

# FINITENESS PROPERTIES OF FORMAL LOCAL HOMOLOGY MODULES

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**ABSTRACT.** Let  $(R, \mathfrak{m})$  be a commutative Noetherian ring,  $\mathfrak{a}$  an ideal of  $R$  and  $M$  an Artinian  $R$ -module. In this paper, we investigate the structure of the formal local homology. We prove several results concerning finiteness properties of formal local homology module.

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## 1. INTRODUCTION

Throughout this paper, we assume that  $(R, \mathfrak{m})$  is a commutative Noetherian local ring which have non-zero identity. Let  $\mathfrak{a}$  be an ideal of  $R$  and  $M$  an  $R$ -module. It is well known that the  $\mathfrak{a}$ -adic completion functor  $\Lambda_{\mathfrak{a}}$  is defined by  $\Lambda_{\mathfrak{a}}(M) = \varprojlim_t M/\mathfrak{a}^t M$  (see [8], [9]). In [4], N. T. Coung and T. T. Nam defined the local homology module  $H_i^{\mathfrak{a}}(M)$  with respect to  $\mathfrak{a}$  by

$$H_i^{\mathfrak{a}}(M) = \varprojlim_n \mathrm{Tor}_i^R(R/\mathfrak{a}^n, M).$$

If we use  $L_i^{\mathfrak{a}}(M)$  to denote the  $i$ -th left derived module of  $\Lambda_{\mathfrak{a}}(M)$ , then  $H_i^{\mathfrak{a}}(M) = L_i^{\mathfrak{a}}(M)$  for the Artinian  $R$ -modules (see [4]). For an Artinian  $R$ -module  $M$ , in [2], the  $i$ -th formal local homology module of  $M$  with respect to  $\mathfrak{a}$  is defined by  $\varinjlim_n H_i^{\mathfrak{a}}((0 :_M \mathfrak{a}^n))$  and it is investigated its structure. While the formal local cohomology modules are studied in great detail not so much is known about the formal local homology modules (see [1], [5], [7]).

In this paper, for an integer  $i$  and an Artinian  $R$ -module  $M$ , let  $\mathfrak{F}_i^{\mathfrak{a}}(M)$  be the formal local homology module of  $M$  with respect to  $\mathfrak{a}$ . Let  $t$  be a non-negative integer. It is shown that the local homology module  $H_i^{\mathfrak{a}}(M)$  is Artinian for all  $i < t$  if and only if there is some non-negative integer  $s$  such that  $\mathfrak{a}^s H_i^{\mathfrak{a}}(M) = 0$  for all  $i < t$ , provided that  $M$  be Artinian  $R$ -module (see [4, 4.7]).

We generalize the above result for formal local homology modules. In fact the purpose of this paper is to answer the following question for the formal local homology modules: When are the formal local homology modules finitely generated?

The main aim of this paper is to prove that under some conditions,  $\mathfrak{F}_i^{\mathfrak{a}}(M)$  is finitely generated for all  $i > t$  if and only if  $\mathfrak{a} \subseteq \text{Rad}(\text{Ann}(\mathfrak{F}_i^{\mathfrak{a}}(M)))$  for all  $i > t$ .

We define the formal finiteness dimension and obtain some finiteness properties of the formal local homology modules. We also show that under some conditions, if  $\mathfrak{F}_i^{\mathfrak{a}}(M)$  is finitely generated for all  $i > t$  then  $\text{Hom}(R/\mathfrak{a}, \mathfrak{F}_i^{\mathfrak{a}}(M))$  is finitely generated.

Throughout this paper, for an  $R$ -module  $M$ ,  $E(R/\mathfrak{m})$  denotes the injective envelope of  $R/\mathfrak{m}$  and  $D(M)$  denotes the Matlis duality functor  $\text{Hom}(M, E(R/\mathfrak{m}))$ .

## 2. MAIN RESULTS

Let  $\underline{x} = x_1, \dots, x_r$  be a system of elements of  $R$ , and  $b = \text{Rad}(\underline{x})$ . Let  $C_{\underline{x}}$  denote the Čech complex of  $R$  with respect to  $\underline{x}$ , (see [3], [6]). For an  $R$ -module  $M$  and an ideal  $\mathfrak{a}$ , the direct system of  $R$ -modules  $\{(0 :_M \mathfrak{a}^n)\}_{n \in \mathbb{N}}$  induces a direct system of  $R$ -complexes  $\{\text{Hom}(C_{\underline{x}}, (0 :_M \mathfrak{a}^n))\}_{n \in \mathbb{N}}$ . In [2], the formal local homology module is defined by the following isomorphisms

$$H_i(\varinjlim_n \text{Hom}(C_{\underline{x}}, (0 :_M \mathfrak{a}^n))) \simeq \varinjlim_n H_i^b((0 :_M \mathfrak{a}^n)),$$

for all  $i \in \mathbb{Z}$ , provided that  $M$  be an Artinian  $R$ -module.

**Notation.** Let  $\mathfrak{a}$  be an ideal of a local ring  $(R, \mathfrak{m})$ ,  $M$  be an Artinian  $R$ -module and  $\mathfrak{m} = \text{Rad}(\underline{x})$ . We call  $\mathfrak{F}_i^{\mathfrak{a}}(M) := \varinjlim_n H_i^{\mathfrak{m}}((0 :_M \mathfrak{a}^n))$  the  $i$ -th  $\mathfrak{a}$ -formal local homology.

**Definition 2.1.** Let  $M$  be an Artinian  $R$ -module. For an ideal  $\mathfrak{a}$  of  $R$ , we define the formal finiteness dimension,  $ff^{\mathfrak{a}}(M)$  by

$$ff^{\mathfrak{a}}(M) := \inf\{i \mid \mathfrak{F}_i^{\mathfrak{a}}(M) \text{ is not finitely generated}\}.$$

**Proposition 2.2.** Let  $\mathfrak{a}, \mathfrak{b}$  be two ideals of  $R$  and  $M$  be an Artinian  $R$ -module. If  $\text{Rad}(\mathfrak{a}) = \text{Rad}(\mathfrak{b})$ , then  $ff^{\mathfrak{a}}(M) = ff^{\mathfrak{b}}(M)$ .

*Proof.* By assumption  $\text{Rad}(\mathfrak{a}) = \text{Rad}(\mathfrak{b})$ , it is easy to prove that if  $n \in \mathbb{N}$ , then there is a positive integer  $m$  such that  $(0 :_M \mathfrak{a}^n) = (0 :_M \mathfrak{b}^m)$ . So the result follows by the definition of the formal local homology.  $\square$

**Theorem 2.3.** Let  $\mathfrak{a}$  be an ideal of  $R$  and  $M$  be an Artinian  $R$ -module. Then  $\mathfrak{F}_i^{\mathfrak{a}}(M) \simeq \widehat{\mathfrak{F}}_i^{\mathfrak{a}}(M)$  for all  $i \in \mathbb{Z}$ , where  $\widehat{\phantom{x}}$  is the completion functor with respect to  $\mathfrak{m}$ .

*Proof.* It is a consequence of theorem [2, 2.6] for the particular case  $\mathfrak{b} = \mathfrak{m}$ .  $\square$

**Proposition 2.4.** Let  $\mathfrak{a}$  be an ideal of  $R$  and  $M$  be an Artinian  $R$ -module. Then  $ff^{\mathfrak{a}}(M) = ff^{\widehat{\mathfrak{a}}}(M)$ , where  $\widehat{\phantom{x}}$  is the completion functor with respect to  $\mathfrak{m}$ .

*Proof.* It immediately follows from Theorem 2.3.  $\square$

**Theorem 2.5.** Let  $\mathfrak{a}$  be an ideal of  $R$ . Let  $M$  be an Artinian  $R$ -module and  $x \in \mathfrak{m}$ . Then there is the long exact sequence

$$\dots \rightarrow \widehat{\mathfrak{F}}_i^{(\mathfrak{a}, x)}(M) \rightarrow \mathfrak{F}_i^{\mathfrak{a}}(M) \rightarrow R_x \otimes \mathfrak{F}_i^{\mathfrak{a}}(M) \rightarrow \dots$$

for all  $i \in \mathbb{Z}$ .

*Proof.* It is a consequence of theorem [2, 2.13] for the particular case  $\mathfrak{b} = \mathfrak{m}$ .  $\square$

**Corollary 2.6.** *Let  $x$  be an element of  $\mathfrak{m}$  and  $M$  be an Artinian  $R$ -module. Then there is a short exact sequence*

$$\cdots \rightarrow \varinjlim_n H_i^{\mathfrak{m}}((0 :_M x^n)) \rightarrow H_i^{\mathfrak{m}}(M) \rightarrow R_x \otimes H_i^{\mathfrak{m}}(M) \rightarrow \cdots$$

for all  $i \in \mathbb{Z}$ .

*Proof.* Whenever  $\mathfrak{a} = 0$ , it is a consequence of Theorem 2.5.  $\square$

**Theorem 2.7.** *Let  $\mathfrak{a}$  be an ideal of  $R$  and  $M$  be an Artinian  $R$ -module. Choose  $x \in R$  such that  $x \notin \mathfrak{a}$ . Then  $ff^{\mathfrak{a}}(M) \leq ff^{(\mathfrak{a}, x)}(M) + 1$ .*

*Proof.* By Theorem 2.5, there is the following long exact sequence

$$\cdots \rightarrow R_x \otimes \mathfrak{F}_{i+1}^{\mathfrak{a}}(M) \rightarrow \mathfrak{F}_i^{(\mathfrak{a}, x)}(M) \rightarrow \mathfrak{F}_i^{\mathfrak{a}}(M) \rightarrow \cdots$$

For all  $i < ff^{\mathfrak{a}}(M) - 1$ ,  $\mathfrak{F}_{i+1}^{\mathfrak{a}}(M)$  and  $\mathfrak{F}_i^{\mathfrak{a}}(M)$  are finitely generated. Therefore  $\mathfrak{F}_i^{(\mathfrak{a}, x)}(M)$  is finitely generated and this completes the proof.  $\square$

**Proposition 2.8.** *Let  $M$  be an Artinian  $R$ -module. Then the  $R$ -module  $H_i^{\mathfrak{m}}(M)$  is finitely generated for all  $i \geq 0$ .*

*Proof.* Since  $H_i^{\mathfrak{m}}(M) \simeq H_i^{\hat{\mathfrak{m}}}(M)$  as  $\hat{R}$ -module, so without loss of generality we may assume that  $R$  is a complete local ring. By Matlis duality,  $D(M)$  is finitely generated. Therefore  $H_{\mathfrak{m}}^i(D(M))$  is Artinian (see [3, 7.1.3]). But by [4, 3.3(ii)] follows that

$$H_i^{\mathfrak{m}}(M) \simeq H_i^{\mathfrak{m}}(D(D(M))) \simeq D(H_{\mathfrak{m}}^i(D(M))).$$

Thus  $H_i^{\mathfrak{m}}(M)$  is a finitely generated  $R$ -module.  $\square$

We recall the concept of Krull dimension of an Artinian  $R$ -module  $M$ , denoted by  $\text{Kdim}M$ . Let  $M$  be an Artinian  $R$ -module. When  $M = 0$  we put  $\text{Kdim}M = -1$ . Then by induction, for any ordinal  $\alpha$ , we put  $\text{Kdim}M = \alpha$  when (i)  $\text{Kdim}M < \alpha$  is false, and (ii) for every ascending chain,  $M_0 \subseteq M_1 \subseteq \cdots$  of submodules of  $M$ , there exists a positive integer  $m_0$  such that  $\text{Kdim}(M_{m+1}/M_m) < \alpha$  for all  $m \geq m_0$  (see [10]).

For an ideal  $\mathfrak{a}$  of  $(R, \mathfrak{m})$ , we can define the formal homological dimension of  $M$  with respect to  $\mathfrak{a}$ , by

$$f\text{-cd}(\mathfrak{a}, M) = \sup\{i \in \mathbb{Z} : \mathfrak{F}_i^{\mathfrak{a}}(M) \neq 0\}.$$

**Theorem 2.9.** *Let  $\mathfrak{a}$  be an ideal of  $(R, \mathfrak{m})$ . Then  $\text{Kdim}(0 :_M \mathfrak{a}) = \sup\{i \in \mathbb{Z} : \mathfrak{F}_i^{\mathfrak{a}}(M) \neq 0\}$ , for any non-zero Artinian  $R$ -module  $M$ .*

*Proof.* Since  $\text{Kdim}(0 :_M \mathfrak{a}^n) = \text{Kdim}(0 :_M \mathfrak{a})$  for all  $n \in \mathbb{N}$ , by [4, 4.8],  $\mathfrak{F}_i^{\mathfrak{a}}(M) = 0$  for all  $i > \text{Kdim}(0 :_M \mathfrak{a})$ . Therefore

$$\text{Kdim}(0 :_M \mathfrak{a}) \geq \sup\{i \in \mathbb{Z} : \mathfrak{F}_i^{\mathfrak{a}}(M) \neq 0\}.$$

In order to prove the equality, put  $\text{Kdim}(0 :_M \mathfrak{a}) = s$ . First let we consider the short exact sequence

$$0 \rightarrow (0 :_M \mathfrak{a}^n) \rightarrow (0 :_M \mathfrak{a}^{n+1}) \rightarrow \frac{(0 :_M \mathfrak{a}^{n+1})}{(0 :_M \mathfrak{a}^n)} \rightarrow 0.$$

Because of  $\text{Kdim}((0 :_M \mathfrak{a}^{n+1})/(0 :_M \mathfrak{a}^n)) < s$ ,  $H_{s+1}^{\mathfrak{m}}((0 :_M \mathfrak{a}^{n+1})/(0 :_M \mathfrak{a}^n)) = 0$  that induces a monomorphism  $0 \rightarrow H_s^{\mathfrak{m}}((0 :_M \mathfrak{a}^n)) \rightarrow H_s^{\mathfrak{m}}((0 :_M \mathfrak{a}^{n+1}))$  of non-zero  $R$ -modules for all  $n \in \mathbb{N}$  (see [4, 4.2, 4.10]). So  $\mathfrak{F}_s^{\mathfrak{a}}(M) := \varinjlim H_s^{\mathfrak{m}}((0 :_M \mathfrak{a}^n)) \neq 0$  and the claim is proved.  $\square$

**Theorem 2.10.** *Let  $\mathfrak{a}$  denote an ideal of  $R$ . Let  $M$  be an Artinian  $R$ -module such that  $\text{Kdim } M = d$ . Then  $\mathfrak{F}_d^{\mathfrak{a}}(M)$  is finitely generated.*

*Proof.* Let  $\mathfrak{a} := (x_1, x_2, \dots, x_n)$ . We argue by induction on  $n$ . Let  $n = 1$ . By Corollary 2.6, we have the exact sequence

$$\cdots \rightarrow R_{\mathfrak{a}} \otimes H_{d+1}^{\mathfrak{m}}(M) \rightarrow \mathfrak{F}_d^{\mathfrak{a}}(M) \xrightarrow{\alpha} H_d^{\mathfrak{m}}(M) \rightarrow \cdots$$

By [4, 4.8],  $H_{d+1}^{\mathfrak{m}}(M) = 0$ . So  $\mathfrak{F}_d^{\mathfrak{a}}(M) \simeq \text{Im } \alpha \subseteq H_d^{\mathfrak{m}}(M)$  and the claim follows from Proposition 2.8. Now suppose, inductively, that the result has been proved for  $n - 1$  and let we put  $\mathfrak{b} = (x_1, x_2, \dots, x_{n-1})$ . By Theorem 2.5, there is a long exact sequence

$$\cdots \rightarrow R_{x_n} \otimes \mathfrak{F}_{d+1}^{\mathfrak{b}}(M) \rightarrow \mathfrak{F}_d^{\mathfrak{a}}(M) \xrightarrow{\beta} \mathfrak{F}_d^{\mathfrak{b}}(M) \rightarrow \cdots$$

So, from Theorem 2.9,  $\mathfrak{F}_{d+1}^{\mathfrak{b}}(M) = 0$  and  $\mathfrak{F}_d^{\mathfrak{a}}(M) \simeq \text{Im } \beta \subseteq \mathfrak{F}_d^{\mathfrak{b}}(M)$ . Therefore the induction hypothesis yields that  $\mathfrak{F}_d^{\mathfrak{a}}(M)$  is finitely generated.  $\square$

**Lemma 2.11.** *Let  $\mathfrak{a}$  denote an ideal of  $R$ ,  $M$  an Artinian  $R$ -module and  $N \subset M$  be a summand of  $M$  such that  $\text{Supp}(M/N) \subseteq V(\mathfrak{a})$ . Then there is a natural isomorphism  $\mathfrak{F}_i^{\mathfrak{a}}(M/N) \simeq H_i^{\mathfrak{m}}(N)$  for all  $i$  and there exists a long exact sequence*

$$\cdots \rightarrow H_{i+1}^{\mathfrak{m}}(M/N) \rightarrow \mathfrak{F}_i^{\mathfrak{a}}(N) \rightarrow \mathfrak{F}_i^{\mathfrak{a}}(M) \rightarrow H_i^{\mathfrak{m}}(M/N) \rightarrow \cdots$$

*Proof.* Since  $\text{Supp}(M/N) \subseteq V(\mathfrak{a})$ , it follows that  $M/N$  is annihilated by some power of  $\mathfrak{a}$ . So

$$\mathfrak{F}_{i+1}^{\mathfrak{a}}(M/N) \simeq \varinjlim_n H_i^{\mathfrak{m}}((0 :_{M/N} \mathfrak{a}^n)) \simeq \varinjlim_n H_i^{\mathfrak{m}}(M/N) \simeq H_i^{\mathfrak{m}}(M/N),$$

for all  $i$ . By [2, 2.9], the short exact sequence  $0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$  induces a long exact sequence

$$\cdots \rightarrow \mathfrak{F}_{i+1}^{\mathfrak{a}}(M/N) \rightarrow \mathfrak{F}_i^{\mathfrak{a}}(N) \rightarrow \mathfrak{F}_i^{\mathfrak{a}}(M) \rightarrow \mathfrak{F}_i^{\mathfrak{a}}(M/N) \rightarrow \cdots$$

and, with the isomorphisms above, the claim is proved for all  $i$ .  $\square$

**Theorem 2.12.** *Let  $M$  be an Artinian  $R$ -module and  $t$  be a non-negative integer, such that  $\mathfrak{a} \subseteq \text{Rad}(\text{Ann}(\mathfrak{F}_i^{\mathfrak{a}}(M)))$  for all  $i > t$ . If  $\Lambda_{\mathfrak{a}}(M)$  is projective then  $\mathfrak{F}_i^{\mathfrak{a}}(M)$  is finitely generated for all  $i > t$ .*

*Proof.* We use induction on  $\text{Kdim } M = d$ . Let  $d = 0$ .  $\mathfrak{F}_i^{\mathfrak{a}}(M) = 0$  for all  $i > 0$ , so in this case the claim holds. Now let  $d > 0$  and suppose that the claim holds for all value less than  $d$ . Since  $M$  is Artinian, there exists a positive integer  $s$  such that  $\mathfrak{a}^t M = \mathfrak{a}^s M$  for all  $t \geq s$ . So  $\bigcap_{t \geq 0} \mathfrak{a}^t M = \mathfrak{a}^s M$ ,  $\Lambda_{\mathfrak{a}}(M) = M/\mathfrak{a}^s M$  and we have a short exact sequence of Artinian  $R$ -modules

$$0 \rightarrow \bigcap \mathfrak{a}^t M \rightarrow M \rightarrow \Lambda_{\mathfrak{a}}(M) \rightarrow 0.$$

By Lemma 2.11 we get a long exact sequence of local homology modules

$$\cdots \rightarrow H_{i+1}^m(\Lambda_{\mathfrak{a}}(M)) \rightarrow \mathfrak{F}_i^{\mathfrak{a}}(\bigcap \mathfrak{a}^t M) \rightarrow \mathfrak{F}_i^{\mathfrak{a}}(M) \rightarrow H_i^m(\Lambda_{\mathfrak{a}}(M)) \rightarrow \cdots$$

So by Proposition 2.8 it is enough to prove that  $\mathfrak{F}_i^{\mathfrak{a}}(\bigcap \mathfrak{a}^t M)$  is finitely generated for all  $i > t$ . We can replace  $M$  by  $\bigcap \mathfrak{a}^t M$  and we may assume that  $\mathfrak{a}M = M$ . Since  $M$  is Artinian,  $xM = M$  for some  $x \in \mathfrak{a}$ . Thus, by the hypothesis, there exists a positive integer  $r$  such that  $x^r \mathfrak{F}_i^{\mathfrak{a}}(M) = 0$  for all  $i > t$ . Then by [2, 2.9] the short exact sequence

$$0 \rightarrow (0 :_M x^r) \rightarrow M \xrightarrow{x^r} M \rightarrow 0$$

induces an exact sequence

$$0 \rightarrow \mathfrak{F}_{i+1}^{\mathfrak{a}}(M) \rightarrow \mathfrak{F}_i^{\mathfrak{a}}((0 :_M x^r)) \rightarrow \mathfrak{F}_i^{\mathfrak{a}}(M) \rightarrow 0,$$

for all  $i > t$ . So, by the inductive hypothesis,  $\mathfrak{F}_i^{\mathfrak{a}}((0 :_M x^r))$  is finitely generated for all  $i > t$ . We see that  $\mathfrak{F}_i^{\mathfrak{a}}(M)$  is finitely generated for all  $i > t$ . This finishes the inductive step.  $\square$

**Theorem 2.13.** *Let  $\mathfrak{a}$  be an ideal of  $R$  and  $M$  be an Artinian  $R$ -module. Assume that the integer  $t$  is such that  $\mathfrak{F}_i^{\mathfrak{a}}(M)$  is finitely generated for all  $i > t$ . If  $\Lambda_{\mathfrak{a}}(M)$  is projective then  $(0 :_{\mathfrak{F}_i^{\mathfrak{a}}(M)} \mathfrak{a})$  is finitely generated.*

*Proof.* We use induction on  $\text{Kdim} M = n$ . For  $n = 0$  we have that  $\mathfrak{F}_i^{\mathfrak{a}}(M) = 0$  for all  $i > 0$  and  $\mathfrak{F}_0^{\mathfrak{a}}(M)$  is finitely generated by Theorem 2.10. Now let  $n > 0$  and suppose the claim holds for all values less than  $n$ . By the same argument as in the proof of Theorem 2.12, there is an element  $x \in \mathfrak{a}$ , such that  $xM = M$ . Then the short exact sequence of Artinian modules

$$0 \rightarrow (0 :_M x) \rightarrow M \xrightarrow{x} M \rightarrow 0,$$

implies that

$$\cdots \rightarrow \mathfrak{F}_{i+1}^{\mathfrak{a}}(M) \rightarrow \mathfrak{F}_i^{\mathfrak{a}}((0 :_M x)) \rightarrow \mathfrak{F}_i^{\mathfrak{a}}(M) \rightarrow \cdots$$

It yields that  $\mathfrak{F}_i^{\mathfrak{a}}((0 :_M x))$  is finitely generated for all  $i > t$ . Thus, by the induction hypothesis,  $(0 :_{\mathfrak{F}_i^{\mathfrak{a}}((0 :_M x))} \mathfrak{a})$  is finitely generated. Now consider the exact sequence

$$\cdots \rightarrow \mathfrak{F}_{t+1}^{\mathfrak{a}}(M) \xrightarrow{g} \mathfrak{F}_t^{\mathfrak{a}}((0 :_M x)) \xrightarrow{f} \mathfrak{F}_t^{\mathfrak{a}}(M) \xrightarrow{x} \mathfrak{F}_t^{\mathfrak{a}}(M) \rightarrow \cdots$$

which breaks into two short exact sequences

$$0 \rightarrow \text{Im} g \rightarrow \mathfrak{F}_t^{\mathfrak{a}}((0 :_M x)) \rightarrow \text{Im} f \rightarrow 0,$$

$$0 \rightarrow \text{Im} f \rightarrow \mathfrak{F}_t^{\mathfrak{a}}(M) \xrightarrow{x} \mathfrak{F}_t^{\mathfrak{a}}(M).$$

The first of these sequences induces a long exact sequence

$$\cdots \rightarrow (0 :_{\mathfrak{F}_t^{\mathfrak{a}}((0 :_M x))} \mathfrak{a}) \rightarrow (0 :_{\text{Im} f} \mathfrak{a}) \rightarrow \text{Ext}^1(R/\mathfrak{a}, \text{Im} g) \cdots$$

where  $(0 :_{\mathfrak{F}_t^{\mathfrak{a}}((0 :_M x))} \mathfrak{a})$  and  $\text{Ext}^1(R/\mathfrak{a}, \text{Im} g)$  are finitely generated. The second sequence induces a long exact sequence

$$0 \rightarrow (0 :_{\text{Im} f} \mathfrak{a}) \rightarrow (0 :_{\mathfrak{F}_t^{\mathfrak{a}}(M)} \mathfrak{a}) \xrightarrow{x} (0 :_{\mathfrak{F}_t^{\mathfrak{a}}(M)} \mathfrak{a}) \cdots$$

Since  $x \in \mathfrak{a}$ , it follows that  $(0 :_{\text{Im} f} \mathfrak{a}) \simeq (0 :_{\mathfrak{F}_t^{\mathfrak{a}}(M)} \mathfrak{a})$ . This completes the proof.  $\square$

**Corollary 2.14.** *Let  $M$  be an Artinian  $R$ -module. Then  $(0 :_{\mathfrak{F}_{f-cd(\mathfrak{a},M)}^{\mathfrak{a}}(M)} \mathfrak{a})$  is finitely generated.*

*Proof.* It immediately follows from Theorem 2.13. □

**Theorem 2.15.** *Let  $\mathfrak{a}$  be an ideal of  $R$  and  $M$  be an Artinian  $R$ -module. Let  $t$  be a non-negative integer. Then the following statements are equivalent:*

- (a)  $\mathfrak{m} \subseteq \text{Rad}(\text{Ann}(\mathfrak{F}_i^{\mathfrak{a}}(M)))$  for all  $i > t$ ,
- (b)  $\mathfrak{F}_i^{\mathfrak{a}}(M)$  is finitely generated for all  $i > t$ ,
- (c)  $\mathfrak{F}_i^{\mathfrak{a}}(M) = 0$  for all  $i > t$ .

*Proof.* (a)  $\Rightarrow$  (b). We use induction on  $\text{Kdim}M = d$ . Let  $d = 0$ .  $\mathfrak{F}_i^{\mathfrak{a}}(M) = 0$  for all  $i > 0$ , so in this case the claim holds. Now let  $d > 0$  and suppose that the claim hold for all value less than  $d$ . Since  $(0 :_M \mathfrak{a}^n)$  is Artinian, we can replace it by  $\bigcap_{t>0} \mathfrak{m}^t(0 :_M \mathfrak{a}^n)$  (see [4, 4.5]). We can assume that  $\mathfrak{m}(0 :_M \mathfrak{a}^n) = (0 :_M \mathfrak{a}^n)$  and since  $M$  is Artinian,  $x(0 :_M \mathfrak{a}^n) = (0 :_M \mathfrak{a}^n)$  for some  $x \in \mathfrak{m}$ . From the hypothesis it follows that there exists a positive integer  $s$  such that  $x^s \mathfrak{F}_i^{\mathfrak{a}}(M) = 0$  for all  $i > t$ . Thus for all  $i > t$  the exact sequence

$$0 \rightarrow (0 :_{(0 :_M x^s) \mathfrak{a}^n}) \rightarrow (0 :_M \mathfrak{a}^n) \xrightarrow{x^s} (0 :_M \mathfrak{a}^n) \rightarrow 0,$$

implies that

$$0 \rightarrow \mathfrak{F}_{i+1}^{\mathfrak{a}}(M) \rightarrow \mathfrak{F}_i^{\mathfrak{a}}((0 :_M x^s)) \rightarrow \mathfrak{F}_i^{\mathfrak{a}}(M) \rightarrow 0.$$

So, by the inductive hypothesis,  $\mathfrak{F}_i^{\mathfrak{a}}((0 :_M x^s))$  is finitely generated for all  $i > t$ , and we see that  $\mathfrak{F}_i^{\mathfrak{a}}(M)$  is finitely generated for all  $i > t$ .

(b)  $\Rightarrow$  (c). We use induction on  $\text{Kdim}M = d$ . Let  $d = 0$ .  $\mathfrak{F}_i^{\mathfrak{a}}(M) = 0$  for all  $i > 0$ , so in this case the claim holds. Now let  $d > 0$  and suppose that the claim hold for all value less than  $d$ . As we did in the proof of (a)  $\Rightarrow$  (b), there is the following exact sequence

$$\cdots \rightarrow \mathfrak{F}_{i+1}^{\mathfrak{a}}(M) \rightarrow \mathfrak{F}_i^{\mathfrak{a}}((0 :_M x^s)) \rightarrow \mathfrak{F}_i^{\mathfrak{a}}(M) \rightarrow \cdots$$

So, by the inductive hypothesis,  $\mathfrak{F}_i^{\mathfrak{a}}((0 :_M x^s)) = 0$  for all  $i > t$ . Thus  $\mathfrak{F}_i^{\mathfrak{a}}(M) = x^s \mathfrak{F}_i^{\mathfrak{a}}(M)$  and so,  $\mathfrak{F}_i^{\mathfrak{a}}(M) = 0$  for all  $i > t$ .

(c)  $\Rightarrow$  (a). It is clear. □

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