THE TYPE OF THE BASE RING ASSOCIATED TO A PRODUCT OF TRANSVERSAL POLYMATROIDS

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Abstract. A polymatroid is a generalization of the classical notion of matroid. The main results of this paper are formulas for computing the type of base ring associated to a product of transversal polymatroids. We also present some extensive computational experiments which were needed in order to deduce the formulas. The base ring associated to a product of transversal polymatroids has multiplicity very large in general. At this moment we have examples of base rings with multiplicity of order 10^{15}.

Keywords: Base ring, transversal polymatroid, equations of a cone, canonical module
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1. Introduction

For the algorithms implemented in `Normaliz` see [3], [4], [5] and [7]. This paper is organized as follows. In Section 2 we fix the notation and recall some basic results related to finitely generated rational cones. The notion of polymatroid is a generalization of the classical notion of matroid, see [8], [9], [12], [13] and [20]. Associated with the base B of a discrete polymatroid $\mathcal{P}$ one has a $K-$algebra $K[B]$, called the base ring of $\mathcal{P}$, defined to be the $K-$subalgebra of the polynomial ring in n indeterminates $K[x_1, \ldots, x_n]$ generated by the monomials x^u with $u \in B$. From [12], [19] the algebra $K[B]$ is known to be normal and hence Cohen-Macaulay. The type of normal ring is the minimal number of generators of the canonical module. Danilov Stanley theorem, see [10], [17] gives us a description of the canonicale module in terms of relative interior of the cone.

If A_i are some nonempty subsets of $[n]$ for $1 \leq i \leq m$, $\mathcal{A} = \{A_1, \ldots, A_m\}$, then the set of the vectors $\sum_{k=1}^{m} e_{i_k}$ with $i_k \in A_k$ is the base of a polymatroid, called the transversal polymatroid presented by $\mathcal{A}$. The base ring of a transversal polymatroid presented by $\mathcal{A}$ is the ring

$$K[\mathcal{A}] := K[x_{i_1} \cdots x_{i_m} \mid i_j \in A_j, 1 \leq j \leq m].$$

In Section 4 we study the cone generated by a product of transversal polymatroids and we compute the type of the associated base ring. We end this section with the following conjecture:

Conjecture: Let $n \geq 4$, $A_i \subset [n]$ for any $1 \leq i \leq n$ and $K[\mathcal{A}]$ be the base ring associated to the transversal polymatroid presented by $\mathcal{A} = \{A_1, \ldots, A_n\}$. If the Hilbert series is:

$$H_{K[\mathcal{A}]}(t) \ = \ \frac{1 + h_1\ t + \ldots + h_{n-r}\ t^{n-r}}{(1-t)^n},$$

then we have the following:
1) If $r = 1$, then $\text{type}(K[\mathcal{A}]) = 1 + h_{n-2} - h_1$.
2) If $2 \leq r \leq n$, then $\text{type}(K[\mathcal{A}]) = h_{n-r}$.

The base ring associated to a product of transversal polymatroids has multiplicity very large in general. At this moment we have examples of base rings with multiplicity of order 10^{15}.

I am grateful to B. Ichim for some extensive computational experiments which was needed in order to deduce the formulas contained in the paper.

2. Prelimininaries

In this section we fix the notation and recall some basic results. For details we refer the reader to [1], [6], [2], [17], [18] and [21].

The subsets of elements ≥ 0 in $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ will be referred to by $\mathbb{Z}_+, \mathbb{Q}_+, \mathbb{R}_+$ and the subsets of elements > 0 by $\mathbb{Z}_>, \mathbb{Q}_>, \mathbb{R}_>$.

Fix an integer $n > 0$. If $0 \neq a \in \mathbb{Q}^n$, then H_a will denote the rational hyperplane of $\mathbb{R}^n$ through the origin with normal vector a, that is,

$$H_a = \{x \in \mathbb{R}^n \mid \langle x, a\rangle = 0\},$$

where $\langle\ ,\ \rangle$ is the scalar product in $\mathbb{R}^n$. The two closed rational linear halfspaces bounded by H_a are:

$$H_a^+ = \{x \in \mathbb{R}^n \mid \langle x, a\rangle \geq 0\} \text{ and } H_a^- = H_{-a}^+ = \{x \in \mathbb{R}^n \mid \langle x, a\rangle \leq 0\}.$$

The two open rational linear halfspaces bounded by H_a are:

$$H_a^> = \{x \in \mathbb{R}^n \mid \langle x, a\rangle > 0\} \text{ and } H_a^< = H_{-a}^> = \{x \in \mathbb{R}^n \mid \langle x, a\rangle < 0\}.$$

If $S \subset \mathbb{Q}^n$, then the set

$$\mathbb{R}_+S = \{\sum_{i=1}^{r} a_i v_i \ :\ a_i \in \mathbb{R}_+,\ v_i \in S,\ r \in \mathbb{N}\}$$

is called the *rational cone* generated by S. The *dimension* of a cone is the dimension of the smallest vector subspace of $\mathbb{R}^n$ which contains it.

By the theorem of Minkowski-Weyl, see [2], [11], [21], finitely generated rational cones can also be described as intersection of finitely many rational closed subspaces (of the form H_a^+). We further restrict this presentation to the class of finitely generated rational cones, which will be simply called cones. If a cone C is presented as

$$C = H_{a_1}^+ \cap \ldots \cap H_{a_r}^+$$

such that no $H_{a_i}^+$ can be omitted, then we say that this is an *irredundant representation* of C. If $\dim(C) = n$, then the halfspaces $H_{a_1}^+, \ldots, H_{a_r}^+$ in an irredundant representation of C are uniquely determined and we set

$$\operatorname{relint}(C) = H_{a_1}^> \cap \ldots \cap H_{a_r}^>$$

the *relative interior* of C. If $a_i = (a_{i1}, \ldots, a_{in})$, then we call

$$H_{a_i}(x) := a_{i1}x_1 + \ldots + a_{in}x_n = 0,$$

the *equations of the cone* C.

A hyperplane H is called a supporting hyperplane of a cone C if $C \cap H \neq \emptyset$ and C is contained in one of the closed halfspaces determined by H. If H is a supporting hyperplane of C, then $F = C \cap H$ is called a *proper face* of C. It is convenient to consider also the empty set and C as faces, the *improper faces*. The faces of a cone are themselves cones. A face F with $\dim(F) = \dim(C) - 1$ is called a *facet*. If dim $\mathbb{R}_+S = n$ and F is a facet defined by the supporting hyperplane H, then H is generated as a linear subspace by a linearly independent subset of S.

A cone C is *pointed* if 0 is a face of C. This equivalent to say that $x \in C$ and $-x \in C$ implies $x = 0$. The faces of dimension 1 of a pointed cone are called *extreme rays*.

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In this section we introduce the notion of a discrete polymatroid and the particular case of transversal polymatroid. We further recall some results from [16] on the embedding cone and the type of a particular family of transversal polymatroids.

Discrete polymatroids. Fix an integer $n > 0$ and set $[n] := \{1, 2, \ldots, n\}$. The canonical basis vectors of $\mathbb{R}^n$ will be denoted by $e_1, \ldots, e_n$. For a vector $a \in \mathbb{R}^n$, $a = (a_1, \ldots, a_n)$, we set $\mid a \mid := a_1 + \ldots + a_n$.

A nonempty finite set $B \subset \mathbb{Z}_+^n$ is the *set of bases a discrete polymatroid* $\mathcal{P}$ if:

(a) for every $u,\ v \in B$ one has $\mid u \mid = \mid v \mid$;
(b) (the exchange property) if $u,\ v \in B$, then for all i such that $u_i > v_i$ there exists j such that $u_j < v_j$ and $u + e_j - e_i \in B$.

An element of B is called a *base of the discrete polymatroid* $\mathcal{P}$.

Let K be an infinite field. For $a \in \mathbb{Z}_+^n$, $a = (a_1, \ldots, a_n)$ we denote by $x^a \in K[x_1, \ldots, x_n]$ the monomial $x^a := x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ and we set $\log(x^a) = a$. Associated with the set of bases B of a discrete polymatroid $\mathcal{P}$ one has a $K-$algebra $K[B]$, called the *base ring* of $\mathcal{P}$, defined to be the $K-$subalgebra of the polynomial ring in n indeterminates $K[x_1, x_2, \ldots, x_n]$ generated by the monomials x^u with $u \in B$. From [12], [19] the monoid algebra $K[B]$ is known to be normal and we recall that by a well known result of Danilov and Stanley the canonical module $\omega_{K[B]}$ of $K[B]$, with respect to standard grading, can be expressed as an ideal of $K[B]$ generated by monomials, that is $\omega_{K[B]} = (\{x^a \mid a \in \mathbb{Z}_+ B \cap \text{relint}(\mathbb{R}_+ B)\})$.

Transversal polymatroids. Consider another integer m such that $1 \le m \le n$. If A_i are some nonempty subsets of $[n]$ for $1 \le i \le m$ and $\mathcal{A} = \{A_1, \ldots, A_m\}$, then the set of the vectors $\sum_{k=1}^m e_{i_k}$ with $i_k \in A_k$ is the set of bases of a polymatroid, called the *transversal polymatroid presented by* $\mathcal{A}$. The base ring of the transversal polymatroid presented by $\mathcal{A}$ is the ring

$$K[\mathcal{A}] := K[x_{i_1} \cdots x_{i_m} \mid i_j \in A_j, 1 \le j \le m].$$

We denote by

$$A := \{\log(x_{j_1} \cdots x_{j_n}) \mid j_k \in A_k,\ \text{for all } 1 \le k \le n\} \subset \mathbb{N}^n$$

the set of the exponents of the generators of the associated base ring $K[\mathcal{A}]$. Further, for the transversal polymatroid presented by $\mathcal{A}$ we associate a $(n \times n)$ square tiled by unit subsquares, called *boxes*, colored with white and black as follows: the box of coordinate (i, j) is white if $j \in A_i$, otherwise the box is black. We will call this square the *polymatroidal diagram* associated to the presentation $\mathcal{A} = \{A_1, \ldots, A_n\}$([14],[15]).

In the following we shall restrict our study to a special family of transversal polymatroids. Fix $n \in \mathbb{Z}_+$, $n \ge 3$, $1 \le i \le n-2$ and $1 \le j \le n-1$ and consider the transversal polymatroid presented by $\mathcal{A} = \{A_1 = [n], A_2 = [n] \setminus [i], \ldots, A_{j+1} = [n] \setminus [i], A_{j+2} = [n], \ldots, A_n = [n]\}$.

We recall at this point some previous results contained in [16]. The cone generated by A has the irredundant representation

$$\mathbb{R}_+ A = \bigcap_{a \in N} H_a^+,$$

where $N = \{\nu_i^j\} \bigcup \{e_k \mid 1 \le k \le n\}$ and

$$\nu_i^j := \sum_{k=1}^{i} -j e_k + \sum_{k=i+1}^{n} (n-j) e_k.$$

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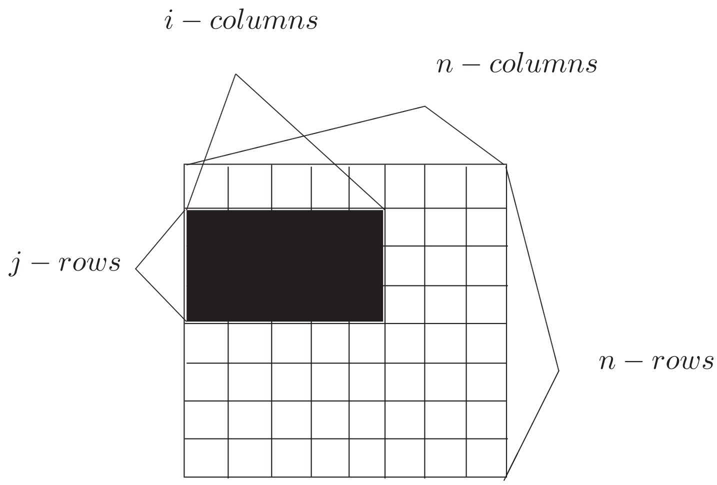

Polymatroidal diagram associated to the presentation

$$\mathcal{A} = \{A_1 = [n], A_2 = [n] \setminus [i], \ldots, A_{j+1} = [n] \setminus [i], A_{j+2} = [n], \ldots, A_n = [n]\}.$$

The extreme rays of the cone $\mathbb{R}_+A$ are given by

$$E := \{ne_k \mid i+1 \leq k \leq n\} \bigcup \{(n-j)e_r + j\ e_s \mid 1 \leq r \leq i \text{ and } i+1 \leq s \leq n\}.$$

The polynomial

$$P_d(k) = \binom{d+k-1}{d-1}$$

counts the number of monomials in degree k over the standard graded polynomial ring $K[x_1, \ldots, x_d]$, i.e. $P_d(k)$ is the Hilbert function of $K[x_1, \ldots, x_d]$. Then

$$P_d(k-d) = \binom{k-1}{d-1} = Q_d(k)$$

counts the number of monomials in degree k for which all the variables have nonzero powers, i.e. $Q_d(k)$ is the Hilbert function of the canonical module $\omega_{K[x_1,\ldots,x_d]} = K[x_1, \ldots, x_d](-d)$.

The main result of [16] is the following theorem.

Theorem 1. *With the above assumptions, the following holds:*

(a) *If $i + j \leq n - 1$, then the type of $K[\mathcal{A}]$ is*

$$\operatorname{type}(K[\mathcal{A}]) = 1 + \sum_{t=1}^{n-i-j-1} Q_i(n+i-j+t)Q_{n-i}(n-i+j-t),$$

(b) *If $i + j \geq n$, then the type of $K[\mathcal{A}]$ is*

$$\operatorname{type}(K[\mathcal{A}]) = \sum_{t=1}^{r(n-j)-i} Q_i(r(n-j)-t)Q_{n-i}(rj+t),$$

where $r = \left\lceil \frac{i+1}{n-j} \right\rceil$ ($\lceil x \rceil$ is the least integer $\geq x$).

Further, from the proof of main theorem in [16], we get the following lemma:

Lemma 2. *The following holds:*

(a) *Suppose* $i+j \leq n-1$. *Let* M *be the set*

$$M = \{\alpha \in \mathbb{Z}_{>}^{n} \mid |(\alpha_1,\ldots,\alpha_i)| = n+i-j+t,$$
$$|(\alpha_{i+1},\ldots,\alpha_n)| = n-i+j-t,\ t \in [n-i-j-1]\}.$$

Then for any $\beta \in \mathbb{Z}_+A \cap \operatorname{relint}(\mathbb{R}_+A)$ *with* $\mid \beta \mid = sn \geq 2n$ *and* $t \in [n-i-j-1]$ *such that* $H_{\nu_i^j}(\beta) = n(n-i-j-t)$ *we can find* $\alpha \in M$ *with* $H_{\nu_i^j}(\alpha) = n(n-i-j-t)$ *and* $\beta - \alpha \in \mathbb{Z}_+A$.

(b) *Suppose* $i+j \geq n$ *and set* $r = \left\lceil \frac{i+1}{n-j} \right\rceil$. *Let* M *be the set*

$$M = \{\alpha \in \mathbb{Z}_{>}^{n} \mid |(\alpha_1,\ldots,\alpha_i)| = r(n-j)-t,$$
$$|(\alpha_{i+1},\ldots,\alpha_n)| = rj+t,\quad t \in [r(n-j)-i]\}.$$

Then for any $\beta \in \mathbb{Z}_+A \cap \operatorname{relint}(\mathbb{R}_+A)$ *with* $\mid \beta \mid = sn \geq rn$ *and* $t \in [r(n-j)-i]$ *such that* $H_{\nu_i^j}(\beta) = nt$ *we can find* $\alpha \in M$ *with* $H_{\nu_i^j}(\alpha) = nt$ *such that* $\beta - \alpha \in \mathbb{Z}_+A$.

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We set

$$A^r = \{\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{Z}_+^n \mid \alpha = \sum_{i=1}^{r} \beta_i \text{ where } \beta_i \in A\}$$

and

$$A^{(r)} = A^r \bigcap \operatorname{relint}(\mathbb{R}_+A).$$

Lemma 3. *The following holds:*

(a) *The cardinal of* A^r *is*

$$\#(A^r) = \sum_{t=0}^{r(n-j)} P_i(t)P_{n-i}(rn-t);$$

(b) *The cardinal of* $A^{(r)}$ *is*

$$\#(A^{(r)}) = \sum_{t=i}^{r(n-j)} Q_i(t)Q_{n-i}(rn-t).$$

Proof. Since the cone generated by A has the irreducible representation

$$\mathbb{R}_+A = \bigcap_{a\in N} H_a^+$$

and the monoid generated by A is normal it follows that

$$A^r = \{\alpha \in \mathbb{Z}_+^n \mid |\alpha| = rn,\ \sum_{k=1}^{i} -j\alpha_k + \sum_{k=i+1}^{n} (n-j)\alpha_k \geq 0\}$$
$$= \{\alpha \in \mathbb{Z}_+^n \mid |\alpha| = rn,\ 0 \leq \alpha_1 + \ldots + \alpha_i \leq r(n-j)\}$$

and

$$A^{(r)} = \{(\alpha \in \mathbb{Z}_{>}^n \mid |\alpha| = rn,\ \sum_{k=1}^{i} -j\alpha_k + \sum_{k=i+1}^{n} (n-j)\alpha_k > 0\}$$
$$= \{(\alpha \in \mathbb{Z}_{>}^n \mid |\alpha| = rn,\ i \leq \alpha_1 + \ldots + \alpha_i < r(n-j)\}.$$

a) For any $0 \leq t \leq r(n-j)$, the equation $\alpha_1 + \ldots + \alpha_i = t$ has $P_i(t)$ distinct nonnegative integer solutions, respectively $\alpha_{i+1} + \ldots + \alpha_n = rn - t$ has $P_{n-i}(rn-t)$ distinct nonnegative integer solutions. Thus, the cardinal of A^r is

$$\#(A^r) = \sum_{t=0}^{r(n-j)} P_i(t) P_{n-i}(rn-t).$$

b) For any $i \leq t \leq r(n-j)-1$, the equation $\alpha_1 + \ldots + \alpha_i = t$ has $Q_i(t)$ distinct nonnegative integer solutions with $\alpha_k \geq 1$, for any $k \in [i]$, respectively $\alpha_{i+1} + \ldots + \alpha_n = rn - t$ has $Q_{n-i}(rn-t)$ distinct nonnegative integer solutions with $\alpha_k \geq 1$ for any $k \in [n] \setminus [i]$. Thus, the cardinal of $A^{(r)}$ is

$$\#(A^{(r)}) = \sum_{t=i}^{r(n-j)} Q_i(t) Q_{n-i}(rn-t).$$

□

4. The cone and the type of the base ring associated to a product of transversal polymatroids

This section contains the main results of this paper. We study the cone generated by a product of transversal polymatroids and the type of the associated base ring.

The product of transversal polymatroids. Fix $n_1, n_2 \in \mathbb{Z}_+$, $n_1, n_2 \geq 3$, $n = n_1 + n_2$, $i_1 \in [n_1 - 2]$, $i_2 \in [n_2 - 2]$, $j_1 \in [n_1 - 1]$ and $j_2 \in [n_2 - 1]$. For the vectors $\alpha \in \mathbb{Z}_+^{n_1}$ and $\beta \in \mathbb{Z}_+^{n_2}$ we denote by $\widetilde{\alpha}, \bar{\beta} \in \mathbb{Z}_+^{n_1+n_2}$ the vectors

$$\widetilde{\alpha} = (\alpha, \underbrace{0, \ldots, 0}_{n_2 \ times}) \in \mathbb{Z}_+^{n_1+n_2} \ , \ \bar{\beta} = (\underbrace{0, \ldots, 0}_{n_1 \ times}, \beta) \in \mathbb{Z}_+^{n_1+n_2}.$$

If $S \subset \mathbb{Z}_+^{n_1}$ and $P \in \subset \mathbb{Z}_+^{n_2}$ we denote by $\widetilde{S}, \bar{P} \in \mathbb{Z}_+^{n_1+n_2}$ the following sets

$$\widetilde{S} = \{\widetilde{\alpha} \mid \alpha \in S\} \text{ and } \bar{P} = \{\bar{\beta} \mid \beta \in P\}.$$

Next, we consider the $K-$algebras $K[\mathcal{A}]$ and $K[\mathcal{B}]$ which are the base rings of the transversal polymatroids presented by $\mathcal{A}$, respectively $\mathcal{B}$, where:

$$\mathcal{A} = \{A_1 = [n_1], A_2 = [n_1] \setminus [i_1], \ldots, A_{j_1+1} = [n_1] \setminus [i_1], A_{j_1+2} = [n_1], \ldots, A_{n_1} = [n_1]\}$$

and

$$\mathcal{B} = \{A_{n_1+1} = [n] \setminus [n_1], A_{n_1+2} = [n] \setminus [n_1 + i_2], \ldots, A_{n_1+j_2+1} = [n] \setminus [n_1 + i_2], \\ A_{n_1+j_2+2} = [n] \setminus [n_1], \ldots, A_{n_1+n_2} = [n] \setminus [n_1]\}.$$

Let

$$A = \{\log(x_{t_1} \cdots x_{t_{n_1}}) \mid j_k \in A_k, \text{ for all } 1 \leq k \leq n_1\} \subset \mathbb{Z}_+^{n_1}$$

be the exponent set of generators of $K-$algebra $K[\mathcal{A}]$ and

$$B = \{\log(x_{t_1} \cdots x_{t_{n_1}}) \mid j_k \in A_k, \text{for all } n_1 + 1 \leq k \leq n_1 + n_2\} \subset \mathbb{Z}_+^{n_2}$$

be the exponent set of generators of $K-$algebra $K[\mathcal{B}]$. We denote by $K[\mathcal{A} \diamond \mathcal{B}]$ the $K-$algebra $K[x^{\widetilde{\alpha}+\bar{\beta}} \mid \alpha \in A, \ \beta \in B]$ and by $A \diamond B$ the exponent set of generators of $K[\mathcal{A} \diamond \mathcal{B}]$.

It is easy to see that $K-$algebra $K[\mathcal{A} \diamond \mathcal{B}]$ is the base ring associated to the transversal polymatroid presented by

$$\mathcal{A} \diamond \mathcal{B} = \{A_1 = [n_1], A_2 = [n_1] \setminus [i_1], \ldots, A_{j_1+1} = [n_1] \setminus [i_1], A_{j_1+2} = [n_1], \ldots, A_{n_1} = [n_1], \\ A_{n_1+1} = [n] \setminus [n_1], A_{n_1+2} = [n] \setminus [n_1 + i_2], \ldots, A_{n_1+j_2+1} = [n] \setminus [n_1 + i_2], \\ A_{n_1+j_2+2} = [n] \setminus [n_1], \ldots, A_{n_1+n_2} = [n] \setminus [n_1]\}.$$

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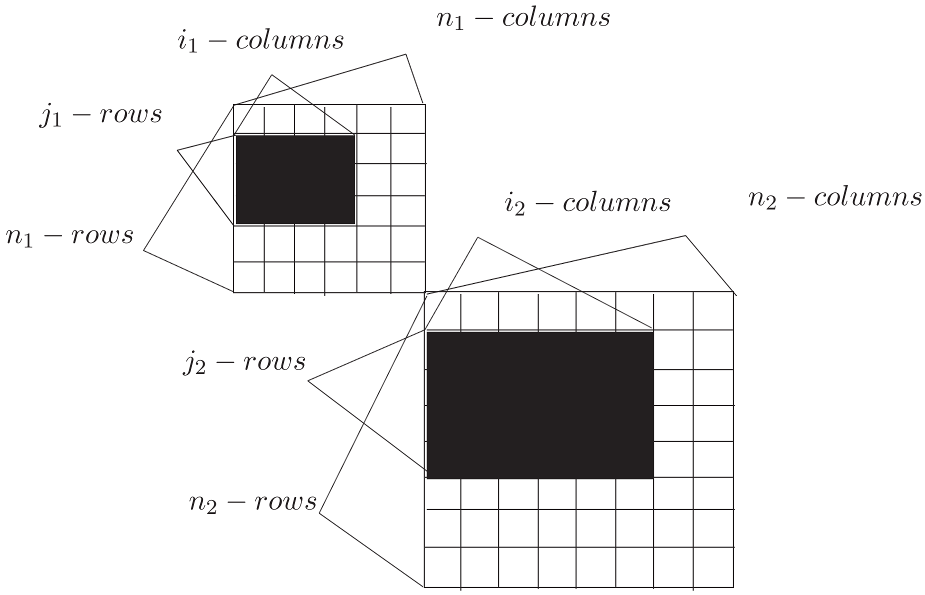

Polymatroidal diagram associated to the presentation $\mathcal{A} \diamond \mathcal{B}$.

The cone generated by a product of transversal polymatroids. The following proposition describes the cone generated by $A \diamond B$.

Proposition 4. *With the notations from above, the cone generated by* $A \diamond B$ *has the irreducible representation*

$$\mathbb{R}_+(A \diamond B) = \Pi \cap \bigcap_{a \in N} H_a^+,$$

where Π *is the hyperplane described by the equation*

$$-n_2 x_1 - \cdots - n_2 x_{n_1} + n_1 x_{n_1+1} + \cdots + n_1 x_{n_1+n_2} = 0$$

and $N = \{\widetilde{\nu}_{i_1}^{j_1}, \bar{\nu}_{i_2}^{j_2}\} \bigcup \{\, e_k \mid 1 \le k \le n\}$.

Proof. Since $A \diamond B = \{\widetilde{\alpha} + \bar{\beta} \mid \alpha \in A,\ \beta \in B\}$ and $\mid \widetilde{\alpha} \mid = n_1$, $\mid \bar{\beta} \mid = n_2$, it is clear that $\mathbb{R}_+(A \diamond B) \subset \Pi$. It is also clear that

$$\mathbb{R}_+(A \diamond B) \subset \mathbb{R}_+(\widetilde{A} \cup \bar{\mathbb{R}}^{n_2}) \bigcap \mathbb{R}_+(\widetilde{\mathbb{R}}^{n_1} \cup \bar{B}).$$

From the irredundant representation presented in [16] (see Section 3) for the cone generated by A and B we deduce that

$$\mathbb{R}_+(\widetilde{A} \cup \bar{\mathbb{R}}^{n_2}) = \bigcap_{a \in \widetilde{N_1}} H_a^+, \quad \mathbb{R}_+(\widetilde{\mathbb{R}}^{n_1} \cup \bar{B}) = \bigcap_{a \in \bar{N}_2} H_a^+$$

where $\widetilde{N_1} = \{\widetilde{\nu}_{i_1}^{j_1}\} \bigcup \{e_k \mid 1 \le k \le n_1\}$ and $\bar{N}_2 = \{\bar{\nu}_{i_2}^{j_2}\} \bigcup \{e_k \mid n_1 + 1 \le k \le n\}$. We get

$$\mathbb{R}_+(A \diamond B) \subset \Pi \cap \bigcap_{a \in N} H_a^+.$$

Let

$$C = \bigcap_{a \in N} H_a^+.$$

It is clear that C is a pointed cone of dimension n so $\Pi \cap C$ is pointed of dimension $n-1$. Consider an extremal ray v of the cone $\Pi \cap C$. Then $v \in \Pi$ so it is not possible that all entries γ_i are 0 for all $1 \le i \le n_1$ or for all $n_1+1 \le i \le n$. Moreover υ is contained in at least $n-2$ hyperplanes H_a so υ is contained in at least $n-4$ hyperplanes of type H_{e_k}.

1) If υ is contained in $n-4$ hyperplanes of type H_{e_k} then $\upsilon \in H_{\bar{\nu}_{i_1}^{j_1}}$ and $\upsilon \in H_{\bar{\nu}_{i_2}^{j_2}}$

2) If υ is contained in $n-3$ hyperplanes of type H_{e_k} then $\upsilon \in H_{\bar{\nu}_{i_1}^{j_1}}$ or $\upsilon \in H_{\bar{\nu}_{i_2}^{j_2}}$

3) If υ is contained in $n-2$ hyperplanes of type H_{e_k} then $\upsilon \notin H_{\bar{\nu}_{i_1}^{j_1}}$ and $\upsilon \notin H_{\bar{\nu}_{i_2}^{j_2}}$

First case.

Let $1 \le k_1 < \ldots < k_{n-4} \le n$ be a sequence of integers and $\{r_1, s_1, r_2, s_2\} = [n] \setminus \{k_1, \ldots, k_{n-4}\}$. If $1 \le r_1 \le i_1$, $i_1+1 \le s_1 \le n_1$, $n_1+1 \le r_2 \le n_1+i_2$ and $n_1+i_2+1 \le s_2 \le n$ then $x = (x_1, \ldots, x_n) \in \mathbb{Z}_+^n$ with $x_t = (n_1-j_1)\delta_{tr_1} + j_1\delta_{ts_1} + (n_2-j_2)\delta_{tr_2} + j_2\delta_{ts_2}$ (δ_{tk} is the Kronecker symbol) is a solution of the system of equations

$$(*) \quad \begin{cases} z_{k_1} = 0 \\ \vdots \\ z_{k_{n-4}} = 0 \\ -j_1\ z_1 - \ldots - j_1\ z_{i_1} + (n_1-j_1)z_{i_1+1} + \ldots + (n_1-j_1)z_{n_1} = 0 \\ -j_2\ z_{n_1+1} - \ldots - j_2\ z_{n_1+i_2} + (n_2-j_2)z_{n_1+i_2+1} + \ldots + (n_2-j_2)z_n = 0. \end{cases}$$

fulfilling also the condition $\Pi(x) = 0$. Else, there exists no solution $x \in \mathbb{Z}_+^n$ for the system of equations $(*)$ with $\Pi(x) = 0$ because either $H_{\bar{\nu}_{i_1}^{j_1}}(x) \ne 0$ or $H_{\bar{\nu}_{i_2}^{j_2}}(x) \ne 0$.

Thus, there are $i_1 i_2 (n_1-i_1)(n_2-i_2)$ sequences $1 \le k_1 < \ldots < k_{n-4} \le n$ such that the system of equations $(*)$ has a solution $x \in \mathbb{Z}_+^n$ with $\Pi(x) = 0$, and they induce the set of extremal rays:

$$\{(n_1-j_1)e_{r_1} + j_1\ e_{s_1} + (n_2-j_2)e_{r_2} + j_2\ e_{s_2} \mid 1 \le r_1 \le i_1,\ i_1+1 \le s_1 \le n_1,$$
$$n_1+1 \le r_2 \le n_1+i_2,\ n_1+i_2+1 \le s_2 \le n\}.$$

Second case.

Let $1 \le k_1 < \ldots < k_{n-3} \le n$ be a sequence of integers and $\{r_1, s_1, p\} = [n] \setminus \{k_1, \ldots, k_{n-3}\}$. If $1 \le r_1 \le i_1$, $i_1+1 \le s_1 \le n_1$ and $n_1+1 \le p \le n$ then $x \in \mathbb{Z}_+^n$ with $x_t = (n_1-j_1)\delta_{tr_1} + j_1\ \delta_{ts_1} + n_2\ \delta_{tp}$ is a solution of the system of equations

$$(**) \quad \begin{cases} z_{k_1} = 0 \\ \vdots \\ z_{k_{n-3}} = 0 \\ -j_1\ z_1 - \ldots - j_1\ z_{i_1} + (n_1-j_1)z_{i_1+1} + \ldots + (n_1-j_1)z_{n_1} = 0. \end{cases}$$

fulfilling also the condition $\Pi(x) = 0$. Else, there exists no solution $x \in \mathbb{Z}_+^n$ for the system of equations $(**)$ with $\Pi(x) = 0$.

Thus, there exist $i_1(n_1-i_1)n_2$ sequences $1 \le k_1 < \ldots < k_{n-3} \le n$ such that the system of equations $(**)$ has a solution $x \in \mathbb{Z}_+^n$ with $\Pi(x) = 0$, and they induce the set of extremal rays:

$$\{(n_1-j_1)e_{r_1} + j_1\ e_{s_1} + n_2 e_p \mid 1 \le r_1 \le i_1,\ i_1+1 \le s_1 \le n_1,\ n_1+1 \le p \le n\}.$$

Analog one obtains the set of extremal rays induced by $\upsilon \in H_{\bar{\nu}_{i_2}^{j_2}}$:

$$\{n_1 e_p + (n_2-j_2)e_{r_2} + j_2\ e_{s_2} \mid 1 \le p \le n_1,\ n_1+1 \le r_2 \le n_1+i_2,\ n_1+i_2+1 \le s_2 \le n\}.$$

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The third case.
It is easy to see that there are $(n_1 - i_1)(n_2 - i_2)$ induced extremal rays in this case:

$$\{n_1 e_r + n_2 e_s \mid i_1 + 1 \leq r \leq n_1,\ n_1 + i_2 + 1 \leq s \leq n\}.$$

In conclusion, $E := \{v_1 + v_2 \mid v_1 \in E_1,\ v_2 \in E_2\}$ is the set of extremal rays of the cone $\Pi \cap C$ where

$$E_1 := \{n_1 e_k \mid i_1 + 1 \leq k \leq n_1\} \bigcup \{(n_1 - j_1)e_r + j_1\ e_s \mid 1 \leq r \leq i_1 \text{ and } i_1 + 1 \leq s \leq n_1\}$$

and

$$E_2 := \{n_2 e_k \mid n_1 + i_1 + 1 \leq k \leq n\} \bigcup \{(n_2 - j_2)e_r + j_2\ e_s \mid n_1 + 1 \leq r \leq n_1 + i_2 \text{ and } n_1 + i_2 + 1 \leq s \leq n\}.$$

It clear that $E \subset \mathbb{R}_+(A \diamond B)$ and we get

$$\mathbb{R}_+(A \diamond B) \supset \Pi \cap \bigcap_{a \in N} H_a^+.$$

□

The type of the base ring. The next theorem is the main result of this paper. It contains formulas for computing the type of the base ring associated to a product of transversal polymatroids.

Theorem 5. *Let $K[\mathcal{A}]$ and $K[\mathcal{B}]$ the base rings of the transversal polymatroids presented by $\mathcal{A}$ and $\mathcal{B}$ from above. Then:*
a) If $i_1 + j_1 \leq n_1 - 1$ and $i_2 + j_2 \leq n_2 - 1$, then the type of $K[\mathcal{A} \diamond \mathcal{B}]$ is

$$\text{type}(K[\mathcal{A} \diamond \mathcal{B}]) = 1 + (\text{type}(K[\mathcal{A}] - 1)Q_2 + (\text{type}(K[\mathcal{B}] - 1)Q_1 - (\text{type}(K[\mathcal{A}] - 1)(\text{type}(K[\mathcal{B}] - 1),$$

where

$$Q_r = \sum_{t=i_r}^{2(n_r - j_r)-1} Q_{i_r}(t) Q_{n_r - i_r}(2n_r - t), \text{ for } r \in [2].$$

b) If $i_1 + j_1 \geq n_1$ and $i_2 + j_2 \geq n_2$ such that $r_1 \leq r_2$ where $r_1 = \left\lceil \frac{i_1+1}{n_1-j_1} \right\rceil$, $r_2 = \left\lceil \frac{i_2+1}{n_2-j_2} \right\rceil$ then the type of $K[\mathcal{A} \diamond \mathcal{B}]$ is

$$\text{type}(K[\mathcal{A} \diamond \mathcal{B}]) = [\sum_{t=i_1}^{r_2(n_1-j_1)-1} Q_{i_1}(t) Q_{n_1 - i_1}(r_2 n_1 - t)] \,\text{type}(K[\mathcal{B}]).$$

c) If $i_1 + j_1 \leq n_1 - 1$, $i_2 + j_2 \geq n_2$ and $r_2 = \left\lceil \frac{i_2+1}{n_2-j_2} \right\rceil$, then the type of $K[\mathcal{A} \diamond \mathcal{B}]$ is

$$\text{type}(K[\mathcal{A} \diamond \mathcal{B}]) = [G + E] \,\text{type}(K[\mathcal{B}]),$$

where

$$G = \sum_{t=0}^{(r_2-1)(n_1-j_1)} P_{i_1}(t) P_{n_1-i_1}((r_2 - 1)n_1 - t),$$

$$E = \sum_{t=1}^{n_1-i_1-j_1-1} Q_{i_1}(i_1 + (r_2 - 1)(n_1 - j_1) + t) Q_{n_1-i_1}(n_1 - i_1 + (r_2 - 1)j_1 - t).$$

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Proof. Since $K[\mathcal{A}\diamond\mathcal{B}]$ is normal ([12]), the canonical module $\omega_{K[\mathcal{A}\diamond\mathcal{B}]}$ of $K[\mathcal{A}\diamond\mathcal{B}]$, with respect to standard grading, can be expressed as an ideal of $K[\mathcal{A}\diamond\mathcal{B}]$ generated by monomials

$$\omega_{K[\mathcal{A}\diamond\mathcal{B}]}=(\{x^a|\ a\in\mathbb{Z}_+(A\diamond B)\cap\mathrm{relint}(\mathbb{R}_+(A\diamond B))\})K[\mathcal{A}\diamond\mathcal{B}],$$

where $A\diamond B$ is the exponent set of the $K-$ algebra $K[\mathcal{A}\diamond\mathcal{B}]$ and $relint(\mathbb{R}_+(A\diamond B))$ denotes the relative interior of $\mathbb{R}_+(A\diamond B)$. By Proposition 4 the cone generated by $A\diamond B$ has the irreducible representation

$$\mathbb{R}_+(A\diamond B)=\Pi\cap\bigcap_{a\in N}H_a^+,$$

where $\Pi: -n_2x_1-\cdots-n_2x_{n_1}+n_1x_{n_1+1}+\cdots+n_1x_{n_1+n_2}=0$,
$N=\{\widetilde{\nu}_{i_1}^{j_1},\bar{\nu}_{i_2}^{j_2},\ e_k\mid 1\le k\le n_1+n_2\}$ and $\{e_i\}_{1\le i\le n_1+n_2}$ is the canonical base of $\mathbb{R}^{n_1+n_2}$.
a) Let $i_1\in[n_1-2]$, $j_1\in[n_1-1]$, $i_2\in[n_2-2]$, $j_2\in[n_2-1]$ be such that $i_1+j_1\le n_1-1$ and $i_2+j_2\le n_2-1$. If we denote by $M_\mathcal{A},M_\mathcal{B}$ the sets

$$M_\mathcal{A}=\{\alpha\in\mathbb{Z}_>^{n_1}\mid|(\alpha_1,\ldots,\alpha_{i_1})|=n_1+i_1-j_1+t,\ |(\alpha_{i_1+1},\ldots,\alpha_{n_1})|=$$
$$n_1-i_1+j_1-t \text{ for any } t\in[n_1-i_1-j_1-1]\},$$
$$M_\mathcal{B}=\{\alpha\in\mathbb{Z}_>^{n_2}\mid|(\alpha_1,\ldots,\alpha_{i_2})|=n_2+i_2-j_2+t,\ |(\alpha_{i_2+1},\ldots,\alpha_{n_2})|=$$
$$n_2-i_2+j_2-t \text{ for any } t\in[n_2-i_2-j_2-1]\}$$

we know from [16] that the canonical module $\omega_{K[\mathcal{A}]}$ of $K[\mathcal{A}]$ (respectively, $\omega_{K[\mathcal{B}]}$ of $K[\mathcal{B}]$) with respect to the standard grading can be expressed as an ideal of $K[\mathcal{A}]$(respectively, $K[\mathcal{B}]$) generated by monomials

$$\omega_{K[\mathcal{A}]}=(\{x_1\cdots x_n,\ x^\alpha|\ \alpha\in M_\mathcal{A}\})K[\mathcal{A}],$$

respectively

$$\omega_{K[\mathcal{B}]}=(\{x_1\cdots x_n,\ x^\alpha|\ \alpha\in M_\mathcal{B}\})K[\mathcal{B}].$$

We will denote by $M_{\mathcal{A}\diamond\mathcal{B}}$ the set

$$M_{\mathcal{A}\diamond\mathcal{B}}=\{\widetilde{\alpha}+\bar{q},\ \bar{\beta}+\widetilde{p}\mid\alpha\in M_\mathcal{A},\ \beta\in M_\mathcal{B},\ p\in A^{(2)},\ q\in B^{(2)}\}.$$

We will show that the canonical module $\omega_{K[\mathcal{A}\diamond\mathcal{B}]}$ of $K[\mathcal{A}\diamond\mathcal{B}]$, with respect to standard grading, can be expressed as an ideal of $K[\mathcal{A}\diamond\mathcal{B}]$, generated by monomials

$$\omega_{K[\mathcal{A}\diamond\mathcal{B}]}=(\{x_1\cdots x_n,\ x^\alpha|\ \alpha\in M_{\mathcal{A}\diamond\mathcal{B}}\})K[\mathcal{A}\diamond\mathcal{B}].$$

This fact is equivalent to show that

$$\mathbb{Z}_+(A\diamond B)\cap\mathrm{relint}(\mathbb{R}_+(A\diamond B))=\{(1,\ldots,1)+\mathbb{Z}_+(A\diamond B)\}\cup\bigcup_{\alpha\in M_{\mathcal{A}\diamond\mathcal{B}}}\{\alpha+\mathbb{Z}_+(A\diamond B)\}.$$

Since for any $\alpha\in M_\mathcal{A},\ \beta\in M_\mathcal{B},\ p\in A^{(2)},\ q\in B^{(2)}$

$$H_{\widetilde{\nu}_{i_1}^{j_1}}(\widetilde{\alpha}+\bar{q})=H_{\nu_{i_1}^{j_1}}(\alpha)=n_1(n_1-i_1-j_1+t)>0,\ H_{\widetilde{\nu}_{i_1}^{j_1}}(\bar{\beta}+\widetilde{p})=H_{\nu_{i_1}^{j_1}}(p)>0$$

and

$$H_{\bar{\nu}_{i_2}^{j_2}}(\bar{\beta}+\widetilde{p})=H_{\nu_{i_2}^{j_2}}(\beta)=n_2(n_2-i_2-j_2+t)>0,\ H_{\bar{\nu}_{i_2}^{j_2}}(\widetilde{\alpha}+\bar{q})=H_{\nu_{i_2}^{j_2}}(q)>0$$

it follows that

$$\mathbb{Z}_+(A\diamond B)\cap\mathrm{relint}(\mathbb{R}_+(A\diamond B))\supseteq\{(1,\ldots,1)+\mathbb{Z}_+(A\diamond B)\}\cup\bigcup_{\alpha\in M_{\mathcal{A}\diamond\mathcal{B}}}\{\alpha+\mathbb{Z}_+(A\diamond B)\}.$$

Let $\gamma\in\mathbb{Z}_+(A\diamond B)\cap\mathrm{relint}(\mathbb{R}_+(A\diamond B))$, then $\gamma_k\ge 1$ for any $k\in[n_1+n_2]$. Since $H_{\widetilde{\nu}_{i_1}^{j_1}}((1,\ldots,1))=n_1(n_1-i_1-j_1)>0$ and $H_{\bar{\nu}_{i_2}^{j_2}}((1,\ldots,1))=n_2(n_2-i_2-j_2)>0$ it follows that $(1,\ldots,1)\in\mathrm{relint}(\mathbb{R}_+(A\diamond B))$. Let $\delta\in{\mathbb{Z}_+}^{n_1+n_2}$, $\delta=\gamma-(1,\ldots,1)$. It is

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clear that $\mathbb{Z}_+(A \diamond B) = \mathbb{Z}_+\widetilde{A} + \mathbb{Z}_+\bar{B}$. So, we have $H_{\widetilde{\nu}_{i_1}^{j_1}}(\delta) = H_{\widetilde{\nu}_{i_1}^{j_1}}(\gamma) - n_1(n_1 - i_1 - j_1) = H_{\nu_{i_1}^{j_1}}(\gamma^{'}) - n_1(n_1 - i_1 - j_1)$ and $H_{\bar{\nu}_{i_2}^{j_2}}(\delta) = H_{\bar{\nu}_{i_2}^{j_2}}(\gamma) - n_2(n_2 - i_2 - j_2) = H_{\nu_{i_2}^{j_2}}(\gamma^{''}) - n_2(n_2 - i_2 - j_2)$ where $\gamma = (\gamma^{'}, \gamma^{''})$, $\gamma^{'} \in \mathbb{Z}_+A$ and $\gamma^{''} \in \mathbb{Z}_+B$

If $H_{\nu_{i_1}^{j_1}}(\gamma^{'}) \geq n_1(n_1 - i_1 - j_1)$ and $H_{\nu_{i_2}^{j_2}}(\gamma^{''}) \geq n_2(n_2 - i_2 - j_2)$ then $H_{\widetilde{\nu}_{i_1}^{j_1}}(\delta) \geq 0$ and $H_{\bar{\nu}_{i_2}^{j_2}}(\delta) \geq 0$. Thus $\delta \in \mathbb{Z}_+(A \diamond B)$ and $\gamma \in \{(1, \ldots, 1) + \mathbb{Z}_+(A \diamond B)\}$. If $H_{\nu_{i_1}^{j_1}}(\gamma^{'}) < n_1(n_1 - i_1 - j_1)$ or $H_{\nu_{i_2}^{j_2}}(\gamma^{''}) < n_2(n_2 - i_2 - j_2)$, then let $t_1 \in [n_1 - i_1 - j_1 - 1]$ and $t_2 \in [n_2 - i_2 - j_2 - 1]$ such that $H_{\nu_{i_1}^{j_1}}(\gamma^{'}) = n_1(n_1 - i_1 - j_1 - t_1)$ or $H_{\nu_{i_2}^{j_2}}(\gamma^{''}) = n_2(n_2 - i_2 - j_2 - t_2)$. Using Lemma 2 we can find $\eta^{'} \in M_{\mathcal{A}}$ with $H_{\nu_{i_1}^{j_1}}(\gamma^{'}) = H_{\nu_{i_1}^{j_1}}(\eta^{'})$ and $\gamma^{'} - \eta' \in \mathbb{Z}_+A$, respectively we can find $\eta^{''} \in M_{\mathcal{B}}$ with $H_{\nu_{i_2}^{j_2}}(\gamma^{''}) = H_{\nu_{i_2}^{j_2}}(\eta^{''})$ and $\gamma^{''} - \eta'' \in \mathbb{Z}_+B$. Thus for any $p \in A^{(2)}$ and $q \in B^{(2)}$ we have $\gamma - (\widetilde{\eta^{'}} + \bar{q}) \in \mathbb{Z}_+(A \diamond B)$, $\gamma - (\bar{\eta''} + \widetilde{p}) \in \mathbb{Z}_+(A \diamond B)$ and so there exists $\alpha \in M_{\mathcal{A}\diamond\mathcal{B}}$ such that $\gamma \in \{\alpha + \mathbb{Z}_+(A \diamond B)\}$. If $H_{\nu_{i_1}^{j_1}}(\gamma^{'}) \geq n_1(n_1 - i_1 - j_1)$ and $H_{\nu_{i_2}^{j_2}}(\gamma^{''}) < n_2(n_2 - i_2 - j_2)$, then $\gamma^{'} \in (1, \ldots, 1) + \mathbb{Z}_+A$ and we can find $\eta^{''} \in M_{\mathcal{B}}$ with $H_{\nu_{i_2}^{j_2}}(\gamma^{''}) = H_{\nu_{i_2}^{j_2}}(\eta^{''})$ and $\gamma^{''} - \eta'' \in \mathbb{Z}_+B$. Thus, $\gamma \in (\widetilde{p} + \bar{\eta^{''}}) + \mathbb{Z}_+(A \diamond B)$, where $p = (\underbrace{2, \ldots, 2}_{n_1 - times})$. So there exists $\alpha \in M_{\mathcal{A}\diamond\mathcal{B}}$ such that $\gamma \in \{\alpha + \mathbb{Z}_+(A \diamond B)\}$. Analog the another case: $H_{\nu_{i_1}^{j_1}}(\gamma^{'}) < n_1(n_1 - i_1 - j_1)$ and $H_{\nu_{i_2}^{j_2}}(\gamma^{''}) \geq n_2(n_2 - i_2 - j_2)$.

Thus

$$\mathbb{Z}_+(A \diamond B) \cap relint(\mathbb{R}_+(A \diamond B)) = \{(1, \ldots, 1) + \mathbb{Z}_+(A \diamond B)\} \cup \bigcup_{\alpha \in M_{\mathcal{A}\diamond\mathcal{B}}} \{\alpha + \mathbb{Z}_+(A \diamond B)\}.$$

So, the canonical module $\omega_{K[\mathcal{A}\diamond\mathcal{B}]}$ of $K[\mathcal{A} \diamond \mathcal{B}]$, with respect to standard grading, can be expressed as an ideal of $K[\mathcal{A} \diamond \mathcal{B}]$, generated by monomials

$$\omega_{K[\mathcal{A}\diamond\mathcal{B}]} = (\{x_1 \cdots x_n,\ x^{\alpha} |\ \alpha \in M_{\mathcal{A}\diamond\mathcal{B}}\})K[\mathcal{A} \diamond \mathcal{B}].$$

The type of $K[\mathcal{A} \diamond \mathcal{B}]$ is the minimal number of generators of the canonical module. So, type$(K[\mathcal{A} \diamond \mathcal{B}]) = 1 + \#(M_{\mathcal{A}\diamond\mathcal{B}})$, where

$$\#(M_{\mathcal{A}\diamond\mathcal{B}}) = \#(M_{\mathcal{A}})\#(B^{(2)}) + \#(M_{\mathcal{B}})\#(A^{(2)}) - \#(M_{\mathcal{A}})\#(M_{\mathcal{B}}).$$

Using lemma 3 and since $\#(M_{\mathcal{A}}) = \text{type}(K[\mathcal{A}]) - 1,\ \#(M_{\mathcal{B}}) = \text{type}(K[\mathcal{B}]) - 1$ we get that

$$\#(M_{\mathcal{A}\diamond\mathcal{B}}) = (\text{type}(K[\mathcal{A}] - 1)Q_2 + (\text{type}(K[\mathcal{B}] - 1)Q_1 - (\text{type}(K[\mathcal{A}] - 1)(\text{type}(K[\mathcal{B}] - 1),$$

where $\#(A^{(2)}) = Q_1,\ \#(B^{(2)}) = Q_2$,

$$Q_r = \sum_{t=i_r}^{2(n_r - j_r) - 1} Q_{i_r}(t) Q_{n_r - i_r}(2n_r - t),\ \text{for } r \in [2].$$

b) Let $i_1 \in [n_1 - 2]$, $j_1 \in [n_1 - 1]$, $i_2 \in [n_2 - 2]$, $j_2 \in [n_2 - 1]$ be such that $i_1 + j_1 \geq n_1$, $i_2 + j_2 \geq n_2$, $r_1 = \left\lceil \frac{i_1 + 1}{n_1 - j_1} \right\rceil$ and $r_2 = \left\lceil \frac{i_2 + 1}{n_2 - j_2} \right\rceil$.

If we denote by $M_{\mathcal{A}}^{'}, M_{\mathcal{B}}^{'}$ the sets

$$M_{\mathcal{A}}^{'} = \{\alpha \in \mathbb{Z}_{>}^{n_1} \mid |(\alpha_1, \ldots, \alpha_{i_1})| = r_1(n_1 - j_1) - t,\ |(\alpha_{i_1+1}, \ldots, \alpha_{n_1})| = r_1 j_1 + t \text{ for any } t \in [r_1(n_1 - j_1) - i_1]\},$$

$$M_{\mathcal{B}}^{'} = \{\alpha \in \mathbb{Z}_{>}^{n_2} \mid |(\alpha_1, \ldots, \alpha_{i_2})| = r_2(n_2 - j_2) - t,\ |(\alpha_{i_2+1}, \ldots, \alpha_{n_2})| =$$

$$r_2j_2 + t \text{ for any } t \in [r_2(n_2 - j_2) - i_2]\}$$

we know from [16] that the canonical module $\omega_{K[\mathcal{A}]}$ of $K[\mathcal{A}]$ (respectively, $\omega_{K[\mathcal{B}]}$ of $K[\mathcal{B}]$) with respect to the standard grading can be expressed as an ideal of $K[\mathcal{A}]$ (respectively, $K[\mathcal{B}]$) generated by monomials

$$\omega_{K[\mathcal{A}]} = (\{x^\alpha |\ \alpha \in M'_{\mathcal{A}}\})K[\mathcal{A}],$$

respectively

$$\omega_{K[\mathcal{B}]} = (\{x^\alpha |\ \alpha \in M'_{\mathcal{B}}\})K[\mathcal{B}].$$

We will denote by $M_{\mathcal{A}\diamond\mathcal{B}}$ the set $M_{\mathcal{A}\diamond\mathcal{B}} = \{\widetilde{p}+\bar{\beta} \mid p \in A^{(r_2)},\ \beta \in M'_{\mathcal{B}}\}$. We will show that the canonical module $\omega_{K[\mathcal{A}\diamond\mathcal{B}]}$ of $K[\mathcal{A}\diamond\mathcal{B}]$, with respect to standard grading, can be expressed as an ideal of $K[\mathcal{A}\diamond\mathcal{B}]$, generated by monomials

$$\omega_{K[\mathcal{A}\diamond\mathcal{B}]} = (\{x^\alpha |\ \alpha \in M_{\mathcal{A}\diamond\mathcal{B}}\})K[\mathcal{A}\diamond\mathcal{B}].$$

This fact is equivalent to show that

$$\mathbb{Z}_+(A\diamond B) \cap \operatorname{relint}(\mathbb{R}_+(A\diamond B)) = \bigcup_{\alpha \in M_{\mathcal{A}\diamond\mathcal{B}}} \{\alpha + \mathbb{Z}_+(A\diamond B)\}.$$

Since for any $p \in A^{(r_2)},\ \beta \in M'_{\mathcal{B}}$ we have $H_{\widetilde{\nu}^{j_1}_{i_1}}(\widetilde{p}+\bar{\beta}) = H_{\nu^{j_1}_{i_1}}(p) > 0$, $H_{\bar{\nu}^{j_2}_{i_2}}(\widetilde{p}+\bar{\beta}) = H_{\nu^{j_2}_{i_2}}(\beta) = n_2t > 0$ it follows that

$$\mathbb{Z}_+(A\diamond B) \cap \operatorname{relint}(\mathbb{R}_+(A\diamond B)) \supseteq \bigcup_{\alpha \in M_{\mathcal{A}\diamond\mathcal{B}}} \{\alpha + \mathbb{Z}_+(A\diamond B)\}.$$

Since $H_{\widetilde{\nu}^{j_1}_{i_1}}((1,\ldots,1)) = n_1(n_1 - i_1 - j_1) \leq 0$ and $H_{\bar{\nu}^{j_2}_{i_2}}((1,\ldots,1)) = n_2(n_2 - i_2 - j_2) \leq 0$ it follows that $(1,\ldots,1) \notin \operatorname{relint}(\mathbb{R}_+(A\diamond B))$. Let $\gamma \in \mathbb{Z}_+(A\diamond B) \cap \operatorname{relint}(\mathbb{R}_+(A\diamond B))$, then $H_{\widetilde{\nu}^{j_1}_{i_1}}(\gamma) > 0,\ H_{\bar{\nu}^{j_2}_{i_2}}(\gamma) > 0$ and $\gamma_k \geq 1$ for any $k \in [n_1+n_2]$. We claim that $|\ \gamma\ | \geq r_2(n_1+n_2)$. Indeed, since $\gamma = (\gamma', \gamma'') \in \mathbb{Z}_+(A\diamond B) \cap \operatorname{relint}(\mathbb{R}_+(A\diamond B)),\ |\gamma| = s(n_1+n_2)$ and $\mathbb{Z}_+(A\diamond B) = \mathbb{Z}_+\widetilde{A} + \mathbb{Z}_+\bar{B}$, it follows that $\gamma' \in \mathbb{Z}_+A,\ \gamma'' \in \mathbb{Z}_+B$ with $|\gamma'| = sn_1,\ |\gamma''| = sn_2$ and

$$H_{\bar{\nu}^{j_2}_{i_2}}(\gamma) = H_{\nu^{j_2}_{i_2}}(\gamma'') = -j_2\sum_{k=1}^{i_2}\gamma''_k + (n_2 - j_2)(sn_2 - \sum_{k=1}^{i_2}\gamma''_k) > 0 \Longleftrightarrow \sum_{k=1}^{i_2}\gamma''_k < (n_2 - j_2)s.$$

Hence $i_2+1 \leq s(n_2-j_2)$ and so $r_2 = \left\lceil \frac{i_2+1}{n_2-j_2} \right\rceil \leq s$. Using Lemma 2 we can find $\eta'' \in M'_{\mathcal{B}}$ with $H_{\nu^{j_2}_{i_2}}(\gamma'') = H_{\nu^{j_2}_{i_2}}(\eta'')$ and $\gamma'' - \eta'' \in \mathbb{Z}_+B$. Since for any $p \in A^{(r_2)}$, we have $H_{\widetilde{\nu}^{j_1}_{i_1}}(\widetilde{p}+\bar{\eta}'') = H_{\nu^{j_1}_{i_1}}(p) > 0, H_{\bar{\nu}^{j_2}_{i_2}}(\widetilde{p}+\bar{\eta}'') = H_{\nu^{j_2}_{i_2}}(\eta'') = n_2t > 0$ it follows that $\gamma \in \widetilde{p} + \bar{\eta}'' + \mathbb{Z}_+(A\diamond B)$. Thus,

$$\mathbb{Z}_+(A\diamond B) \cap \operatorname{relint}(\mathbb{R}_+(A\diamond B)) \subseteq \bigcup_{\alpha \in M_{\mathcal{A}\diamond\mathcal{B}}} \{\alpha + \mathbb{Z}_+(A\diamond B)\}.$$

So, the canonical module $\omega_{K[\mathcal{A}\diamond\mathcal{B}]}$ of $K[\mathcal{A}\diamond\mathcal{B}]$, with respect to standard grading, can be expressed as an ideal of $K[\mathcal{A}\diamond\mathcal{B}]$, generated by monomials

$$\omega_{K[\mathcal{A}\diamond\mathcal{B}]} = (\{x^\alpha |\ \alpha \in M_{\mathcal{A}\diamond\mathcal{B}}\})K[\mathcal{A}\diamond\mathcal{B}].$$

The type of $K[\mathcal{A}\diamond\mathcal{B}]$ is the minimal number of generators of the canonical module. So, $\operatorname{type}(K[\mathcal{A}\diamond\mathcal{B}]) = \#(M_{\mathcal{A}\diamond\mathcal{B}}) = \#(A^{(r_2)})\#(M'_{\mathcal{B}})$. Using Lemma 3 and since $\#(M'_{\mathcal{B}}) = \operatorname{type}(K[\mathcal{B}])$ we get that

$$\operatorname{type}(K[\mathcal{A}\diamond\mathcal{B}]) = [\sum_{t=i_1}^{r_2(n_1-j_1)-1} Q_{i_1}(t)Q_{n_1-i_1}(r_2n_1 - t)]\operatorname{type}(K[\mathcal{B}]).$$

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c) Let $i_1 \in [n_1-2]$, $j_1 \in [n_1-1]$, $i_2 \in [n_2-2]$, $j_2 \in [n_2-1]$, $r_2 = \left\lceil \frac{i_2+1}{n_2-j_2} \right\rceil$ be such that $i_1 + j_1 \leq n_1$ and $i_2 + j_2 \geq n_2$. If we denote by $M_{\mathcal{A}}, M'_{\mathcal{B}}$ the sets

$$M_{\mathcal{A}} = \{\alpha \in \mathbb{Z}^{n_1}_{>} \mid |(\alpha_1, \ldots, \alpha_{i_1})| = n_1 + i_1 - j_1 + t,\ |(\alpha_{i_1+1}, \ldots, \alpha_{n_1})| = n_1 - i_1 + j_1 - t \text{ for any } t \in [n_1 - i_1 - j_1 - 1]\},$$

$$M'_{\mathcal{B}} = \{\alpha \in \mathbb{Z}^{n_2}_{>} \mid |(\alpha_1, \ldots, \alpha_{i_2})| = r_2(n_2 - j_2) - t,\ |(\alpha_{i_2+1}, \ldots, \alpha_{n_2})| = r_2 j_2 + t \text{ for any } t \in [r_2(n_2 - j_2) - i_2]\}$$

we know from [16] that the canonical module $\omega_{K[\mathcal{A}]}$ of $K[\mathcal{A}]$ (respectively, $\omega_{K[\mathcal{B}]}$ of $K[\mathcal{B}]$) with respect to the standard grading can be expressed as an ideal of $K[\mathcal{A}]$(respectively, $K[\mathcal{B}]$) generated by monomials

$$\omega_{K[\mathcal{A}]} = (\{x_1 \cdots x_n,\ x^{\alpha} \mid \alpha \in M_{\mathcal{A}}\})K[\mathcal{A}],$$

respectively

$$\omega_{K[\mathcal{B}]} = (\{x^{\alpha} \mid \alpha \in M'_{\mathcal{B}}\})K[\mathcal{B}].$$

We will denote by $M_{\mathcal{A}\diamond\mathcal{B}}$ the set $M_{\mathcal{A}\diamond\mathcal{B}} = \{\widetilde{\alpha} + \bar{\beta} \mid \beta \in M'_{\mathcal{B}},\ \alpha = (1, \ldots, 1) + \alpha' \text{ with } \alpha' \in A^{r_2-1} \text{ or } \alpha = \gamma + \alpha'' \text{ with } \alpha'' \in A^{r_2-2},\ \gamma \in M_{\mathcal{A}}\}$. We will show that the canonical module $\omega_{K[\mathcal{A}\diamond\mathcal{B}]}$ of $K[\mathcal{A} \diamond \mathcal{B}]$, with respect to standard grading, can be expressed as an ideal of $K[\mathcal{A} \diamond \mathcal{B}]$, generated by monomials

$$\omega_{K[\mathcal{A}\diamond\mathcal{B}]} = (\{x^{a} \mid a \in M_{\mathcal{A}\diamond\mathcal{B}}\})K[\mathcal{A} \diamond \mathcal{B}].$$

This fact is equivalent to show that

$$\mathbb{Z}_+(A \diamond B) \cap \operatorname{relint}(\mathbb{R}_+(A \diamond B)) = \bigcup_{a \in M_{\mathcal{A}\diamond\mathcal{B}}} \{a + \mathbb{Z}_+(A \diamond B)\}.$$

Since for any $\beta \in M'_{\mathcal{B}}$ and $\alpha \in \mathbb{Z}^{n_1}_{+}$ such that $\alpha = (1, \ldots, 1) + \alpha'$ with $\alpha' \in A^{r_2-1}$ or $\alpha = \gamma + \alpha''$ with $\gamma \in M_{\mathcal{A}},\ \alpha'' \in A^{r_2-2}$ we have $H_{\widetilde{\nu}^{j_1}_{i_1}}(\widetilde{\alpha} + \bar{\beta}) = H_{\nu^{j_1}_{i_1}}(\alpha) = H_{\nu^{j_1}_{i_1}}(1, \ldots, 1) + H_{\nu^{j_1}_{i_1}}(\alpha') = n_1(n_1 - i_1 - j_1) + H_{\nu^{j_1}_{i_1}}(\alpha') > 0$ or $H_{\widetilde{\nu}^{j_1}_{i_1}}(\widetilde{\alpha} + \bar{\beta}) = H_{\nu^{j_1}_{i_1}}(\alpha) = H_{\nu^{j_1}_{i_1}}(\gamma) + H_{\nu^{j_1}_{i_1}}(\alpha'') = n_1(n_1 - i_1 - j_1 - t) + H_{\nu^{j_1}_{i_1}}(\alpha'') > 0$ and $H_{\bar{\nu}^{j_2}_{i_2}}(\widetilde{\alpha} + \bar{\beta}) = H_{\nu^{j_2}_{i_2}}(\beta) = n_2 t > 0$ for any $t \in [n_1 - i_1 - j_1 - 1]$, it follows that

$$\mathbb{Z}_+(A \diamond B) \cap \operatorname{relint}(\mathbb{R}_+(A \diamond B)) \supseteq \bigcup_{a \in M_{\mathcal{A}\diamond\mathcal{B}}} \{a + \mathbb{Z}_+(A \diamond B)\}.$$

Since $H_{\widetilde{\nu}^{j_1}_{i_1}}((1, \ldots, 1)) = n_1(n_1 - i_1 - j_1) > 0$ and $H_{\bar{\nu}^{j_2}_{i_2}}((1, \ldots, 1)) = n_2(n_2 - i_2 - j_2) \leq 0$ it follows that $(1, \ldots, 1) \notin \operatorname{relint}(\mathbb{R}_+(A \diamond B))$. Let $\gamma \in \mathbb{Z}_+(A \diamond B) \cap \operatorname{relint}(\mathbb{R}_+(A \diamond B))$, then $H_{\widetilde{\nu}^{j_1}_{i_1}}(\gamma) > 0,\ H_{\bar{\nu}^{j_2}_{i_2}}(\gamma) > 0$ and $\gamma_k \geq 1$ for any $k \in [n_1 + n_2]$. We claim that $|\ \gamma\ | \geq r_2(n_1+n_2)$. Indeed, since $\gamma = (\gamma', \gamma'') \in \mathbb{Z}_+(A\diamond B) \cap relint(\mathbb{R}_+(A\diamond B))$, $|\gamma| = s(n_1+n_2)$ and $\mathbb{Z}_+(A \diamond B) = \mathbb{Z}_+\widetilde{A} + \mathbb{Z}_+\bar{B}$, it follows that $\gamma' \in \mathbb{Z}_+A,\ \gamma'' \in \mathbb{Z}_+B$ with $|\gamma'| = sn_1,\ |\gamma''| = sn_2$ and

$$H_{\bar{\nu}^{j_2}_{i_2}}(\gamma) = H_{\nu^{j_2}_{i_2}}(\gamma'') = -j_2 \sum_{k=1}^{i_2} \gamma''_k + (n_2 - j_2)(sn_2 - \sum_{k=1}^{i_2} \gamma''_k) > 0 \Longleftrightarrow \sum_{k=1}^{i_2} \gamma''_k < (n_2 - j_2)s.$$

Hence $i_2 + 1 \leq s(n_2 - j_2)$ and so $r_2 = \left\lceil \frac{i_2+1}{n_2-j_2} \right\rceil \leq s$. Since $H_{\nu^{j_1}_{i_1}}((1, \ldots, 1)) = n_1(n_1 - i_1 - j_1) > 0$ and for any $\delta \in M_{\mathcal{A}}$ we have $H_{\nu^{j_1}_{i_1}}(\delta) = n_1(n_1 - i_1 - j_1 - t) > 0$ it follows that for $\gamma' \in \mathbb{Z}_+A \cap \operatorname{relint}(\mathbb{R}_+A)$ such that $|\gamma'| = sn_1$ with $s \geq r_2$ there exists $\alpha' \in A^{r_2-1}$and $\alpha'' \in A^{r_2-2}$ such that $\gamma' \in (1, \ldots, 1) + \alpha' + \mathbb{Z}_+A$ or $\gamma' \in \delta + \alpha'' + \mathbb{Z}_+A$. Using

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Lemma 2 we can find $\eta^{''} \in M^{'}_{\mathcal{B}}$ such that $H_{\nu^{j_2}_{i_2}}(\gamma^{''}) = H_{\nu^{j_2}_{i_2}}(\eta^{''})$ and $\gamma^{''} - \eta^{''} \in \mathbb{Z}_+ B$. Thus, $\gamma = (\gamma^{'}, \gamma^{''}) \in ((1, \ldots, 1) + \alpha^{'}, \eta^{''}) + \mathbb{Z}_+(A \diamond B)$ with $\alpha^{'} \in A^{r_2-1}$, $\eta^{''} \in M^{'}_{\mathcal{B}}$ or $\gamma = (\gamma^{'}, \gamma^{''}) \in (\delta + \alpha^{''}, \eta^{''}) + \mathbb{Z}_+(A \diamond B)$ with $\delta \in M_{\mathcal{A}}$, $\alpha^{''} \in A^{r_2-2}$, $\eta^{''} \in M^{'}_{\mathcal{B}}$ and so

$$\mathbb{Z}_+(A \diamond B) \cap \mathrm{relint}(\mathbb{R}_+(A \diamond B)) \subseteq \bigcup_{a \in M_{\mathcal{A}\diamond\mathcal{B}}} \{a + \mathbb{Z}_+(A \diamond B)\}.$$

The canonical module $\omega_{K[\mathcal{A}\diamond\mathcal{B}]}$ of $K[\mathcal{A} \diamond \mathcal{B}]$, with respect to standard grading, can be expressed as an ideal of $K[\mathcal{A} \diamond \mathcal{B}]$, generated by monomials

$$\omega_{K[\mathcal{A}\diamond\mathcal{B}]} = (\{x^a | \ a \in M_{\mathcal{A}\diamond\mathcal{B}}\})K[\mathcal{A} \diamond \mathcal{B}].$$

The type of $K[\mathcal{A} \diamond \mathcal{B}]$ is the minimal number of generators of the canonical module,

$$\mathrm{type}(K[\mathcal{A}\diamond\mathcal{B}]) = \#(M_{\mathcal{A}\diamond\mathcal{B}}) = [\#(A^{r_2-1}) + \#(\{M_{\mathcal{A}} + A^{r_2-2}\} \setminus \{(1, \ldots, 1) + A^{r_2-1}\})]\#(M^{'}_{\mathcal{B}}).$$

We denote

$$E^{(r_2-2)} = \{\alpha \in \mathbb{Z}_+^{r_2 n_1} \mid \alpha_k \geq 1, \quad \alpha_1 + \ldots + \alpha_{i_1} = i_1 + (r_2 - 1)(n_1 - j_1) + t,$$
$$\alpha_{i_1+1} + \ldots + \alpha_{n_1} = n_1 - i_1 + (r_2 - 1)j_1 - t, \text{ for any } k \in [n] \text{ and } t \in [n_1 - i_1 - j_1 - 1]\}.$$

It is easy to see that $E^{(r_2-2)} \supseteq \{M_{\mathcal{A}} + A^{r_2-2}\} \setminus \{(1, \ldots, 1) + A^{r_2-1}\}$. Since for any $\alpha \in E^{(r_2-2)}$ we have $\alpha_1 + \ldots + \alpha_{i_1} = n_1 + i_1 - j_1 + t + (r_2 - 2)(n_1 - j_1)$, $\alpha_{i_1+1} + \ldots + \alpha_{n_1} = n_1 - i_1 + j_1 - t + (r_2 - 2)j_1$ for $t \in [n_1 - i_1 - j_1 - 1]$ and the set $\{(n_1 - j_1)e_r + j_1 \ e_s \mid 1 \leq r \leq i_1 \text{ and } i_1 + 1 \leq s \leq n_1\} \subset A$ are extremal rays of the cone $\mathbb{R}_+ A$ it follows that $\{M_{\mathcal{A}} + A^{r_2-2}\} \setminus \{(1, \ldots, 1) + A^{r_2-1}\} = E^{(r_2-2)}$. For any $1 \leq t \leq n_1 - i_1 - j_1 - 1$, the equation $\alpha_1 + \ldots + \alpha_{i_1} = i_1 + (r_2 - 1)(n_1 - j_1) + t$ has $Q_{i_1}(i_1 + (r_2 - 1)(n_1 - j_1) + t)$ distinct nonnegative integer solutions with $\alpha_k \geq 1$, for any $k \in [i_1]$, respectively $\alpha_{i_1+1} + \ldots + \alpha_{n_1} = n_1 - i_1 + (r_2 - 1)j_1 - t$ has $Q_{n_1-i_1}(n_1 - i_1 + (r_2 - 1)j_1 - t)$ distinct nonnegative integer solutions with $\alpha_k \geq 1$ for any $k \in [n_1] \setminus [i_1]$. Thus, the cardinal of $E^{(r_2-2)}$ is

$$\#(E^{(r_2-2)}) = \sum_{t=1}^{n_1-i_1-j_1-1} Q_{i_1}(i_1 + (r_2 - 1)(n_1 - j_1) + t)Q_{n_1-i_1}(n_1 - i_1 + (r_2 - 1)j_1 - t).$$

So,

$$\mathrm{type}(K[\mathcal{A} \diamond \mathcal{B}]) = [\#(A^{r_2-1}) + \#(E^{(r_2-2)})] \, \mathrm{type}(K[\mathcal{B}]).$$

□

Corollary 6. *Let $K[\mathcal{A}]$ and $K[\mathcal{B}]$ the base rings of the transversal polymatroids presented by $\mathcal{A}$ and $\mathcal{B}$ and $K[\mathcal{A} \diamond \mathcal{B}]$ the base ring of the transversal polymatroid presented by $\mathcal{A} \diamond \mathcal{B}$, then: $K[\mathcal{A} \diamond \mathcal{B}]$ is Gorenstein ring if and only if $K[\mathcal{A}]$ and $K[\mathcal{B}]$ are Gorenstein rings.*

Next we will give some examples.

Let $\mathcal{A} = \{A_1, \ldots, A_5\}$, $\mathcal{B} = \{A_6, \ldots, A_{12}\}$ and $\mathcal{A} \diamond \mathcal{B} = \{A_1, \ldots, A_{12}\}$, where $A_1 = A_3 = A_4 = A_5 = [5]$, $A_2 = [5] \setminus [2]$, $A_6 = A_9 = A_{10} = A_{11} = A_{12} = [12] \setminus [5]$, $A_7 = A_8 = [12] \setminus [8]$.

The type of $K[\mathcal{A} \diamond \mathcal{B}]$ is

$$\mathrm{type}(K[\mathcal{A} \diamond \mathcal{B}]) = 1 + (7 - 1)1680 + (113 - 1)126 - (7 - 1)(113 - 1) = 23521,$$

where

$$\mathrm{type}(K[\mathcal{A}]) = 7, \ \mathrm{type}(K[\mathcal{B}]) = 113, \ Q_1 = 126, Q_2 = 1680.$$

The Hilbert series of $K[\mathcal{A} \diamond \mathcal{B}]$ is

$$H_{K[\mathcal{A}\diamond\mathcal{B}]}(t) \ = \ \frac{1 + 188149t + 32250295t^2 + \ldots + 34608475t^8 + 211669t^9 + t^{10}}{(1 - t)^{11}}.$$

Note that $\mathrm{type}(K[\mathcal{A} \diamond \mathcal{B}]) = 1 + h_9 - h_1 = 23521$.

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Let $\mathcal{A} = \{A_1, \ldots, A_7\}$, $\mathcal{B} = \{A_8, \ldots, A_{15}\}$ and $\mathcal{A} \diamond \mathcal{B} = \{A_1, \ldots, A_{15}\}$, where $A_1 = A_6 = A_7 = [7]$, $A_2 = A_3 = A_4 = A_5 = [7] \setminus [5]$, $A_8 = A_{15} = [15] \setminus [7]$, $A_9 = A_{10} = A_{11} = A_{12} = A_{13} = A_{14} = [15] \setminus [13]$.

The type of $K[\mathcal{A} \diamond \mathcal{B}]$ is

$$\text{type}(K[\mathcal{A} \diamond \mathcal{B}]) = (\sum_{t=5}^{11} \binom{t-1}{4}\binom{27-t}{1})169 = 1327326,$$

where

$$\text{type}(K[\mathcal{B}]) = 169.$$

The Hilbert series of $K[\mathcal{A} \diamond \mathcal{B}]$ is

$$H_{K[\mathcal{A}\diamond\mathcal{B}]}(t) = \frac{1 + 62818t + 12287443t^2 + \ldots + 91435344t^9 + 1327326t^{10}}{(1-t)^{14}}.$$

Note that $\text{type}(K[\mathcal{A} \diamond \mathcal{B}]) = h_{10} = 1327326$.

Let $\mathcal{A} = \{A_1, \ldots, A_8\}$, $\mathcal{B} = \{A_9, \ldots, A_{16}\}$ and $\mathcal{A} \diamond \mathcal{B} = \{A_1, \ldots, A_{16}\}$, where $A_1 = A_4 = A_5 = A_6 = A_7 = A_8 = [8]$, $A_2 = A_3 = [8] \setminus [3]$, $A_9 = A_{16} = [16] \setminus [8]$, $A_{10} = A_{11} = A_{12} = A_{13} = A_{14} = A_{15} = [16] \setminus [14]$.

The type of $K[\mathcal{A} \diamond \mathcal{B}]$ is

$$\text{type}(K[\mathcal{A} \diamond \mathcal{B}]) = (2572125 + 42630)169 = 441893595,$$

where

$$\text{type}(K[\mathcal{A}]) = 226,\ type(K[\mathcal{B}]) = 169,\ G = 2572125,\ E = 42630.$$

The Hilbert series of $K[\mathcal{A} \diamond \mathcal{B}]$ is

$$H_{K[\mathcal{A}\diamond\mathcal{B}]}(t) = \frac{1 + 1266825t + 661717155t^2 + \ldots + 32407888815t^{10} + 441893595t^{11}}{(1-t)^{15}}.$$

Note that $\text{type}(K[\mathcal{A} \diamond \mathcal{B}]) = h_{11} = 441893595$.

We end this section with the following conjecture:

Conjecture: Let $n \geq 4$, $A_i \subset [n]$ for any $1 \leq i \leq n$ and $K[\mathcal{A}]$ be the base ring associated to the transversal polymatroid presented by $\mathcal{A} = \{A_1, \ldots, A_n\}$. If the Hilbert series is:

$$H_{K[\mathcal{A}]}(t) = \frac{1 + h_1\ t + \ldots + h_{n-r}\ t^{n-r}}{(1-t)^n},$$

then we have the following:
1) If $r = 1$, then $\text{type}(K[\mathcal{A}]) = 1 + h_{n-2} - h_1$.
2) If $2 \leq r \leq n$, then $\text{type}(K[\mathcal{A}]) = h_{n-r}$.

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