OPERATOR h-PREINVEX CLASS FOR CONTINUOUS FUNCTIONS OF SELFADJOINT OPERATORS

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Abstract. In this paper, firstly we define a new class of functions, namely, operator h-preinvex function. Secondly, we give some algebraic properties of this class. Finally we obtain new inequalities via Hermite-Hadamard type inequality for operator h-preinvex function.

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1. Introduction and Preliminaries

We know the (1.1) inequality in literature as Hermite-Hadamard,

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

where $f:\mathbb{R}\to\mathbb{R}$ is any convex function, $a,b\in\mathbb{R}$. It satisfies approximates of the mean value of $f:[a,b]\to\mathbb{R}$ continuous convex function. Let $A,B\in B(H)$ be selfadjoint operators, where H is a Hilbert space, $B(H)$ is all bounded operators from H to H. Then for every $x\in H$

$$A\leq B \qquad \text{means that} \qquad \langle Ax,x\rangle \leq \langle Bx,x\rangle$$

or

$$B\leq A \qquad \text{means that} \qquad \langle Bx,x\rangle \leq \langle Ax,x\rangle$$

Let $(H,\langle\cdot,\cdot\rangle)$ be a complex Hilbert space and A be a selfadjoint operator on it. The Gelfand transformation sets up a Φ *-isometrical isomorphism $C(Sp(A))$ among $C^*(A)$ the $C^*(A)$-algebra $C(Sp(A))$ is $C^*(A)$-algebra of all continuous complex-valued functions on spectrum A.

Let $f,g\in C(Sp(A))$ and $\alpha,\beta\in\mathbb{C}$

(1) $\Phi(\alpha f+\beta g)=\alpha\Phi(f)+\beta\Phi(g)$;
(2) $\Phi(fg)=\Phi(f)\Phi(g)\, and \Phi(f^*)=\Phi(f)^*$;
(3) $\|\Phi(f)\|=\|f\|:=sup_{t\in Sp(A)}|f(t)|$;
(4) $\Phi(f_0)=1\, and\, \Phi(f_1)=A,\, where f_0(t)=1\, and f_1(t)=t,\, for\ t\in Sp(A)$.

If f is a continuous complex-valued functions on $C(Sp(A))$, the element $\Phi(f)$ of $C^*(A)$ is denoted by $f(A)$, and we call it the continuous functional calculus for a bounded selfadjoint operator A. And then $f(A)$ is defined by $\Phi(f)$. Under these conditions, if for all $t\in Sp(A)$,

$$f(t)\geq 0 \qquad \text{then} \qquad f(A)\geq 0,$$

namely $f(A)$ is positive on H.

Let $f, g : Sp(A) \to \mathbb{R}$ be two functions. If for every $t \in Sp(A)$

$$f(t) \le g(t), \qquad \text{then} \qquad f(A) \le g(A)$$

in the operator order $B(H)$.

Now we give some known concepts and results.

Definition 1.1. [4] *Let $f, g : I \subseteq \mathbb{R} \to \mathbb{R}$ two functions.Then, we said to f and g are similarly ordered functions on I, if the following inequality holds for all $x, y \in I$*

$$0 \le \big[f(x) - f(y)\big]\big[g(x) - g(y)\big].$$

Definition 1.2. [4] *I and J are intervals in $\mathbb{R}, (0,1) \subseteq J$ and functions h and f are real non-negative functions defined on J and I respectively.In this case let $h : J \to \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $f : I \to \mathbb{R}$ is an h-convex function if f is non-negative and for all $x, y \in I$, $\alpha \in (0,1)$ we have*

$$f(\alpha x + (1-\alpha)y) \le h(\alpha)f(x) + h(1-\alpha)f(y).$$

Definition 1.3. [7] *Let F be a nonempty closed set in $\mathbb{R}^n$ let $f : F \to \mathbb{R}$ be a continuous function and let $\eta(.,.) : F \times F \to \mathbb{R}^n$ be a continuous bi function. Then a set F is said to be invex set with respect to $\eta(.,.)$, if for every $x, y \in F$ and $t \in [0,1]$*

$$y + t\eta(x, y) \in F.$$

Remark 1.4. [1] *It is obvious that every convex set is invex with respect to the map $\eta(x, y) = x - y$, but there exist invex sets which are not convex.*

Let X be a real vector space and $F \subseteq X$ be an invex set with respect to $\eta : F \times F \to X$. For every $x, y \in F$ the η-path P_{xv} joining the points x and $v := x + \eta(y, x)$ is defined as follows

$$P_{xv} := \{z : z = x + t\eta(y, x), t \in [0,1]\}.$$

The mapping η is said to be satisfies the condition (C)[3] if for every $x, y \in F$ and $t \in [0,1]$,

$$(C) \qquad \begin{cases} \eta(y, y + t\eta(x, y)) = -t\eta(x, y) \\ \eta(x, y + t\eta(x, y)) = (1-t)\eta(x, y). \end{cases}$$

Remark 1.5. *For every $x, y \in F$ and every $t_1, t_2 \in [0,1]$ from condition (C) we have*

$$\eta(y + t_2\eta(x, y), y + t_1\eta(x, y)) = (t_2 - t_1)\eta(x, y).$$

You can see [3] *and* [8] *for details.*

Definition 1.6. [2] *Let $F \subseteq B(H)_{sa}$ be an invex set with respect to $\eta : F \times F \to B(H)_{sa}$, where $B(H)_{sa}$ is denoted by all bounded self adjoint operators set from H to H. Then the continuous function $f : \mathbb{R} \to \mathbb{R}$ is said to be operator preinvex with respect to η on F, if for every $A, B \in F$ and $t \in [0,1]$*

$$f(A + t\eta(B, A)) \le (1-t)f(A) + tf(B)$$

in the operator order in $B(H)$.

Remark 1.7. [2] *Every operator convex function is on operator preinvex function with respect to the map $\eta(A, B) = A - B$ but converse does not holds.*

Example 1.8. [2] *The function $f(x) = -|x|$ is not a convex function, but it is a preinvex function with respect to η, where*

$$\eta(A, B) := \begin{cases} A - B, & A.B < 0 \\ B - A, & A.B > 0 \end{cases}$$

S.-H. Wang and X.-M. Liu [5] introduced the concept of operator s-preinvex function. They established some new Hermite-Hadamard type inequalities for operator s-preinvex functions, and provided the estimates of both sides of Hermite-Hadamard type inequality in which some operator s-preinvex functions of positive selfadjoint operators in Hilbert space was involved. And then, S.-H. Wang and X.-W.Sun [6] similarly introduced the concept of operator α-preinvex function and some inequalities.

So, in this paper firstly we define a new class of operator preinvex function, i.e operator h-preinvex function. Secondly we give some algebraic properties of this class. And finally we obtain some new inequalities of this class via Hermite-Hadamard type inequality.

2. MAIN RESULTS

Definition 2.1. *Let $(0,1) \subseteq I, h : I \to \mathbb{R}$ and $S \subseteq B(H)_{sa}$ be an invex set with respect to $\eta : S \times S \to B(H)_{sa}$. Then the continuous function $f : S \to \mathbb{R}$ said to be operator h- preinvex with respect to η on S, if for every $A, B \in S$ and $t \in (0,1)$*

$$f(A + t\eta(B, A)) \leq h(t)f(A) + h(1-t)f(B)$$

in the operator order in $B(H)$.

Proposition 2.2. *Let $S \subseteq B(H)_{sa}$ be an invex set respect to $\eta : S \times S \to B(H)_{sa}$ and $f : S \to \mathbb{R}$ be a continuous function. Assume that η satisfies condition (C) on S. Then for every $A, B \in S$ and $V = A + \eta(B, A)$ the function f is operator h-preinvex with respect to η on $\eta - path$ P_{AV} if and only the function*

$$\varphi_{x,A,B} : [0,1] \to \mathbb{R}$$

defined by

$$\varphi_{x,A,B}(t) :=< f(A + t\eta(B, A)x, x) >$$

is $h - convex$ on $[0,1]$ for every $x \in H, \| x \| = 1$

Proof. Suppose that $x \in H, ||x|| = 1$ and $\varphi_{x,A,B}$ is h-convex on [0,1]. For $t_1, t_2 \in [0,1]$ we defined by

$$C_1 := A + t_1\eta(B, A) \in P_{AV}$$
$$C_2 := A + t_1\eta(B, A) \in P_{AV}$$

we have for $\lambda \in [0,1]$

$$< f(C_1 + \lambda\eta(C_2, C_1)x, x) >=< f(A + t_1\eta(B, A) + \lambda\eta(A + t_2\eta(B, A), A + t_1\eta(B, A)))x, x >$$

Due to η satisfies condition (C) on S, we can write following

$$\begin{aligned} < f(C_1 + \lambda\eta(C_2, C_1)x, x) > &= < f(A + t_1\eta(B, A) + \lambda(t_2 - t_1)\eta(B, A))x, x > \\ &= < f(A + t_1\eta(B, A) + \lambda t_2\eta(B, A) - \lambda t_1\eta(B, A))x, x > \\ &= < f(A + (1-\lambda)t_1\eta(B, A) + \lambda t_2\eta(B, A))x, x > \\ &= < f(A + [(1-\lambda)t_1 + \lambda t_2]\eta(B, A))x, x > \\ &= \varphi_{x,A,B}((1-\lambda)t_1 + \lambda t_2) \end{aligned}$$

Since $\varphi_{x,A,B}$ is h-convex on $[0,1]$, we get

$$\begin{aligned} \varphi_{x,C_1,C_2}(\lambda) &= \varphi_{x,A,B}((1-\lambda)t_1 + \lambda t_2) \\ &\leq h(\lambda)\varphi_{x,A,B}(t_1) + h(1-\lambda)\varphi_{x,A,B}(t_2) \\ &\leq h(\lambda) < f(C_1)x, x > + h(1-\lambda) < f(C_2)x, x > \end{aligned}$$

Note:

$$\varphi_{x,A,B}(t_1) :=< f(A + t_1\eta(B, A))x, x >=< f(C_1)x, x >$$
$$\varphi_{x,A,B}(t_2) :=< f(A + t_2\eta(B, A))x, x >=< f(C_2)x, x >

Consequently, f is operator $h - preinvex$ with respect to η on $\eta - path\ P_{AV}$.

Inversely, Assume that for all $A, B \in S$ and $V = A + \eta(B, A)$, f is operator h-preinvex with respect to η on η-path P_{AV},then we will show that the function

$$\varphi_{x,A,B} : [0, 1] \to \mathbb{R}$$

defined by

$$\varphi_{x,A,B}(t) :=< f(A + t\eta(B, A))x, x >$$

is h-convex on $[0, 1]$ for every $x \in H, ||x|| = 1$.

Thus, since f is operator h-preinvex with respect to η on $\eta - path\ P_{AV}$ for $t_1, t_2 \in [0, 1]$ and $A, B \in S$, $\lambda \in [0, 1]$, we have

$$\begin{aligned} \varphi_{x,A,B}((1-\lambda)t_1 + \lambda t_2) &= < f(A + [(1-\lambda)t_1 + \lambda t_2]\eta(B, A))x, x > \\ &= < f(A + t_1\eta(B, A) - \lambda t_1\eta(B, A) + \lambda t_2\eta(B, A))x, x > \\ &= < f(A + t_1\eta(B, A) + (t_2 - t_1)\lambda\eta(B, A))x, x > \\ &= < f(A + t_1\eta(B, A) + \lambda[(t_2 - t_1)\eta(B, A)])x, x > \end{aligned}$$

Because η satisfies condition (C) on S, we can write below equality,

$$\varphi_{x,A,B}((1-\lambda)t_1 + \lambda t_2) =< f(A + t_1\eta(B, A) + \lambda[\eta(A + t_2\eta(B, A), A + t_1\eta(B, A))]) > .$$

In the circumstances, if we choose

$$A^* := A + t_1\eta(B, A) \qquad \text{and} \qquad B^* := A + t_2\eta(B, A),$$

then we can write followings

$$\begin{aligned} \varphi_{x,A,B}((1-\lambda)t_1 + \lambda t_2) &= < f(A^* + \lambda\eta(B^*, A^*))x, x > \\ &\leq h(\lambda) < f(A^*)x, x > +h(1-\lambda) < f(B^*)x, x > \\ &\leq h(\lambda) < f(A + t_1\eta(B, A))x, x > \\ &\quad +h(1-\lambda) < f(A + t_2\eta(B, A))x, x > \\ &\leq h(\lambda)\varphi_{x,A,B}(t_1) + h(1-\lambda)\varphi_{x,A,B}(t_2) \end{aligned}$$

Thereby proof is completed. □

Theorem 2.3. *Let f and g be two operator h-preinvex functions, and similarly ordered. If for all $t \in [0, 1], h(t) + h(1-t) \leq 1$ inequality satisfies, then $f.g$ is operator h-preinvex function.*

Proof. Since f and g are operator h-preinvex, we have

$$f(A + t\eta(B, A)) \leq h(1-t)f(A) + h(t)f(B)$$

$$g(A + t\eta(B, A)) \leq h(1-t)g(A) + h(t)g(B).$$

Now we multiply above inequalities side by side,

$$\begin{aligned} f(A + t\eta(B, A))g(A + t\eta(B, A)) &\leq [h(1-t)f(A) + h(t)f(B)][h(1-t)g(A) + h(t)g(B)] \\ &= [h(1-t)]^2 f(A)g(A) + h(t)h(1-t)[f(A)g(B) + f(B)g(A)] \\ &\quad +[h(t)]^2 f(B)g(B). \end{aligned}$$

Due to f and g similarly ordered, we can write the following inequality

$$\begin{aligned} f(A + t\eta(B, A))g(A + t\eta(B, A)) &= (f.g)(A + t\eta(B, A)) \\ &\leq [h(1-t)]^2 f(A)g(A) + h(t)h(1-t)[f(A)g(A) + f(B)g(B)] \\ &\quad +[h(t)]^2 f(B)g(B) \\ &= [h(1-t)f(A)g(A) + h(t)f(B)g(B)][h(t) + h(1-t)] \\ &\leq h(1-t)f(A)g(A) + h(t)f(B)g(B). \end{aligned}$$

Because of $h(t)+h(1-t)\leq 1$, we obtain following ,

$$\begin{aligned}(f.g)(A+t\eta(B,A)) &\leq h(1-t)f(A)g(A)+h(t)f(B)g(B)\\ &= h(1-t)(f.g)(A)+h(t)(f.g)(B)\end{aligned}$$

Then (fg) is operator h-preinvex function. So we have desire result. □

Theorem 2.4. *Let f and g be two operator h-preinvex functions, and similarly ordered. If for all $t\in[0,1], h(t)+h(1-t)\leq 1$ inequality satisfies, then $f.g$ is operator h-preinvex function.*

Proof. Since f and g are operator h-preinvex, we have

$$f(A+t\eta(B,A))\leq h(1-t)f(A)+h(t)f(B)$$

$$g(A+t\eta(B,A))\leq h(1-t)g(A)+h(t)g(B)$$

Now, we multiply above inequalities side by side,

$$\begin{aligned}f(A+t\eta(B,A))g(A+t\eta(B,A)) &\leq [h(1-t)f(A)+h(t)f(B)][h(1-t)g(A)+h(t)g(B)]\\ &= [h(1-t)]^2f(A)g(A)+h(t)h(1-t)[f(A)g(B)+f(B)g(A)]\\ &\quad +[h(t)]^2f(B)g(B).\end{aligned}$$

Due to f and g similarly ordered, we can write the following inequality

$$\begin{aligned}f(A+t\eta(B,A))g(A+t\eta(B,A)) &= (f.g)(A+t\eta(B,A))\\ &\leq [h(1-t)]^2f(A)g(A)+h(t)h(1-t)[f(A)g(A)+f(B)g(B)]\\ &\quad +[h(t)]^2f(B)g(B)\\ &= [h(1-t)f(A)g(A)+h(t)f(B)g(B)][h(t)+h(1-t)]\\ &\leq h(1-t)f(A)g(A)+h(t)f(B)g(B).\end{aligned}$$

Because of $h(t)+h(1-t)\leq 1$, we obtain following,

$$\begin{aligned}(f.g)(A+t\eta(B,A)) &\leq h(1-t)f(A)g(A)+h(t)f(B)g(B)\\ &= h(1-t)(f.g)(A)+h(t)(f.g)(B)\end{aligned}$$

Then (fg) is operator h-preinvex. So we have desire result. □

Theorem 2.5. *Let f is operator h-preinvex function. Then the following inequality holds,*

(2.1) $$f(A+(1-t)\eta(B,A))\leq [h(t)+h(1-t)][f(A)+f(B)]-f(A+t\eta(B,A))$$

or if we take

$$x:=A+t\eta(B,A),$$

then we have

$$A+(1-t)\eta(B,A)=2A+\eta(B,A)-x.$$

So we get the below inequality instead of (2.1)

$$f(2A+\eta(B,A)-x)\leq [h(t)+h(1-t)][f(A)+f(B)]-f(x).$$

Proof. Since $f:I\to\mathbb{R}$ is operator h-preinvex, $x=A+t\eta(B,A)\in I$. Then

$$\begin{aligned}f(2A+\eta(B,A)-x) &= f(2A+\eta(B,A)-(A+t\eta(B,A)))\\ &= f(A+(1-t)\eta(B,A))\end{aligned}$$

From operator h-preinvexity of f, we get following

(2.2) $$f(2A+\eta(B,A)-x)\leq h(t)f(A)+h(1-t)f(B).$$

Now we write another equality

$$\begin{aligned} h(t)f(A)+h(1-t)f(B) &= [h(t)+h(1-t)][f(A)+f(B)] \\ &\quad -[h(1-t)f(A)+h(t)f(B)] \\ &= [h(t)+h(1-t)][f(A)+f(B)]-f(A+t\eta(B,A)) \\ &= [h(t)+h(1-t)][f(A)+f(B)]-f(x) \end{aligned}$$

Namely,

$$h(t)f(A)+h(1-t)f(B)=[h(t)+h(1-t)][f(A)+f(B)]-f(x) \tag{2.3}$$

If we put (2.3) in (2.2), we can obtain desire result. So the proof is completed. □

Theorem 2.6. *Let $f: I \subseteq \mathbb{R} \to (0,\infty)$ be a operator h-preinvex with $A < A+\eta(B,A), h(\frac{1}{2}) \neq 0$ and $w: [A, A+\eta(B,A)] \to \mathbb{R}$ is a non-negative, integrable function and symmetric about $A+\frac{1}{2}\eta(B,A)$. Then we get the following inequality,*

$$\begin{aligned} \frac{1}{2h(\frac{1}{2})} f(\frac{2A+\eta(B,A)}{2}) \int_A^{A+\eta(B,A)} w(x)dx &\leq \int_A^{A+\eta(B,A)} f(x)w(x)dx \\ &\leq \frac{f(A)+f(B)}{2}(h(t)+h(1-t)) \int_A^{A+\eta(B,A)} w(x)dx \end{aligned}$$

Proof. According to the theorem's assertion, we get followings

$$\frac{1}{2h(\frac{1}{2})} f(\frac{2A+\eta(B,A)}{2}) \int_A^{A+\eta(B,A)} w(x)dx = \frac{1}{2h(\frac{1}{2})} \int_A^{A+\eta(B,A)} f(\frac{2A+\eta(B,A)}{2}) w(x)dx$$

$$\begin{aligned} &= \frac{1}{2h(\frac{1}{2})} \int_A^{A+\eta(B,A)} f(\frac{2A+\eta(B,A)-x+x}{2}) w(x)dx \\ &\leq \frac{1}{2h(\frac{1}{2})} \int_A^{A+\eta(B,A)} [h(\frac{1}{2})\{f(2A+\eta(B,A)-x)+f(x)\}] w(x)dx \\ &= \frac{1}{2}\int_A^{A+\eta(B,A)} f(2A+\eta(B,A)-x)w(2A+\eta(B,A)-x)dx \\ &\quad +\frac{1}{2}\int_A^{A+\eta(B,A)} f(x)w(x)dx \\ &= \int_A^{A+\eta(B,A)} f(x)w(x)dx \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2}\int_A^{A+\eta(B,A)} f(2A+\eta(B,A)-x)w(2A+\eta(B,A)-x)dx \\ &\quad +\frac{1}{2}\int_A^{A+\eta(B,A)} f(x)w(x)dx \\ &= \frac{1}{2}\int_A^{A+\eta(B,A)} f(2A+\eta(B,A)-x)w(x)dx \\ &\quad +\frac{1}{2}\int_A^{A+\eta(B,A)} f(x)w(x)dx \end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{2}\int_A^{A+\eta(B,A)} [(h(t)+h(1-t))[f(A)+f(B)]-f(x)]w(x)dx \\
&+\frac{1}{2}\int_A^{A+\eta(B,A)} f(x)w(x)dx \\
&\le \frac{f(A)+f(B)}{2}(h(t)+h(1-t))\int_A^{A+\eta(B,A)} w(x)dx
\end{aligned}
$$

So the proof is finished. □

Theorem 2.7. *Let $f : I \subseteq \mathbb{R} \to (0,\infty)$ and $w : I \subseteq \mathbb{R} \to (0,\infty)$ be operator h_1-preinvex and h_2-preinvex functions respectively with $A < A+\eta(B,A), h_1(\frac{1}{2}) \neq 0, h_2(\frac{1}{2}) \neq 0$. If the $\eta(.,.)$ satisfies condition (C), then the following inequality holds,*

$$
\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})} f(\frac{2A+\eta(B,A)}{2})w(\frac{2A+\eta(B,A)}{2}) - \frac{1}{\eta(B,A)}\int_A^{A+\eta(B,A)} f(x)w(x)dx
$$

$$
\le M(A,B)\int_0^1 h_1(t)h_2(1-t)dt + N(A,B)\int_0^1 h_1(t)h_2(t)dt
$$

where

$$
\begin{aligned}
M(A,B) &= f(A)w(A)+f(B)w(B) \\
N(A,B) &= f(A)w(B)+f(B)w(A)
\end{aligned}
$$

Proof. Since f and w are operator h_1-prinvex and h_2-preinvex functions respectively, and η satisfies condition (C), we have

$$
\begin{aligned}
f(\frac{2A+\eta(B,A)}{2})w(\frac{2A+\eta(B,A)}{2}) &= f(A+(1-t)\eta(B,A)+\frac{1}{2}\eta(A+t\eta(B,A),A+(1-t)\eta(B,A))) \\
&\quad \times w(A+(1-t)\eta(B,A)+\frac{1}{2}\eta(A+t\eta(B,A),A+(1-t)\eta(B,A))) \\
&\le h_1(\frac{1}{2})h_2(\frac{1}{2})[f(A+t\eta(B,A))+f(A+(1-t)\eta(B,A))] \\
&\quad \times[w(A+t(B,A))+w(A+(1-t)\eta(B,A))] \\
&= h_1(\frac{1}{2})h_2(\frac{1}{2})[f(A+t\eta(B,A))w(A+t\eta(B,A)) \\
&\quad +f(A+(1-t)\eta(B,A))w(A+(1-t)\eta(B,A))] \\
&\quad +h_1(\frac{1}{2})h_2(\frac{1}{2})[h_1(t)h_2(1-t)+h_1(1-t)h_2(t)]M(A,B) \\
&\quad +[h_1(t)h_2(t)+h_1(1-t)h_2(1-t)]N(A,B).
\end{aligned}
$$

Integrating above inequality with respect to t on $[0,1]$ we get

$$
f(\frac{2A+\eta(B,A)}{2})w(\frac{2A+\eta(B,A)}{2}) - \frac{2h_1(\frac{1}{2})h_2(\frac{1}{2})}{\eta(B,A)}\int_A^{A+\eta(B,A)} f(x)w(x)dx
$$

$$
\le 2h_1(\frac{1}{2})h_2(\frac{1}{2})[M(A,B)\int_0^1 h_1(t)h_2(1-t)dt + N(A,B)\int_0^1 h_1(t)h_2(t)dt]
$$

From the above inequality, we can obtain the required result. □

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