_1}} x_{i,j}^{a(i,j)} \prod_{\substack{2 \leq i \leq 3s_j + 1, \\ 1 \leq j \leq k_2}} y_{i,j}^{b(i,j)} \prod_{\substack{1, 2 \leq i \leq 3t_j + 2, \\ 1 \leq j \leq k_3}} z_{i,j}^{c(i,j)}.$$

Consider the natural partial order on  $\mathbb{N}^n$  which is given by componentwise comparison, i.e. if  $a = (a_1, \dots, a_n)$  and  $b = (b_1, \dots, b_n)$ , then  $a \geq b$  if and only if  $a_i \geq b_i$  for all  $i = 1, \dots, n$ . With respect to the partial order induced on  $P$ , it becomes a poset where  $a \geq a'$  if and only if  $x^{a'} | x^a$ .

Let  $\mathcal{P} : P = \bigcup_{i=1}^r [F_i, G_i]$  be a partition of  $P$ . We denote  $\text{sdepth}(\mathcal{P}) = \min\{|G_i| : 1 \leq i \leq r\}$ . Also, we define the Stanley depth of  $P$ , to be the number

$$\text{sdepth}(P) = \max\{\text{sdepth}(\mathcal{P}) \mid \mathcal{P} \text{ is a partition of } P\}.$$

Herzog, Vlădoiu and Zheng proved in [6] that  $\text{sdepth}(\frac{S}{I}) = \text{sdepth}(P)$ . Now, for  $d \in \mathbb{N}$  and  $\sigma \in P$ , we denote

$$\mathcal{P}_d := \{a \in P : |a| = d\} \quad \text{and} \quad \mathcal{P}_{d,\sigma} := \{a \in \mathcal{P}_d : x^\sigma | x^a\},$$

where for  $a = (a(1), a(2, 1), \dots, a(3r_1, 1), \dots, a(2, k_1), \dots, a(3r_{k_1}, k_1), b(2, 1), \dots, b(3s_1 + 1, 1), \dots, b(2, k_2), \dots, b(3s_{k_2} + 1, k_2), c(2, 1), \dots, c(3t_1 + 2, 1), \dots, c(2, k_3), \dots, c(3t_{k_3} + 2, k_3)) \in \mathbb{N}^n$ ,  $|a| := a(1) + \sum_{j=1}^{k_1} \sum_{i=2}^{3r_j} a(i, j) +$

$$\sum_{j=1}^{k_2} \sum_{i=2}^{3s_j+1} b(i, j) + \sum_{j=1}^{k_3} \sum_{i=2}^{3t_j+2} c(i, j).$$

With these notations, we are able to prove the following result.

**Theorem 3.4.** *Let  $G$  be a graph as in Theorem 3.3 and  $I$  be its edge ideal. Then*

$$sdepth\left(\frac{S}{I}\right) \leq \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i + 1.$$

*Proof.* Firstly, we claim: if  $\sigma \in \mathcal{P}$  such that  $\mathcal{P}_{d,\sigma} = \emptyset$ , then  $sdepth(\mathcal{P}) < d$ .

Indeed, let  $\mathcal{P} : P = \bigcup_{i=1}^r [F_i, G_i]$  be a partition of  $P$  with  $sdepth(\mathcal{P}) = sdepth(P)$ . Since  $\sigma \in P$ , it follows that  $\sigma \in [F_i, G_i]$  for some  $i$ . If  $|G_i| \geq d$ , then it follows that  $\mathcal{P}_{d,\sigma} \neq \emptyset$ , since there exist some subsets in the interval  $[F_i, G_i]$  of cardinality  $d$  which contain  $\sigma$ , a contradiction! Thus,  $|G_i| < d$  and therefore  $sdepth(\mathcal{P}) < d$ .

We set  $d = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i + 1$  and  $\sigma = (a(1), a(2, 1), \dots, a(3r_1, 1), \dots, a(2, k_1), \dots, a(3r_{k_1}, k_1), b(2, 1), \dots, b(3s_1+1, 1), \dots, b(2, k_2), \dots, b(3s_{k_2}+1, k_2), c(2, 1), \dots, c(3t_1+2, 1), \dots, c(2, k_3), \dots, c(3t_{k_3}+2, k_3)) \in \mathbb{N}^n$ , where  $a(1) = 1$ , for any  $1 \leq i \leq k_1$ ,  $1 \leq j \leq r_i - 1$ ,  $a(l, i) = \begin{cases} 1 : l = 3j + 1 \text{ or } l = 3r_i \\ 0 : \text{otherwise} \end{cases}$ , for any  $1 \leq i \leq k_2$ ,  $1 \leq j \leq s_i$ ,  $b(l, i) = \begin{cases} 1 : l = 3j + 1 \\ 0 : \text{otherwise} \end{cases}$ , and for any  $1 \leq i \leq k_3$ ,  $1 \leq j \leq t_i$ ,  $c(l, i) = \begin{cases} 1 : l = 3j + 1 \\ 0 : \text{otherwise} \end{cases}$ . We obtain that  $\mathcal{P}_{d+1,\sigma} = \emptyset$ . Indeed, if monomial

$$u = x_1 \prod_{i=1}^{k_1} (x_{3r_i, i} \prod_{j=1}^{r_i-1} x_{3j+1, i}) \prod_{i=1}^{k_2} (\prod_{j=1}^{s_i} y_{3j+1, i}) \prod_{i=1}^{k_3} (\prod_{j=1}^{t_i} z_{3j+1, i}),$$

one can easily see that if  $a = (a(1), a(2, 1), \dots, a(3r_1, 1), \dots, a(2, k_1), \dots, a(3r_{k_1}, k_1), b(2, 1), \dots, b(3s_1+1, 1), \dots, b(2, k_2), \dots, b(3s_{k_2}+1, k_2), c(2, 1), \dots, c(3t_1+2, 1), \dots, c(2, k_3), \dots, c(3t_{k_3}+2, k_3)) \in \mathbb{N}^n$  such that  $a(l, i) \neq 0$  for some  $l \notin \{3j+1, 3r_i \mid 1 \leq j \leq r_i - 1, 1 \leq i \leq k_1\}$  or  $b(l, i) \neq 0$  for some  $l \notin \{3j+1 \mid 1 \leq j \leq s_i, 1 \leq i \leq k_2\}$  or  $c(l, i) \neq 0$  for some  $l \notin \{3j+1 \mid 1 \leq j \leq t_i, 1 \leq i \leq k_3\}$ , then  $u \cdot x^a \in I$ . Therefore, by previous remark,  $sdepth\left(\frac{S}{I}\right) = sdepth(\mathcal{P}) \leq d = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i + 1$ , as required.  $\square$

#### 4. DEPTH AND STANLEY DEPTH OF THE EDGE IDEALS OF SOME $m$ -CYCLIC GRAPHS WITH A COMMON VERTEX

In this section, we will give some formulas for depth and Stanley depth of the quotient rings of the edge ideals of some  $m$ -cyclic graphs with a common vertex. We assume that  $G$  is the  $m$ -cyclic graph formed by joining  $m$  cycles  $C_{3r_1}, \dots, C_{3r_{k_1}}, C_{3s_1+1}, \dots, C_{3s_{k_2}+1}, C_{3t_1+2}, \dots, C_{3t_{k_3}+2}$  at a common vertex, where  $k_1 + k_2 + k_3 = m$  and  $k_i \geq 0$  for  $i = 1, 2, 3$ . We adopt the following notation to edges of graph  $G$ :  $E(C_{3r_i, i}) = \{x_1 x_{2, i}, x_{2, i} x_{3, i}, \dots, x_{3r_i, i} x_1\}$  for any  $1 \leq i \leq k_1$  and  $r_1 \leq r_2 \leq \dots \leq r_{k_1}$ ,  $E(C_{3s_i+1, i}) = \{x_1 y_{2, i}, y_{2, i} y_{3, i}, \dots, y_{3s_i+1, i} x_1\}$  for any  $1 \leq i \leq k_2$  and  $s_1 \leq \dots \leq s_{k_2}$ ,  $E(C_{3t_i+2, i}) = \{x_1 z_{2, i}, z_{2, i} z_{3, i}, \dots, z_{3t_i+2, i} x_1\}$  for all  $1 \leq i \leq k_3$  and  $t_1 \leq t_2 \leq \dots \leq t_{k_3}$ . Let  $K$  be any field,  $S = K[x_1, x_{2, 1}, \dots, x_{3r_1, 1}, \dots, x_{2, k_1}, \dots, x_{3r_{k_1}, k_1}, y_{2, 1}, \dots, y_{3s_1+1, 1}, \dots, y_{2, k_2}, \dots, y_{3s_{k_2}+1, k_2}, z_{2, 1}, \dots, z_{3t_1+2, 1}, \dots, z_{2, k_3}, \dots, z_{3t_{k_3}+2, k_3}]$  the polynomial ring. Then the edge ideal of graph  $G$  is  $J = (x_1 x_{2, 1}, x_{2, 1} x_{3, 1}, \dots, x_{3r_1, 1} x_1, \dots, x_1 x_{2, k_1}, x_{2, k_1} x_{3, k_1}, \dots, x_{3r_{k_1}, k_1} x_1, x_1 y_{2, 1}, y_{2, 1} y_{3, 1}, \dots, y_{3s_1+1, 1} x_1, \dots, x_1 y_{2, k_2}, y_{2, k_2} y_{3, k_2}, \dots, y_{3s_{k_2}+1, k_2} x_1, x_1 z_{2, 1}, z_{2, 1} z_{3, 1}, \dots, z_{3t_1+2, 1} x_1, \dots, x_1 z_{2, k_3}, z_{2, k_3} z_{3, k_3}, \dots, z_{3t_{k_3}+2, k_3} x_1)$ .

**Example 4.1.** The following graph  $G$  is the union of 5 circle graphs  $C_3, C_4, C_5, C_6$  and  $C_7$  with a common vertex  $x_1$ .

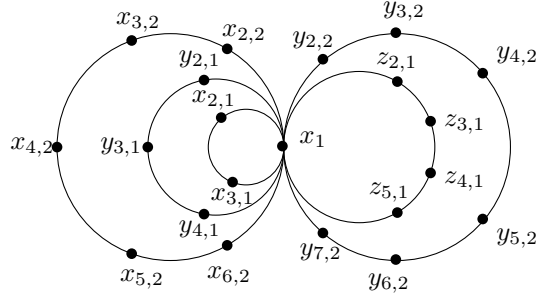


Figure 2

The edge ideal of graph  $G$  is  $J = (x_1x_{2,1}, x_{2,1}x_{3,1}, x_{3,1}x_1, x_1x_{2,2}, x_{2,2}x_{3,2}, x_{3,2}x_{4,2}, x_{4,2}x_{5,2}, x_{5,2}x_{6,2}, x_{6,2}x_1, x_1y_{2,1}, y_{2,1}y_{3,1}, y_{3,1}y_{4,1}, y_{4,1}x_1, x_1y_{2,2}, y_{2,2}y_{3,2}, y_{3,2}y_{4,2}, y_{4,2}y_{5,2}, y_{5,2}y_{6,2}, y_{6,2}y_{7,2}, y_{7,2}x_1, x_1z_{2,1}, z_{2,1}z_{3,1}, z_{3,1}z_{4,1}, z_{4,1}z_{5,1}, z_{5,1}x_1)$ .

Now, we prove the main results of this section.

**Theorem 4.2.** *Let  $G$  be a graph consisting of the union of cycle graphs  $C_{3r_1}, \dots, C_{3r_{k_1}}, C_{3s_1+1}, \dots, C_{3s_{k_2}+1}$  and  $C_{3t_1+2}, \dots, C_{3t_{k_3}+2}$  with a common vertex  $x_1$ , where  $k_i \geq 0$  for  $i = 1, 2, 3$ . Let  $J$  be its edge ideal. Then*

$$sdepth\left(\frac{S}{J}\right) \geq depth\left(\frac{S}{J}\right) = \begin{cases} \sum_{i=1}^{k_2} s_i, & \text{if } k_1 = k_3 = 0, \\ \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i - k_1 + 1, & \text{otherwise.} \end{cases}$$

In particular,  $S/J$  satisfies the Stanley conjecture.

*Proof.* Notice that  $(J : x_1) = (x_{2,1}, \dots, x_{2,k_1}, x_{3r_1,1}, \dots, x_{3r_{k_1},k_1}, y_{2,1}, \dots, y_{2,k_2}, y_{3s_1+1,1}, \dots, y_{3s_{k_2}+1,k_2}, z_{2,1}, \dots, z_{2,k_3}, z_{3t_1+2,1}, \dots, z_{3t_{k_3}+2,k_3}, x_{3,1}x_{4,1}, \dots, x_{3r_1-2,1}x_{3r_1-1,1}, \dots, x_{3,k_1}x_{4,k_1}, \dots, x_{3r_{k_1}-2,k_1}x_{3r_{k_1}-1,k_1}, y_{3,1}y_{4,1}, \dots, y_{3s_1-1,1}y_{3s_1,1}, \dots, y_{3,k_2}y_{4,k_2}, \dots, y_{3s_{k_2}-1,k_2}y_{3s_{k_2},k_2}, z_{3,1}z_{4,1}, \dots, z_{3t_1,1}z_{3t_1+1,1}, \dots, z_{3,k_3}z_{4,k_3}, \dots, z_{3t_{k_3},k_3}z_{3t_{k_3}+1,k_3})$  and  $(J, x_1) = (x_{2,1}x_{3,1}, \dots, x_{3r_1-1,1}x_{3r_1,1}, \dots, x_{2,k_1}x_{3,k_1}, \dots, x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1}, y_{2,1}y_{3,1}, \dots, y_{3s_1,1}y_{3s_1+1,1}, \dots, y_{2,k_2}y_{3,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2}, z_{2,1}z_{3,1}, \dots, z_{3t_1+1,1}z_{3t_1+2,1}, \dots, z_{2,k_3}z_{3,k_3}, \dots, z_{3t_{k_3}+1,k_3}z_{3t_{k_3}+2,k_3}, x_1)$ , thus we get that

$$\begin{aligned} \frac{S}{(J : x_1)} &\cong \frac{K[x_{3,1}, \dots, x_{3r_1-1,1}]}{(x_{3,1}x_{4,1}, \dots, x_{3r_1-2,1}x_{3r_1-1,1})} \otimes_K \cdots \otimes_K \frac{K[x_{3,k_1}, \dots, x_{3r_{k_1}-1,k_1}]}{(x_{3,k_1}x_{4,k_1}, \dots, x_{3r_{k_1}-2,k_1}x_{3r_{k_1}-1,k_1})} \\ &\otimes_K \frac{K[y_{3,1}, \dots, y_{3s_1,1}]}{(y_{3,1}y_{4,1}, \dots, y_{3s_1-1,1}y_{3s_1,1})} \otimes_K \cdots \otimes_K \frac{K[y_{3,k_2}, \dots, y_{3s_{k_2},k_2}]}{(y_{3,k_2}y_{4,k_2}, \dots, y_{3s_{k_2}-1,k_2}y_{3s_{k_2},k_2})} \\ &\otimes_K \frac{K[z_{3,1}, \dots, z_{3t_1+1,1}]}{(z_{3,1}z_{4,1}, \dots, z_{3t_1,1}z_{3t_1+1,1})} \otimes_K \cdots \otimes_K \frac{K[z_{3,k_3}, \dots, z_{3t_{k_3}+1,k_3}]}{(z_{3,k_3}z_{4,k_3}, \dots, z_{3t_{k_3},k_3}z_{3t_{k_3}+1,k_3})} \otimes_K K[x_1], \end{aligned}$$

and

$$\begin{aligned} \frac{S}{(J, x_1)} &\cong \frac{K[x_{2,1}, \dots, x_{3r_1,1}]}{(x_{2,1}x_{3,1}, \dots, x_{3r_1-1,1}x_{3r_1,1})} \otimes_K \cdots \otimes_K \frac{K[x_{2,k_1}, \dots, x_{3r_{k_1},k_1}]}{(x_{2,k_1}x_{3,k_1}, \dots, x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1})} \\ &\otimes_K \frac{K[y_{2,1}, \dots, y_{3s_1+1,1}]}{(y_{2,1}y_{3,1}, \dots, y_{3s_1,1}y_{3s_1+1,1})} \otimes_K \cdots \otimes_K \frac{K[y_{2,k_2}, \dots, y_{3s_{k_2}+1,k_2}]}{(y_{2,k_2}y_{3,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})} \end{aligned}$$



$$\otimes_K \frac{K[z_{2,1}, \dots, z_{3t_1+2,1}]}{(z_{2,1}z_{3,1}, \dots, z_{3t_1+1,1}z_{3t_1+2,1})} \otimes_K \cdots \otimes_K \frac{K[z_{2,k_3}, \dots, z_{3t_{k_3}+2,k_3}]}{(z_{2,k_3}z_{3,k_3}, \dots, z_{3t_{k_3}+1,k_3}z_{3t_{k_3}+2,k_3})}.$$

Therefore, by Lemmas 2.6, 2.7, 3.2 and [13, Proposition 2.2.20, Theorem 2.2.21], we obtain that  $\text{sdepth}(\frac{S}{J:x_1}) \geq \text{depth}(\frac{S}{J:x_1}) = \sum_{i=1}^{k_1} \lceil \frac{3r_i-3}{3} \rceil + \sum_{i=1}^{k_2} \lceil \frac{3s_i-2}{3} \rceil + \sum_{i=1}^{k_3} \lceil \frac{3t_i-1}{3} \rceil + 1 = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i - k_1 + 1$ ,

and  $\text{sdepth}(\frac{S}{(J,x_1)}) \geq \text{depth}(\frac{S}{(J,x_1)}) = \sum_{i=1}^{k_1} \lceil \frac{3r_i-1}{3} \rceil + \sum_{i=1}^{k_2} \lceil \frac{3s_i}{3} \rceil + \sum_{i=1}^{k_3} \lceil \frac{3t_i+1}{3} \rceil = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i + k_3$ .

Using Lemma 2.5 on the short exact sequence

$$(3) \quad 0 \longrightarrow S/(J : x_1) \xrightarrow{\cdot x_1} S/J \longrightarrow S/(J, x_1) \longrightarrow 0,$$

we conclude that  $\text{sdepth}(\frac{S}{J}) \geq \begin{cases} \sum_{i=1}^{k_2} s_i, & \text{if } k_1 = k_3 = 0, \\ \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i - k_1 + 1, & \text{otherwise.} \end{cases}$

If  $k_1 \neq 0$  or  $k_3 \neq 0$ , then  $\text{depth}(\frac{S}{(J,x_1)}) \geq \text{depth}(\frac{S}{J:x_1})$ . Using Lemma 2.4 on the short exact sequence

$$(3), \text{ it follows that } \text{depth}(\frac{S}{J}) = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i - k_1 + 1.$$

Assume that  $k_1 = k_3 = 0$ . We claim that there exists the  $S$ -module isomorphism

$$\begin{aligned} \frac{(J:x_1)}{J} &\cong y_{2,1} \left( \frac{K[y_{4,1}, \dots, y_{3s_1+1,1}, y_{2,2}, \dots, y_{3s_2+1,2}, \dots, y_{2,k_2}, \dots, y_{3s_{k_2}+1,k_2}]}{(y_{4,1}y_{5,1}, \dots, y_{3s_1,1}y_{3s_1+1,1}, y_{2,2}y_{3,2}, \dots, y_{2,k_2}y_{3,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})} \right) [y_{2,1}] \\ &\oplus y_{3s_1+1,1} \left( \frac{K[y_{3,1}, \dots, y_{3s_1-1,1}, y_{2,2}, \dots, y_{3s_2+1,2}, \dots, y_{2,k_2}, \dots, y_{3s_{k_2}+1,k_2}]}{(y_{3,1}y_{4,1}, \dots, y_{3s_1-2,1}y_{3s_1-1,1}, \dots, y_{2,k_2}y_{3,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})} \right) [y_{3s_1+1,1}] \\ &\oplus y_{2,2} \left( \frac{K[y_{3,1}, \dots, y_{3s_1,1}, y_{4,2}, \dots, y_{3s_2+1,2}, \dots, y_{2,k_2}, \dots, y_{3s_{k_2}+1,k_2}]}{(y_{3,1}y_{4,1}, \dots, y_{3s_1-1,1}y_{3s_1,1}, y_{4,2}y_{5,2}, \dots, y_{3s_2,2}y_{3s_2+1,2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})} \right) [y_{2,2}] \\ &\oplus y_{3s_2+1,2} \left( \frac{K[y_{3,1}, \dots, y_{3s_1,1}, y_{3,2}, \dots, y_{3s_2-1,2}, \dots, y_{2,k_2}, \dots, y_{3s_{k_2}+1,k_2}]}{(y_{3,1}y_{4,1}, \dots, y_{3,2}y_{4,2}, \dots, y_{3s_2-2,2}y_{3s_2-1,2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})} \right) [y_{3s_2+1,2}] \\ &\oplus \cdots \\ &\oplus y_{2,k_2} \left( \frac{K[y_{3,1}, \dots, y_{3s_1,1}, \dots, y_{3,k_2-1}, \dots, y_{3s_{k_2-1},k_2-1}, y_{4,k_2}, \dots, y_{3s_{k_2}+1,k_2}]}{(y_{3,1}y_{4,1}, \dots, y_{3s_1-1,1}y_{3s_1,1}, \dots, y_{4,k_2}y_{5,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})} \right) [y_{2,k_2}] \\ &\oplus y_{3s_{k_2}+1,k_2} \left( \frac{K[y_{3,1}, \dots, y_{3s_1,1}, \dots, y_{3,k_2-1}, \dots, y_{3s_{k_2-1},k_2-1}, y_{3,k_2}, \dots, y_{3s_{k_2}-1,k_2}]}{(y_{3,1}y_{4,1}, \dots, y_{3s_1-1,1}y_{3s_1,1}, \dots, y_{3,k_2}y_{4,k_2}, \dots, y_{3s_{k_2}-2,k_2}y_{3s_{k_2}-1,k_2})} \right) [y_{3s_{k_2}+1,k_2}] \end{aligned}$$

Indeed, if  $u \in (J : x_1)$  is a monomial such that  $u \notin J$ , then there exists some  $i \in \{1, \dots, k_2\}$  such that  $y_{j,i} \mid u$ , where  $j = 2$  or  $3s_i + 1$ .

If  $y_{2,1} \mid u$ , then we can write  $u$  as  $u = y_{2,1}^\alpha v$  with  $\alpha \geq 1$  and  $y_{2,1} \nmid v$ . Since  $u \notin J$ , we have that  $v \in K[y_{4,1}, \dots, y_{3s_1+1,1}, y_{2,2}, \dots, y_{3s_2+1,2}, \dots, y_{2,k_2}, \dots, y_{3s_{k_2}+1,k_2}]$  and  $v \notin (y_{4,1}y_{5,1}, \dots, y_{3s_1,1}y_{3s_1+1,1}, y_{2,2}y_{3,2}, \dots, y_{3s_2,2}y_{3s_2+1,2}, \dots, y_{2,k_2}y_{3,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})$ . Similarly, if  $y_{3s_1+1,1} \mid u$  and  $y_{2,1} \nmid u$ , then  $u = y_{3s_1+1,1}^\alpha v$  with  $\alpha \geq 1$  and  $v \in K[y_{3,1}, \dots, y_{3s_1-1,1}, y_{2,2}, \dots, y_{3s_2+1,2}, \dots, y_{2,k_2}, \dots, y_{3s_{k_2}+1,k_2}]$  and  $v \notin (y_{3,1}y_{4,1}, \dots, y_{3s_1-2,1}y_{3s_1-1,1}, y_{2,2}y_{3,2}, \dots, y_{3s_2,2}y_{3s_2+1,2}, \dots, y_{2,k_2}y_{3,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})$ . If  $u$  is a monomial such that  $y_{2,2} \mid u$ ,  $y_{2,1} \nmid u$  and  $y_{3s_1+1,1} \nmid u$ , then we have that  $u = y_{2,2}^\alpha v$  with  $\alpha \geq 1$  and  $v \in K[y_{3,1}, \dots, y_{3s_1,1}, y_{4,2}, \dots, y_{3s_2+1,2}, \dots, y_{2,k_2}, \dots, y_{3s_{k_2}+1,k_2}]$  and  $v \notin (y_{3,1}y_{4,1}, \dots, y_{3s_1-1,1}y_{3s_1,1}, y_{4,2}y_{5,2}, \dots, y_{3s_2,2}y_{3s_2+1,2}, \dots, y_{2,k_2}y_{3,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})$ . Other cases can be shown in a similar way as the above.

Therefore, by [13, Proposition 2.2.20, Theorem 2.2.21] and Lemma 2.6, it follows that  $\text{depth}(\frac{J:x_1}{J}) = \sum_{i=1}^{k_2-1} \lceil \frac{3s_i-2}{3} \rceil + \lceil \frac{3s_{k_2}-3}{3} \rceil + 1 = \sum_{i=1}^{k_2} s_i$ .

Now, using Lemma 2.4 on the short exact sequence

$$(4) \quad 0 \longrightarrow (J : x_1)/J \longrightarrow S/J \longrightarrow S/(J : x_1) \longrightarrow 0,$$

this completes the proof.  $\square$

**Theorem 4.3.** *Let  $G$  be a graph as in Theorem 4.2 and  $J$  be its edge ideal. Then*

$$s\text{depth}(\frac{S}{J}) \leq \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i - k_1 + 1.$$

*Proof.* Let  $d = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i - k_1 + 1$  and  $\sigma = (a(1), a(2, 1), \dots, a(3r_1, 1), \dots, a(2, k_1), \dots, a(3r_{k_1}, k_1), b(2, 1), \dots, b(3s_1+1, 1), \dots, b(2, k_2), \dots, b(3s_{k_2}+1, k_2), c(2, 1), \dots, c(3t_1+2, 1), \dots, c(2, k_3), \dots, c(3t_{k_3}+2, k_3)) \in \mathbb{N}^n$ , where  $a(1) = 1$ , for any  $1 \leq i \leq k_1$ ,  $1 \leq j \leq r_i - 1$ ,  $a(l, i) = \begin{cases} 1 : l = 3j + 1 \\ 0 : \text{otherwise} \end{cases}$ , for any  $1 \leq i \leq k_2$ ,  $1 \leq j \leq s_i - 1$ ,  $b(l, i) = \begin{cases} 1 : l = 3j + 1 \text{ or } l = 3s_i \\ 0 : \text{otherwise} \end{cases}$ , and for any  $1 \leq i \leq k_3$ ,  $1 \leq j \leq t_i$ ,  $c(l, i) = \begin{cases} 1 : l = 3j + 1 \\ 0 : \text{otherwise} \end{cases}$ . From the proof of Theorem 3.4, it is enough to prove that  $\mathcal{P}_{d+1, \sigma} = \emptyset$ . Indeed, if monomial

$$u = x_1 \prod_{i=1}^{k_1} \left( \prod_{j=1}^{r_i-1} x_{3j+1, i} \right) \prod_{i=1}^{k_2} \left( y_{3s_i, i} \prod_{j=1}^{s_i-1} y_{3j+1, i} \right) \prod_{i=1}^{k_3} \left( \prod_{j=1}^{t_i} z_{3j+1, i} \right),$$

one can easily see that if  $a = (a(1), a(2, 1), \dots, a(3r_1, 1), \dots, a(2, k_1), \dots, a(3r_{k_1}, k_1), b(2, 1), \dots, b(3s_1+1, 1), \dots, b(2, k_2), \dots, b(3s_{k_2}+1, k_2), c(2, 1), \dots, c(3t_1+2, 1), \dots, c(2, k_3), \dots, c(3t_{k_3}+2, k_3)) \in \mathbb{N}^n$  such that  $a(l, i) \neq 0$  for some  $l \notin \{3j + 1 \mid 1 \leq j \leq r_i - 1, 1 \leq i \leq k_1\}$  or  $b(l, i) \neq 0$  for some  $l \notin \{3j + 1, 3s_i \mid 1 \leq j \leq s_i, 1 \leq i \leq k_2\}$  or  $c(l, i) \neq 0$  for some  $l \notin \{3j + 1 \mid 1 \leq j \leq t_i, 1 \leq i \leq k_3\}$ , then  $u \cdot x^a \in I$ . Therefore  $\mathcal{P}_{d+1, \sigma} = \emptyset$ , thus we obtain the required result.  $\square$

**Theorem 4.4.** *Let  $G$  be a graph as in Theorem 4.2. Then*

$$s\text{depth}(\frac{J}{I}) \geq \text{depth}(\frac{J}{I}) = \begin{cases} \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_3} t_i - k_1 + 2, & \text{if } k_2 = 0, \\ \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i - k_1 + 1, & \text{otherwise.} \end{cases}$$

*In particular,  $J/I$  satisfies the Stanley conjecture.*

*Proof.* We have the S-module isomorphism:

$$\begin{aligned} \frac{J}{I} &\cong \bigoplus_{i=1}^{k_1} x_1 x_{3r_i, i} \left( \frac{K[x_{3,1}, \dots, x_{3r_1-1,1}, \dots, x_{3,i-1}, \dots, x_{3r_{i-1}-1, i-1}, x_{3,i}, \dots, \widehat{x_{3r_{i-1}, i}}, x_{3,i+1}, \dots, x_{3r_{i+1}, i+1}, \dots, x_{3, k_1}, \dots, x_{3r_{k_1}, k_1}]}{(x_{3,1}x_{4,1}, \dots, x_{3r_1-2,1}x_{3r_1-1,1}, \dots, x_{3,i}x_{4,i}, \dots, x_{3r_{i-3}, i}x_{3r_{i-2}, i}, x_{3,i+1}x_{4,i+1}, \dots, x_{3r_{k_1}-1, k_1}x_{3r_{k_1}, k_1})} \right) \\ &\otimes_K \frac{K[y_{3,1}, \dots, y_{3s_1+1,1}, \dots, y_{3,k_2}, \dots, y_{3s_{k_2}+1, k_2}]}{(y_{3,1}y_{4,1}, \dots, y_{3s_1-1,1}y_{3s_1+1,1}, \dots, y_{3s_{k_2}-1, k_2}y_{3s_{k_2}+1, k_2})} \otimes_K \frac{K[z_{3,1}, \dots, z_{3t_1+2,1}, \dots, z_{3,k_3}, \dots, z_{3t_{k_3}+2, k_3}]}{(z_{3,1}z_{4,1}, \dots, z_{3t_1+1,1}z_{3t_1+2,1}, \dots, z_{3s_{k_3}+1, k_3}z_{3t_{k_3}+2, k_3})} \left[ x_1, x_{3r_i, i} \right] \\ &\oplus \left( \bigoplus_{i=1}^{k_2} x_1 y_{3s_i+1, i} \left( \frac{K[y_{3,1}, \dots, y_{3s_1,1}, \dots, y_{3,i-1}, \dots, y_{3s_{i-1}, i-1}, y_{3,i}, \dots, \widehat{y_{3s_i, i}}, y_{3,i+1}, \dots, y_{3s_{i+1}, i+1}, \dots, y_{3, k_2}, \dots, y_{3s_{k_2}+1, k_2}]}{(y_{3,1}y_{4,1}, \dots, y_{3s_1-1,1}y_{3s_1+1,1}, \dots, y_{3,i}y_{4,i}, \dots, y_{3s_{i-2}, i}y_{3s_{i-1}, i}, y_{3,i+1}y_{4,i+1}, \dots, y_{3s_{k_2}-1, k_2}y_{3s_{k_2}+1, k_2})} \right) \otimes_K \right. \end{aligned}$$

$$\begin{aligned} & \frac{K[x_{3,1}, \dots, x_{3r_1-1,1}, \dots, x_{3,k_1}, \dots, x_{3r_{k_1}-1,k_1}]}{(x_{3,1}x_{4,1}, \dots, x_{3r_1-2,1}x_{3r_1-1,1}, \dots, x_{3r_{k_1}-2,k_1}x_{3r_{k_1}-1,k_1})} \otimes_K \frac{K[z_{3,1}, \dots, z_{3t_1+2,1}, \dots, z_{3,k_3}, \dots, z_{3t_{k_3}+2,k_3}]}{(z_{3,1}z_{4,1}, \dots, z_{3t_1+1,1}z_{3t_1+2,1}, \dots, z_{3s_{k_3}+1,k_3}z_{3t_{k_3}+2,k_3})} (x_1, y_{3s_i+1, i}) \\ & \oplus \left( \bigoplus_{i=1}^{k_3} x_1 z_{3t_i+2, i} \left( \frac{K[z_{3,1}, \dots, z_{3t_1+1,1}, \dots, z_{3,i-1}, \dots, z_{3t_{i-1}+1,i-1}, z_{3,i}, \dots, z_{3t_i+1,i}, z_{3,i+1}, \dots, z_{3t_{i+1}+2,i+1}, \dots, z_{3,k_3}, \dots, z_{3t_{k_3}+2,k_3}]}{(z_{3,1}z_{4,1}, \dots, z_{3t_1+1,1}z_{3t_1+2,1}, \dots, z_{3,i}z_{4,i}, \dots, z_{3t_{i-1}+1,i}z_{3t_i,i}, z_{3,i+1}z_{4,i+1}, \dots, z_{3t_{k_3}+1,k_3}z_{3t_{k_3}+2,k_3})} \otimes_K \right. \right. \\ & \left. \left. \frac{K[x_{3,1}, \dots, x_{3r_1-1,1}, \dots, x_{3,k_1}, \dots, x_{3r_{k_1}-1,k_1}]}{(x_{3,1}x_{4,1}, \dots, x_{3r_1-2,1}x_{3r_1-1,1}, \dots, x_{3r_{k_1}-2,k_1}x_{3r_{k_1}-1,k_1})} \otimes_K \frac{K[y_{3,1}, \dots, y_{3s_1+1,1}, \dots, y_{3,k_2}, \dots, y_{3s_{k_2}+1,k_2}]}{(y_{3,1}y_{4,1}, \dots, y_{3s_1-1,1}y_{3s_1,1}, \dots, y_{3s_{k_2}-1,k_2}y_{3s_{k_2},k_2})} \right) (x_1, z_{3t_i+2, i}), \end{aligned}$$

where  $x_{i,0} = y_{j,0} = z_{l,0} = 0$  for  $3 \leq i \leq k_1$ ,  $3 \leq j \leq k_2$  and  $3 \leq l \leq k_3$ .

Indeed, let  $u \in J$  be a monomial such that  $u \notin I$ , then  $x_1 x_{3r_i, i} | u$  or  $x_1 y_{3s_j+1, j} | u$  or  $x_1 z_{3t_l+2, l} | u$  for some  $1 \leq i \leq k_1$  or  $1 \leq j \leq k_2$  or  $1 \leq l \leq k_3$ .

If  $x_1 x_{3r_1, 1} | u$ , then we can write  $u$  as  $u = x_1^\alpha x_{3r_1, 1}^\beta v$  with  $\alpha, \beta \geq 1$ ,  $x_1 \nmid v$  and  $x_{3r_1, 1} \nmid v$ . Since  $u \notin I$ , we have that  $v \in K[x_{3,1}, \dots, x_{3r_1-2,1}, x_{3,2}, \dots, x_{3r_2,2}, \dots, x_{3,k_1}, \dots, x_{3r_{k_1}-1,k_1}, y_{3,1}, \dots, y_{3s_1+1,1}, \dots, y_{3,k_2}, \dots, y_{3s_{k_2}+1,k_2}, z_{3,1}, \dots, z_{3t_1+2,1}, \dots, z_{3,k_3}, \dots, z_{3t_{k_3}+2,k_3}]$  and  $v \notin (x_{3,1}x_{4,1}, \dots, x_{3r_1-3,1}x_{3r_1-2,1}, x_{3,2}x_{4,2}, \dots, x_{3r_2-1,2}x_{3r_2,2}, \dots, x_{3,k_1}x_{4,k_1}, \dots, x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1}, y_{3,1}y_{4,1}, \dots, y_{3s_1,1}y_{3s_1+1,1}, \dots, y_{3,k_2}y_{4,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2}, z_{3,1}z_{4,1}, \dots, z_{3t_1+1,1}z_{3t_1+2,1}, \dots, z_{3,k_3}z_{4,k_3}, \dots, z_{3t_{k_3}+1,k_3}z_{3t_{k_3}+2,k_3})$ . Similarly, if  $x_1 x_{3r_2, 2} | u$  and  $x_1 x_{3r_1, 1} \nmid u$ , then  $u = x_1^\alpha x_{3r_2, 2}^\beta v$  with  $\alpha, \beta \geq 1$  and  $v \in K[x_{3,1}, \dots, x_{3r_1-1,1}, x_{3,2}, \dots, x_{3r_2-2,2}, x_{3,3}, \dots, x_{3r_3,3}, \dots, x_{3,k_1}, \dots, x_{3r_{k_1}-1,k_1}, y_{3,1}, \dots, y_{3s_1+1,1}, \dots, y_{3,k_2}, \dots, y_{3s_{k_2}+1,k_2}, z_{3,1}, \dots, z_{3t_1+2,1}, \dots, z_{3,k_3}, \dots, z_{3t_{k_3}+2,k_3}]$  and  $v \notin (x_{3,1}x_{4,1}, \dots, x_{3r_1-2,1}x_{3r_1-1,1}, x_{3,2}x_{4,2}, \dots, x_{3r_2-3,2}x_{3r_2-2,2}, x_{3,3}x_{4,3}, \dots, x_{3r_3-1,3}x_{3r_3,3}, \dots, x_{3,k_1}x_{4,k_1}, \dots, x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1}, y_{3,1}y_{4,1}, \dots, y_{3s_1,1}y_{3s_1+1,1}, \dots, y_{3,k_2}y_{4,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2}, z_{3,1}z_{4,1}, \dots, z_{3t_1+1,1}z_{3t_1+2,1}, \dots, z_{3,k_3}z_{4,k_3}, \dots, z_{3t_{k_3}+1,k_3}z_{3t_{k_3}+2,k_3})$ . Other cases can be shown in a similar way as the above.

Therefore, by Lemmas 2.4–2.7, 3.2 and [13, Proposition 2.2.20, Theorem 2.2.21], if  $k_2 \neq 0$ , then  $\text{sdepth}(\frac{J}{I}) \geq \text{depth}(\frac{J}{I}) = \min\left\{\sum_{i=1}^{k_1-1} \lceil \frac{3r_i-3}{3} \rceil + \lceil \frac{3r_{k_1}-4}{3} \rceil + \sum_{i=1}^{k_2} \lceil \frac{3s_i-1}{3} \rceil + \sum_{i=1}^{k_3} \lceil \frac{3t_i}{3} \rceil, \sum_{i=1}^{k_1} \lceil \frac{3r_i-3}{3} \rceil + \sum_{i=1}^{k_2-1} \lceil \frac{3s_i-2}{3} \rceil + \lceil \frac{3s_{k_2}-3}{3} \rceil + \sum_{i=1}^{k_3} \lceil \frac{3t_i}{3} \rceil, \sum_{i=1}^{k_1} \lceil \frac{3r_i-3}{3} \rceil + \sum_{i=1}^{k_2} \lceil \frac{3s_i-2}{3} \rceil + \sum_{i=1}^{k_3-1} \lceil \frac{3t_i-1}{3} \rceil + \lceil \frac{3t_{k_3}-2}{3} \rceil\right\} + 2 = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i - k_1 + 1.$

If  $k_2 = 0$ , then  $\text{sdepth}(\frac{J}{I}) \geq \text{depth}(\frac{J}{I}) = \min\left\{\sum_{i=1}^{k_1-1} \lceil \frac{3r_i-3}{3} \rceil + \lceil \frac{3r_{k_1}-4}{3} \rceil + \sum_{i=1}^{k_3} \lceil \frac{3t_i}{3} \rceil, \sum_{i=1}^{k_1} \lceil \frac{3r_i-3}{3} \rceil + \sum_{i=1}^{k_3-1} \lceil \frac{3t_i-1}{3} \rceil + \lceil \frac{3t_{k_3}-2}{3} \rceil\right\} + 2 = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_3} t_i - k_1 + 2$ . This completes the proof.  $\square$

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SCHOOL OF MATHEMATICAL SCIENCES, SOOCHOW UNIVERSITY, SUZHOU 215006, P. R. CHINA  
*E-mail address:* zhuguangjun@suda.edu.cn