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BOUNDED INTEGRO COMPOSITION OPERATORS ON ORLICZ SPACES

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Abstract

In this paper we study bounded integro composition operators on Orlicz spaces.

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1 Introduction

Let (X, s, μ) be a σ-finite measure space. A measurable transformation $T : (X, s) \to (X, s)$ is called non singular, if $\mu(T^{-1}(E)) = 0$, whenever $\mu(E) = 0$ for each measurable subset E of X. If T is non-singular, then the measure μT^{-1} is absolutely continuous with respect to the measure μ. Therefore by the Radon Nikodym theorem, there exists a positive measurable function f_0 such that $\mu(T^{-1}(E)) = \int_E f_0 d\mu$. The function f_0 is called the Radon Nikodym derivative of the measure μT^{-1} with respect to the measure μ. A bounded projection operator $E : L_p(X, s, \mu) \to L_p(X, T^{-1}(s), \mu)$ is known as the expectation operator or the conditional expectation. The properties of the expectation operator can be found in Parthasarathy [9]. If T is non singular measurable transformation and if f_0 is an essentially bounded measurable function, then the operator $C_T : L_p(\mu) \to L_p(\mu)$ defined by $C_T f = foT, \ \forall f \in L_p(\mu)$ is a bounded operator (see Singh [13]). The operator C_T is called a composition operator induced by T. A measurable function $K : X \times X \to \mathbb{R}$ is called a kernel function.

A convex function $\Phi : \mathbb{R} \to \mathbb{R}^+$ is called a Young function if it satisfies the following properties:
(i) $\Phi(x) = \Phi(-x)$ for every $x \in \mathbb{R}$,

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(ii) $\Phi(0) = 0$,
(iii) $\lim_{x\to\infty} \Phi(x) = \infty$.

With each Young function Φ we can associate another Young function $\Psi : \mathbb{R} \to \mathbb{R}^+$ which is defined by $\Psi(y) = \sup\{x|y| - \Phi(x) : x \geq 0\}$ for each $y \in \mathbb{R}$. The function Ψ is called the complementary function of Φ. Suppose $X \subset \mathbb{R}$. Define $L_\Phi(\mu) = \{f|f : X \to \mathbb{R}$ is measurable function and $\int_X \Phi(\alpha|f|)d\mu < \infty$ for some $\alpha > 0\}$. For $f \in L_\Phi(\mu)$, if we define

$$||f||_\Phi = \inf\left\{\epsilon > 0 : \int_X \Phi\left(\frac{|f|}{\epsilon}\right) d\mu \leq 1\right\},$$

then $L_\Phi(\mu)$ is a Banach space under the norm $||.||_\Phi$. If $\Phi(x) = |x|^p$ for every $x \in \mathbb{R}$, then $L_\Phi(\mu) = L_p(\mu)$, the well known Banach space of p^{th} integrable functions defined on X. The Holder's inequality for Orlicz spaces is stated as follows:

If $f \in L_\Phi(\mu)$ and $g \in L_\Psi(\mu)$, with (Φ, Ψ) as a normalized complementary Young pair, then

$$\int_X |fg|d(\mu) \leq 2||f||_\Phi||g||_\Psi$$

Let $\{\mu_n\}$ be a sequence of strictly positive real numbers. Suppose $X = \mathbb{N}$, the set of natural numbers. Let μ be the measure on $P(\mathbb{N})$, the power set of $\mathbb{N}$, defined by $\mu(E) = \sum_{n\in E} \mu_n$. Then

$$L_\Phi(\mathbb{N}) = \ell^\mu_\Phi(\mathbb{N}) = \left\{f|f : \mathbb{N} \to \mathbb{C} \text{ and } \sum_{n=1}^{\infty} \Phi(\frac{|f_n|}{\alpha})\mu_n < \infty \text{ for some } \alpha > 0\right\}$$

The space $\ell^\mu_\Phi(\mathbb{N})$ is known as weighted Orlicz sequence space.

If $T : X \to X$ is a measurable transformation and $K : X \times X \to \mathbb{R}$ is the kernel function, then the bounded linear operators $R^K_T : L_\Phi(\mu) \to L_\Phi(\mu)$ and $L^K_T : L_\Phi(\mu) \to L_\Phi(\mu)$ defined by

$$(R^K_T f)(x) = \int K(x, y)f(T(y))d\mu(y) \quad for\ every\ f \in L_\Phi(\mu)$$

and

$$(L^K_T f)(x) = \int K(T(x), y)f(y)d\mu(y) \quad for\ every\ f \in L_\Phi(\mu)$$

are known as integro composition operators.

For literature concerning Orlicz spaces, composition operators, integral operators, integro composition operators we refer to Rao [10], Gupta, Komal and Suri [6], Kuffner [7], Cowen [2], Singh and Komal [11], Singh and Manhas [12], Bloom and Kerman [1], Gupta and Komal ([3],[4],[5]), Lyubic [8], Stepanov [14] and Whitley [15].

2 Bounded Integro Composition Operators on Weighted Orlicz Sequence Spaces

In this section we obtain a sufficient condition for an integro composition operator to be bounded. Let $K : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ and $T : \mathbb{N} \to \mathbb{N}$ be two mappings. Set

$$K_0(m,n) = \begin{cases} \sum\limits_{p \in T^{-1}(n)} \dfrac{K(m,p)\mu_p}{\mu_n}, & \text{if } T^{-1}(n) \neq \phi \\ 0 & , \quad \text{if } T^{-1}(n) = \phi \end{cases}$$

For each $m \in \mathbb{N}$, let $K_0^m : \mathbb{N} \to \mathbb{N}$ be defined by $K_0^m(n) = K_0(m,n)$ and $\beta : \mathbb{N} \to \mathbb{R}$ be defined by $\beta(m) = ||K_0^m||_\Psi$.

Theorem 2.1 *Let $\beta \in \ell_\Phi^\mu(\mathbb{N})$. Then the integro composition operator $R_T^K : \ell_\Phi^\mu(\mathbb{N}) \to \ell_\Phi^\mu(\mathbb{N})$ is a bounded operator.*

Proof: Take $f \in \ell_\Phi^\mu(\mathbb{N})$. Consider

$$\begin{aligned}
\int_{\mathbb{N}} \Phi\left(\frac{|(R_T^K f))(m)|}{2||\beta||_\Phi ||f||_\Phi}\right)\mu_m &= \int_{\mathbb{N}} \Phi\left(\frac{|\sum\limits_{n=1}^{\infty} K_0(m,n) f(T(n))\mu_n|}{2||\beta||_\Phi ||f||_\Phi}\right)\mu_m \\
&\leq \int_{\mathbb{N}} \Phi\left(\frac{\sum\limits_{n=1}^{\infty} \sum\limits_{p \in T^{-1}(n)} |K(m,p) f(T(p))|\mu_p}{2||\beta||_\Phi ||f||_\Phi}\right)\mu_m \\
&= \int_{\mathbb{N}} \Phi\left(\frac{\sum\limits_{n=1}^{\infty} \sum\limits_{p \in T^{-1}(n)} |K(m,p) f(n)|\mu_p}{2||\beta||_\Phi ||f||_\Phi}\right)\mu_m \\
&= \int_{\mathbb{N}} \Phi\left(\frac{\sum\limits_{n=1}^{\infty} |K_0(m,n) f(n)|\mu_n}{2||\beta||_\Phi ||f||_\Phi}\right)\mu_m \\
&\leq \int_{\mathbb{N}} \Phi\left(\frac{2||K_0^m||_\Psi}{2||\beta||_\Phi} \frac{||f||_\Phi}{||f||_\Phi}\right)\mu_m \\
&\qquad \text{(by using Hölder's inequality)} \\
&= \int_{\mathbb{N}} \Phi\left(\frac{||K_0^m||_\Psi}{||\beta||_\Phi}\right)\mu_m \\
&= \int_{\mathbb{N}} \Phi\left(\frac{\beta(m)}{||\beta||_\Phi}\right)\mu_m \\
&\leq 1
\end{aligned}$$

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Hence $||R_T^K f||_\Phi \leq 2||\beta||_\Phi ||f||_\Phi$ for every $f \in \ell_\Phi^\mu(\mathbb{N})$. This proves that R_T^K is a bounded operator. □

Example 2.2 Let $T : \mathbb{N} \to \mathbb{N}$ be defined by $T(n) = n+2$ for all $n \in \mathbb{N}$. Let $\Phi : \mathbb{N} \to \mathbb{R}$ be defined by $\Phi(n) = \frac{n^2}{2}$. Then its complementary function $\Psi(n)$ is given by $\Psi(n) = \frac{n^2}{2}$. Suppose $K : \mathbb{N}\times\mathbb{N} \to \mathbb{R}$ is defined by $K(m,n) = \frac{1}{2^{m+n}}$. Then, for $\mu_n = \frac{1}{2^n}$, we get

$$\begin{aligned}
||K_0^m||_\Psi &= \inf\left\{\epsilon > 0 : \sum_{n=1}^{\infty} \Psi\left(\frac{K_0^m(n)}{\epsilon}\right)\mu_n \leq 1\right\} \\
&= \inf\left\{\epsilon > 0 : \sum_{n=3}^{\infty} \Psi\left(\frac{\sum\limits_{p\in T^{-1}(n)} K(m,p)\mu_p}{\epsilon\mu_n}\right)\mu_n \leq 1\right\} \\
&= \inf\left\{\epsilon > 0 : \sum_{n=3}^{\infty} \Psi\left(\frac{K(m,n-2)\mu_{n-2}}{\epsilon\mu_n}\right)\mu_n \leq 1\right\} \\
&= \inf\left\{\epsilon > 0 : \sum_{n=1}^{\infty} \Psi\left(\frac{K(m,n)\mu_n}{\epsilon\mu_{n+2}}\right)\mu_{n+2} \leq 1\right\} \\
&= \inf\left\{\epsilon > 0 : \sum_{n=1}^{\infty} \Psi\left(\frac{\frac{1}{2^{m+n}}}{\epsilon\cdot\frac{1}{2^{n+2}}}\cdot\frac{1}{2^n}\right)\frac{1}{2^{n+2}} \leq 1\right\} \\
&= \inf\left\{\epsilon > 0 : \sum_{n=1}^{\infty} \Psi\left(\frac{4}{\epsilon 2^{m+n}}\right)\cdot\frac{1}{2^{n+2}} \leq 1\right\} \\
&= \inf\left\{\epsilon > 0 : \sum_{n=1}^{\infty} \frac{16}{2\epsilon^2 2^{2m+2n}}\cdot\frac{1}{2^{n+2}} \leq 1\right\} \\
&= \inf\left\{\epsilon > 0 : \frac{2}{\epsilon^2 2^{2m}}\sum_{n=1}^{\infty}\frac{1}{2^{3n}} \leq 1\right\} \\
&= \inf\left\{\epsilon > 0 : \frac{2}{\epsilon^2 2^{2m}}.\frac{1}{7} \leq 1\right\} \\
&= \frac{1}{\sqrt{7.2^{2m-1}}}
\end{aligned}$$

Also

$$\begin{aligned}
||\beta||_\Phi &= \inf\{\epsilon > 0 : \sum_{m=1}^{\infty} \Phi(\frac{|\beta(m)|}{\epsilon}) \leq 1\} \\
&= \inf\{\epsilon > 0 : \sum_{m=1}^{\infty} \frac{\beta^2(m)}{2\epsilon^2} \leq 1\} \\
&= \inf\{\epsilon > 0 : \frac{1}{7\epsilon^2}\sum_{m=1}^{\infty}\frac{1}{2^{2m}} \leq 1\} \\
&= \frac{1}{\sqrt{21}} < \infty.
\end{aligned}$$

Hence R_T^K is a bounded operator in view of Theorem 2.1.

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3 Bounded Integro Composition Operators on Orlicz Spaces

In this section we shall discuss the boundedness of various types of integro composition operators acting on Orlicz spaces. Let μ be any measure on s. Suppose Φ_1,Φ_2 are two Young functions on X. For $x \in X$, let $K^x : X \to \mathbb{R}$ be defined by $K^x(y) = K(x, y)$. Suppose $\beta : X \to \mathbb{R}$ is defined by $\beta(x) = ||K^x||_{\Psi_1}$

Theorem 3.1 *Suppose $\beta \in L_{\Phi_2}(\mu_2 T^{-1})$. Then $L_T^K : L_{\Phi_1}(\mu_1) \to L_{\Phi_2}(\mu_2)$ is a bounded operator.*

Proof: Take $f \in L_{\Phi_1}(\mu_1)$. Write $t_1 = ||f||_{\Phi_1}$, $t_2 = ||\beta||_{\Phi_2}, \mu_2 T^{-1}$. Consider

$$\int \Phi_2 \left(\frac{|(L_T^K f)(x)|}{2t_1t_2} \right) d\mu_2(x) = \int \Phi_2 \left(\frac{|\int K(T(x), y) f(y) d\mu_1(y)|}{2t_1t_2} \right) d\mu_2(x)$$

$$\begin{aligned}
&\leq \int \Phi_2 \left(\frac{||2K^{T(x)}||_{\Psi_1}}{2t_2} \frac{||f||_{\Phi_1}}{t_1} \right) d\mu_2(x) \\
&\qquad \text{(by using Hölder's inequality)} \\
&\leq \int \Phi_2 \left(\frac{\beta(T(x))}{t_2} \right) d\mu_2(x) \\
&= \int \Phi_2 \left(\frac{\beta(x)}{t_2} \right) d\mu_2 T^{-1}(x) \\
&\leq 1.
\end{aligned}$$

Hence $||L_T^K f||_{\Phi_2} \leq 2||\beta||_{\Phi_2, \mu_2 T^{-1}} ||f||_{\Phi_1}$ for every $f \in L_{\Phi_1}(\mu_1)$. This proves that L_T^K is a bounded operator. □

Example 3.2 Let $T : \mathbb{R} \to \mathbb{R}$ be defined by $T(x) = \frac{\pi}{2}x$. Then $\frac{d\mu T^{-1}(x)}{d\mu(x)} = \frac{2}{\pi}$. Take $\mu_1 = \mu_2 = \mu$, the Lebesgue measure and $\Phi_1(x) = \Phi_2(x) = \frac{x^2}{2} = \Phi(x)$. Then $\Psi(x) = \frac{x^2}{2}$.

Suppose $K : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $K(x, y) = e^{\frac{-(x^2+y^2)}{4}}$. Now

$$\begin{aligned}
\beta(x) &= ||K^x||_\Psi \\
&= \inf \left\{ \epsilon > 0 : \int \Psi \left(\frac{K(x, y)}{\epsilon} \right) d\mu(y) \leq 1 \right\} \\
&= \inf \left\{ \epsilon > 0 : \int \Psi \left(\frac{e^{-\frac{(x^2+y^2)}{4}}}{\epsilon} \right) d\mu(y) \leq 1 \right\} \\
&= \inf \left\{ \epsilon > 0 : \int \frac{e^{-\frac{(x^2+y^2)}{2}}}{2\epsilon^2} d\mu(y) \leq 1 \right\} \\
&= \inf \left\{ \epsilon > 0 : \frac{e^{-\frac{x^2}{2}}}{2\epsilon^2} . \sqrt{2\pi} \leq 1 \right\} \\
&= \sqrt{\sqrt{\frac{\pi}{2}} e^{-\frac{x^2}{2}}}
\end{aligned}$$

Therefore,

$$
\begin{aligned}
||\beta||_{\Phi,d\mu T^{-1}} &= \inf\left\{\epsilon > 0 : \int \Phi\left(\frac{\beta(x)}{\epsilon}\right) d\mu T^{-1}(x) \le 1\right\} \\
&= \inf\left\{\epsilon > 0 : \int \frac{\beta^2(x)}{2\epsilon^2} d\mu T^{-1}(x) \le 1\right\} \\
&= \inf\left\{\epsilon > 0 : \frac{1}{2\epsilon^2} \times \sqrt{\frac{\pi}{2}} \times \frac{2}{\pi} \times \sqrt{2\pi} \le 1\right\} \\
&= 1
\end{aligned}
$$

Hence L_T^k is a bounded operator in view of Theorem 3.1.

For each $x \in X$, define $K_T^x : X \to \mathbb{R}$ by $K_T^x(y) = E(K^x)oT^{-1}(y)f_o(y)$ and $\beta(x) = ||K_T^x||_{\Psi_1}$, where Ψ_1 is the complementary function of Φ_1.

Theorem 3.3 *Suppose $\beta \in L_{\Phi_2}(\mu_2)$. Then the integro composition operator $R_T^K : L_{\Phi_1}(\mu_1) \to L_{\Phi_2}(\mu_2)$ is a bounded operator.*

Proof: For $f \in L_{\Phi_1}(\mu_1)$. Write $t_1 = ||f||_{\Phi_1}$ and $t_2 = ||\beta||_{\Phi_2}$. Consider

$$\int \Phi_2\left(\frac{|(R_T^K f)(x)|}{2t_1t_2}\right) d\mu_2(x)$$

$$
\begin{aligned}
&= \int \Phi_2\left(\frac{|\int K(x,y)f(T(y)d\mu_1(y)|}{2t_1t_2}\right) d\mu_2(x) \\
&\le \int \Phi_2\left(\frac{\int |K(x,y)||f(T(y))|d\mu_1(y)}{2t_1t_2}\right) d\mu_2(x) \\
&= \int \Phi_2\left(\frac{|E(K^x)oT^{-1}(y)f_o(y)|}{2t_2}\frac{|f(y)|}{t_1}d\mu_1(y)\right) d\mu_2(x) \\
&\le \int \Phi_2\left(\frac{2||E(K^x)oT^{-1}f_o||_{\Psi_1}}{2t_2}\frac{||f||_{\Phi_1}}{t_1}\right) d\mu_2(x) \\
&\qquad \text{(By using Hölder's Inequality)} \\
&\le \int \Phi_2\left(\frac{\beta(x)}{t_2}\right) d\mu_2(x) \\
&\le 1
\end{aligned}
$$

Hence $||R_T^K f||_{\Phi_2} \le 2||\beta||_{\Phi_2}||f||_{\Phi_1}$ for every $f \in L_{\Phi_1}(\mu_1)$, and R_T^K is a bounded operator. □

Theorem 3.4 *Suppose $\beta \in L_{\Phi_2}(\mu_2)$, where $\beta(x) = ||K^x||_{\Psi_1}$. Then $I_K : L_{\Phi_1}(\mu_1) \to L_{\Phi_2}(\mu_2)$, $(I_K f)(x) = \int K(x,y)f(y)d\mu_1(y)$, is a bounded operator.*

Proof: Take $f \in L_{\Phi_1}(\mu_1)$. Write $t_1 = ||f||_{\Phi_1}$ and $t_2 = ||\beta||_{\Phi_2}$.

Consider

$$\begin{aligned}
\int \Phi_2\left(\frac{|(I_K f)(x)|}{2t_1t_2}\right)d\mu_2(x) &= \int \Phi_2\left(\frac{|\int K(x,y)f(y)d\mu_1(y)|}{2t_1t_2}\right)d\mu_2(x)\\
&\leq \int \Phi_2\left(\frac{\int |K(x,y)|}{2t_2}\frac{|f(y)|}{t_1}d\mu_1(y)\right)d\mu_2(x)\\
&\leq \int \Phi_2\left(2\frac{(||K^x||_{\Psi_1}}{2t_2}\frac{||f||_{\Phi_1}}{t_1}\right)d\mu_2(x)\\
&\qquad\text{(By using Hölder's Inequality)}\\
&\leq \int \Phi_2\left(\frac{\beta(x)}{t_2}\right)d\mu_2(x)\\
&\leq 1
\end{aligned}$$

Hence

$$||I_K f||_{\Phi_2} \leq 2||\beta||_{\Phi_2}||f||_{\Phi_1} \quad \text{for every } f \in L_{\Phi_1}(\mu_1).$$

This proves that I_K is a bounded operator. □

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