AN ABELIAN GROUP OF A CLASS OF MITTAG - LEFFLER FUNCTIONS

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Abstract. In this study, we have considered a class of celebrated Mittag - Leffler functions. A suitable Hadamard type of product has been defined in this class. With respect to this product, it is found that the class forms an infinite torsion - free Abelian group and a subclass of it turns out to be a divisible group.

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1. Introduction

The Mittag - Leffler function (ML function) $E_\alpha(z)$ [10] is a special function of a complex variable z which depends on one parameter $\alpha \in \mathbb{C}$ (the set of complex numbers) defined by the power series as follows,

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, z \in \mathbb{C}, \Re(\alpha) > 0. \tag{1.1}$$

Clearly, the function $E_\alpha(z)$ is the generalization of the exponential functions as $e^z = E_1(z)$ (see [15], [16]). Many authors like, Wiman [19], Humbert [4] and Humbert and Agrawal [5] have analyzed and studied its various properties.

Also, for two parameters α and $\beta \in \mathbb{C}$, a generalized Mittag - Leffler function has been studied in the following form (see Wiman [19], Humbert [4] and Humbert and Agrawal [5] and Peng and Li [12])

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \Re(\alpha), \Re(\beta) > 0. \tag{1.2}$$

For $\beta = 1, E_{\alpha,\beta}(z)$ becomes $E_{\alpha,1}(z) = E_\alpha(z)$.

In 1971, Prabhakar [14] has introduced the three parameter ML function as follows;

$$E^{\nu}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\nu)_k z^k}{\Gamma(\alpha k + \beta)}, \Re(\alpha), \Re(\beta) > 0, (\nu)_k = \frac{\Gamma(\nu + k)}{\Gamma(\nu)} \forall k \in \{0, 1, 2, \ldots\}. \tag{1.3}$$

Evidently, the Eqns. (1.3), (1.2) and (1.1) give the equality $E^1_{\alpha,1}(z) = E_{\alpha,1}(z) = E_\alpha(z)$.

It is known that the functions defined in the Eqns. (1.1), (1.2) and (1.3) are entire functions of order $\rho = \frac{1}{\Re(\alpha)}$ and type $\tau = 1$, provided that $\Re(\alpha) > 0$. For $\Re(\alpha) = 0$, these functions are not entire functions but analytic having radius of convergence $\sigma = e^{\frac{\pi}{2}|Im(\alpha)|}$, (see, [14], [16]).

The Mittag - Leffler functions given in the Eqns. (1.1), (1.2) and (1.3) have been generalized in several directions. A class of generating functions more general than (1.3) has been considered and generalized extensively to its various forms and applied tremendously in solving various scientific and physical problems involving fractional calculus and differential equations (see [6], [7], [11], [13], [17] and [18]).

It is to be noted that for each complex number $\alpha = \alpha_1 + i\alpha_2, i = \sqrt{(-1)}$, $\Re(\alpha) = \alpha_1 > 0$, we can get a Mittag - Leffler function as defined in Eqn. (1.1) and conversely.

Thus the right half plane without y-axis denoted as $\mathbb{C}_+$ here and onwards is in one to one correspondence with the class of all ML functions of type (1.1) and vice-versa. We denote the class of all $E_\alpha(z)$ functions by the symbol Ω_α. Similarly, the class of all ML functions $E_{\alpha,\beta}$ of type (1.2) will be denoted by $\Omega_{\alpha,\beta}$, and that of type (1.3) by the $\Omega^{\nu}_{\alpha,\beta}$. We are mainly concerned with two classes viz Ω_α and $\Omega_{\alpha,\beta}$ in our following study:

2. Ω_α AS AN ALGEBRAIC STRUCTURE

In the year 2010, Peng and Li [12] disprove that a structure for all t and $s \geq 0$ under the operation $E_\alpha(a(t+s)^\alpha) = E_\alpha(at^\alpha)E_\alpha(as^\alpha)$ has the semi group property, where a is a constant. Here, the operation is taken over the variables t and s. Below, in our work, we take the operation under the parameter α and prove that the class Ω_α, under the operation $\oplus$ as defined below, is a *semi group*:

For all $E_\alpha, E_\beta \in \Omega_\alpha$, where, $\alpha = \alpha_1 + i\alpha_2, \beta = \beta_1 + i\beta_2$, are such that $\Re(\alpha) = \alpha_1 > 0$ and $\Re(\beta) = \beta_1 > 0$. Define an operation $\oplus$ as under

$$E_\alpha \oplus E_\beta = E_{\alpha+\beta}, \text{ where, } E_{\alpha+\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma((\alpha+\beta)k+1)}, \tag{2.1}$$

and $\alpha + \beta = (\alpha_1 + \beta_1) + i(\alpha_2 + \beta_2)$ is the usual addition of two complex numbers. Then, it can easily seen that,

(i) $E_{\alpha+\beta} \in \Omega_\alpha$ (as $\alpha_1 > 0, \beta_1 > 0$ implies that $\Re(\alpha+\beta) = (\alpha_1+\beta_1) > 0$), so Ω_α is closed with respect to the operation $\oplus$, defined in (2.1).

(ii) $\forall E_\alpha, E_\beta, E_\gamma \in \Omega_\alpha, (E_\alpha \oplus E_\beta) \oplus E_\gamma = E_{(\alpha+\beta)+\gamma} = E_{\alpha+(\beta+\gamma)}$
$= E_\alpha \oplus (E_\beta \oplus E_\gamma)$.

That is associative law holds in Ω_α.

Therefore, $(\Omega_\alpha, \oplus)$ is a *semi group* [1]. Further as we see that $E_0(0 = 0 + i0) \notin \Omega_\alpha$, so the semi group$(\Omega_\alpha, \oplus)$ is not a *monoid* [1]. Therefore, no question arises of its being a group.

It is also remarkable that if we follow the modified formula suggested by MacRobert [9, p. 374], that if $|\arg z| \leq \pi - \delta, \delta$ being a positive number such that $0 < \delta < \pi$, then

$$|\Gamma(z+\nu)| \leq M|z|^{g-1/2}\exp[x\log|z| - y\mathrm{amp}z - x],$$

where, $z = x + iy, M$ is a positive constant independent of z and $g = \Re(\nu)$, and $\lim_{z\to 0}|\Gamma(z+1)| = 1$.

Then, when $\beta \to \alpha$ and by the Hadamard product operation, we may find

$$|E_\alpha \oplus E_\beta| = |E_{\alpha+\beta}(z)| = \left|\sum_{k=0}^{\infty} \frac{1}{\Gamma((\alpha+\beta)k+1)} z^k\right| \approx \left|\sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta k+1)} z^k\right|,$$

as when $\beta \to \alpha$, i.e. $\beta = \alpha + \theta, 0 \leq \theta < 1$, we have $\sqrt{\alpha\beta} \approx \frac{\alpha+\beta}{2}$.

Since then, $\lim_{\beta\to\alpha}|\Gamma(\alpha k+1)\Gamma(\beta k+1)| = \lim_{\beta\to\alpha} M_1M_2|\alpha\beta|^{1/2}k\ \exp[\{\alpha_1 k\log|\alpha k| - \alpha_2 k\mathrm{amp}\alpha k - \alpha_1 k\} + \{\beta_1 k\log|\beta k| - \beta_2 k\mathrm{amp}\beta k - \beta_1 k\}] \approx \lim_{\beta\to\alpha} M_{12}(\alpha+\beta)k\exp[(\alpha_1+\beta_1)k\log|(\alpha+\beta)k| - (\alpha_2+\beta_2)k\mathrm{amp}(\alpha+\beta)k - (\alpha_1+\beta_1)k] = \lim_{\beta\to\alpha}|\Gamma((\alpha+\beta)k+1)|, \alpha = \alpha_1 + i\alpha_2, \beta = \beta_1 + i\beta_2, M_{12}$ is a positive constant independent of α and β; $\Re(\alpha) > 0$ and $|\arg\alpha| \leq \pi - \delta, \delta > 0 < \pi, \beta \to \alpha$.

Next, if we define a product operation $\odot$ in usual manner in the set Ω_α. That is

$$E_\alpha \odot E_\beta = E_{\alpha.\beta}, \text{ where, } E_{\alpha.\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma((\alpha.\beta)k+1)}, \tag{2.2}$$

and $\alpha.\beta = (\alpha_1 + i\alpha_2)(\beta_1 + i\beta_2) = (\alpha_1\beta_1 - \alpha_2\beta_2) + i(\alpha_1\beta_2 + \alpha_1\beta_2)$, the usual product of two complex numbers, we see that $E_{\alpha.\beta}$ may not belong to Ω_α as, for $\alpha_1 > 0$ and $\beta_1 > 0$, $\Re(\alpha.\beta) = (\alpha_1\beta_1 - \alpha_2\beta_2)$ is not always positive. Hence $(\Omega_\alpha, \odot)$ is not even an *algebraic structure*.

Thus the class Ω_α does not form a group under the operations $\oplus$ or $\odot$ as defined in Eqns. (2.1) and (2.2) in ordinary sense.

So, in order to make this class Ω_α a group, we modify (shorten) the class Ω_α as well as the operations given in the Eqns. (2.1) and (2.2) so that this class becomes a group. For that we proceed as below:

3. Subclass M_α

Let us consider a subclass M_α of Ω_α which consists of those ML functions E_α, $\alpha = \alpha_1 + i\alpha_2$ such that $\alpha_1 > 0$ and $\alpha_2 \neq 0$.

That is

$$M_\alpha = \{E_\alpha : E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \alpha \in \mathbb{C}, \Re(\alpha) > 0 \text{ and } Im(\alpha) \neq 0\}. \tag{3.1}$$

Note that this subclass M_α corresponds to those complex numbers $\alpha = (\alpha_1 + i\alpha_2)$ of right half plane which lie anywhere in the right half plane but not on x-axis or on y-axis.

In the class M_α, we define a Hadamard type product $\otimes$ in the following manner:

For all $E_\alpha, E_\beta \in M_\alpha$,

$$E_\alpha \otimes E_\beta = E_{\alpha\otimes\beta}, \text{ where, } E_{\alpha\otimes\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma((\alpha\otimes\beta)k + 1)}, \tag{3.2}$$

and $\alpha \otimes \beta = (\alpha_1 + i\alpha_2) \otimes (\beta_1 + i\beta_2) = \alpha_1\beta_1 + i\alpha_2\beta_2$.

Theorem 3.1. *$(M_\alpha, \otimes)$ is a torsion - free Abelian group.*

Proof. (i) By definition (3.2), we get $\alpha \otimes \beta = \alpha_1\beta_1 + i\alpha_2\beta_2$, clearly,
$\Re(\alpha \otimes \beta) = \alpha_1\beta_1 > 0$ and $Im(\alpha \otimes \beta) = \alpha_2\beta_2 \neq 0$, $\forall\ E_\alpha, E_\beta \in M_\alpha$. This shows that $E_{\alpha\otimes\beta} \in M_\alpha$. So closure property in M_α is satisfied.

$(ii)\forall\ E_\alpha, E_\beta, E_\gamma \in M_\alpha$, it can easily be checked that
$(E_\alpha \otimes E_\beta) \otimes E_\gamma = E_\alpha \otimes (E_\beta \otimes E_\gamma)$. So that the associativity holds in M_α.

(iii) Let $e = 1 + i$, then $E_e = E_{1+i} \in M_\alpha$ such that $E_\alpha \otimes E_e = E_{\alpha\otimes e} = E_\alpha$, because
$a \otimes e = (\alpha_1 + i\alpha_2) \otimes (1 + i1) = \alpha_1.1 + i\alpha_2.1 = \alpha_1 + i\alpha_2 = \alpha$.

Hence, $E_e = E_{1+i}$ is the identity element in M_α.

(iv) Let $\alpha = \alpha_1 + i\alpha_2$, $\alpha_1 > 0$ and $\alpha_2 \neq 0$ and so that $E_\alpha \in M_\alpha$. Let $\beta = \beta_1 + i\beta_2$ be such that $\alpha \otimes \beta = \alpha_1\beta_1 + i\alpha_2\beta_2 = 1 + i$, which implies $\alpha_1\beta_1 = 1$ and $\alpha_2\beta_2 = 1$. From that we get $\beta_1 = 1/\alpha_1$ and $\beta_2 = 1/\alpha_2$. So the inverse element of $\alpha = \alpha_1 + i\alpha_2$ is $\beta = \frac{1}{\alpha_1} + i\frac{1}{\alpha_2} = \alpha^{-1}$ (say) exists.

Obviously, $E_\beta = E_{\alpha^{-1}} \in M_\alpha$ and $E_\alpha \otimes E_{\alpha^{-1}} = E_{1+i} = E_e$.

Thus we see that each element E_α of M_α is invertible in M_α with respect to the operation $\otimes$ under consideration.

(v) Further it is easy to see that $E_\alpha \otimes E_\beta = E_{\alpha\otimes\beta} = E_{\beta\otimes\alpha} = E_\beta \otimes E_\alpha$ holds in M_α.

Thus, the commutative property holds in M_α.

Hence, $(M_\alpha, \otimes)$ is an Abelian group of infinite order [1], in which every element other than identity E_e (identity E_e is of order 1) is of infinite order, so that M_α is a torsion-free Abelian group [3]. □

Similarly, as above, we can show that the subclass $M_{\alpha,\beta}$ of the class $\Omega_{\alpha,\beta}$ of all ML functions $E_{\alpha,\beta}$ as given in Eqn. (1.2) of two parameters α, β is also an infinite torsion-free Abelian group w.r.t. the operation

$$E_{\alpha,\beta} \otimes E_{\gamma,\delta} = E_{(\alpha\otimes\gamma),(\beta\otimes\delta)}.$$

Remarks

(1) M_α is an abelian group therefore all its subgroup will be normal and the centre $Z(M_\alpha)$ of the group M_α will coincide with M_α itself. That is $Z(M_\alpha) = M_\alpha$.

(2) The Cartesian product $(M_\alpha \times M_\alpha)$ consists of all ordered pair (E_α, E_β) where, $E_\alpha, E_\beta \in M_\alpha$. An operation of composition in $(M_\alpha \times M_\alpha)$ is defined as follows;

$$(E_\alpha, E_\beta) \otimes (E_\gamma, E_\delta) = (E_{\alpha\otimes\gamma}, E_{\beta\otimes\delta}).$$

Then, the Cartesian product $(M_\alpha \times M_\alpha)$ is also an Abelian group. The diagonal subgroup $D_\alpha = \{(E_\alpha, E_\alpha) : E_\alpha \in M_\alpha\}$ of $(M_\alpha \times M_\alpha)$ is a normal subgroup of it.

4. Divisivility in M_α

In the following, we will discuss the divisibility property in M_α and show that M_α is not a divisible group [1], but a sub class D_α of M_α is a divisible group, where,

$$D_\alpha = \{E_\alpha : E_\alpha \in M_\alpha, \Re(\alpha) > 0 \text{ and } Im(\alpha) > 0\}. \tag{4.1}$$

It can easily be checked that D_α is a subgroup of M_α, therefore, it is an Abelian group on its own right.

Definition 4.1. *An Abelian group G is divisible if for any element $g \in G$ and that any non-zero integer n, there exists a $h \in G$ such that $h^n = g$.*

In other words, a group G is divisible in which for every element $g \in G$ and for every non-zero integer n, the equation $x^n = g$ is solvable in G (see [1] and [3]).

Theorem 4.2. *$(M_\alpha, \otimes)$ is not a divisible group, but $(D_\alpha, \otimes)$ is.*

Proof. Consider that n be a non - zero positive integer and $E_\alpha \in M_\alpha$. Let $E_\beta(\beta = \beta_1 + i\beta_2)$ is such that $E_\beta^n = E_\alpha$.

But $E_\beta^n = E_\beta \otimes E_\beta \otimes \ldots \otimes E_\beta(n - times) = E_{\beta^n}$, where, $\beta^n = \beta \otimes \beta \ldots \otimes \beta(n - times)$.

Hence,
$\Rightarrow E_{\beta^n} = E_\alpha \Rightarrow \beta^n = \alpha$
$\Rightarrow \beta_1^n + i\beta_2^n = \alpha_1 + i\alpha_2$ (note that $\beta^n = (\beta_1 + i\beta_2)^n = \beta_1^n + i\beta_2^n by(3.2)$)
$\Rightarrow \beta_1^n = \alpha_1, \beta_2^n = \alpha_2$.
Therefore, we get

$$\beta_1 = (\alpha_1)^{1/n} \text{ and } \beta_2 = (\alpha_2)^{1/n}. \tag{4.2}$$

Or if n is negative integer, then $E_\beta^{-n} = E_\alpha \Rightarrow E_{1/\beta}^n = E_\alpha$ or $E_{(1/\beta)^n} = E_\alpha \Rightarrow (1/\beta)^n = \alpha$.
Thus, $\Rightarrow (1/\beta_1)^n + i(1/\beta_2)^n = \alpha_1 + i\alpha_2$, {by definition (3.2)}.
Then, we get (4.2) as $\beta_1 = (1/\alpha_1)^{1/n}$ and $\beta_2 = (1/\alpha_2)^{1/n}$ etc.

Considering only real roots in (4.2), we see that as $\alpha_1 > 0, \Rightarrow \beta_1 = (\alpha_1)^{1/n}$ or $(1/\alpha_1)^{1/n}$ exist and it is positive one. But as $\alpha_2 \neq 0 \Rightarrow \beta_2 = (\alpha_2)^{1/n}$ or $(1/\alpha_2)^{1/n}$ may not give a real root as when α_2 be negative. For example if we take $n = 2$ and $\alpha = 5 - 3i$, then $E_\alpha \in M_\alpha$ (as $\alpha_1 = 5 > 0$ and $\alpha_2 = -3 \neq 0$).

Then $\beta_1 = (5)^{1/2} or (1/5)^{1/2} > 0$, but $\beta_2 = (-3)^{1/2}$ or $(-1/3)^{1/2} = \pm\sqrt{3}i$ or $\pm (1/\sqrt{3})i \notin \mathbb{R}$ (The set of real numbers). So we can always not get a $E_\beta \in M_\alpha$ such that $E_\beta^n = E_\alpha$. Therefore, M_α is not a divisible group.

But if we consider, the class D_α as defined in (4.1) above then (4.2) gives
$\beta_1 = (\alpha_1)^{1/n}$ or $(1/\alpha_1)^{1/n}$ that exists and give real positive root as $\alpha_1 > 0$. Also,
$\beta_2 = (\alpha_2)^{1/n}$ or $(1/\alpha_2)^{1/n}$ exists and give real positive root as $\alpha_2 > 0$.
$\Rightarrow E_\beta \in D_\alpha$.

Therefore for each $E_\alpha \in D_\alpha$ and for a non-zero integer n, we get a $E_\beta \in D_\alpha$ such that $E_\beta^n = E_\alpha$, so D_α is a divisible group [3]. □

It is remarkable that any divisible Abelian group is a direct sum of a number of copies of $\mathbb{Q}$ (set of rational numbers) and $Z(p^\infty) = \mathbb{Q}_p/\mathbb{Z}_p$ (p - Prüfer group, p is the prime number) (see [1]), thus unlike general infinite Abelian group, divisible group can be classified separately. Also the divisible groups are the injective objects in the category of Abelian groups which underline many applications of homological algebra, we shall study the divisible group D_α separately in our next paper.

5. Application and directions for further research work

In this section, we apply our theory given in previous sections 1 - 4, to obtain following results:

Theorem 5.1. *$\forall E_\alpha, E_\beta \in \Omega_\alpha$, the Cartesian product $(\Omega_\alpha \times \Omega_\alpha)$ consists of all ordered pair (E_α, E_β). An operation of composition in $(\Omega_\alpha \times \Omega_\alpha)$ is defined by*

$$(E_\alpha, E_\beta) \oplus (E_\gamma, E_\delta) = (E_{\alpha+\gamma}, E_{\beta+\delta}), where, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \tag{5.1}$$

also, $\alpha + \gamma$ and $\beta + \delta$ are usual additions of two complex numbers.

Then, in this composition every element satisfies the fractional derivative equation

$${}_{0^+}{}^{C}D_t^{\alpha}\,{}_{0^+}{}^{C}D_t^{\gamma} Y_{\alpha+\gamma}(t,\lambda) = \lambda Y_{\alpha+\gamma}(t,\lambda),\ \Re(\alpha) > 0, \Re(\gamma) > 0, \lambda \in \mathbb{R}, \alpha + \gamma \in [l-1, l], \tag{5.2}$$

for some $k, l \in \mathbb{N}$, $l \leq k$; $Y_{\alpha+\gamma}(t,\lambda) \in \mathbb{C}^k[a,b]$, for some $a < b$; here, $Y_{\alpha+\gamma}(t,\lambda) = E_{\alpha+\gamma}(\lambda t^{\alpha+\gamma}), t \geq 0, {}_{0^+}{}^{C}D_t^{\alpha}$ is the Caputo derivative defined in ([2, p. 49], [7]).

Hence then, $({}_{0^+}{}^{C}D_t^{\alpha+\gamma} E_{\alpha+\gamma}, {}_{0^+}{}^{C}D_t^{\beta+\delta} E_{\beta+\delta}) \in \Omega_\alpha$.

Proof. For $\Re(\alpha) > 0, \Re(\gamma) > 0, t \geq 0$, we are familiar that ${}_{0^+}{}^{C}D_t^{\alpha}\,{}_{0^+}{}^{C}D_t^{\gamma} f(t) = {}_{0^+}{}^{C}D_t^{\alpha+\gamma} f(t)$ (see Lemma 3.13 of Diethelm [2, p. 56], provided that $f \in \mathbb{C}^k[a,b]$, for some $a < b$, and some $k, l \in \mathbb{N}$, $l \leq k$, $\alpha, \alpha+\gamma \in [l-1, l]$). Then, on supposing $\alpha + \gamma = \eta \in [l-1, l]$, and applying it in left hand side of Eqn. (5.2) and again making an appeal to the Theorem 4.3 of Deithelm [2, p.70] to get

$${}_{0^+}{}^{C}D_t^{\eta} Y_\eta(t,\lambda) = \lambda Y_\eta(t,\lambda), where, Y_\eta(t,\lambda) = E_\eta(\lambda t^\eta), t \geq 0. \tag{5.3}$$

Hence, with the aid of Eqn. (5.3), we easily find right hand side of Eqn. (5.2). And thus apply the theory of section 2 and remark 2 in Eqns. (5.2) and Eqn. (5.3) to get

$$(\lambda E_{\alpha+\gamma}, \lambda E_{\beta+\delta}) \in \Omega_\alpha \text{ and since then } ({}_{0^+}{}^{C}D_t^{\alpha+\gamma} E_{\alpha+\gamma}, {}_{0^+}{}^{C}D_t^{\beta+\delta} E_{\beta+\delta}) \in \Omega_\alpha.$$

□

5.2 The multi-parameter generalized Mittag - Leffler functions defined in [7] as follows,

$$E_{\alpha,\beta}^{(\delta)}(z,\zeta) = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} \frac{(\delta)_{n+l}}{n!l!\Gamma(\alpha n + \beta l + 1)} z^n \zeta^l, \forall \alpha, \beta, z, \zeta \in \mathbb{C}, \tag{5.4}$$

where, $\Re(\alpha) > 0, \Re(\beta) > 0$, $Im(\alpha) > 0, Im(\beta) > 0$ and the Mittag - Leffler function of multi-parameter as defined in [11] is given as under

$$E_{\alpha,\beta;\gamma}^{(\delta)}(z,\zeta) = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} \frac{(\delta)_{n+l}}{n!l!\Gamma(\alpha n + \beta l + \gamma)} z^n \zeta^l, \forall \alpha, \beta, \gamma, z, \zeta \in \mathbb{C}, \tag{5.5}$$

where, $\Re(\alpha) > 0, \Re(\beta) > 0$, $\Re(\gamma) > 0, Im(\alpha) > 0, Im(\beta) > 0$, may have divisible Abelian groups $D_{\alpha,\beta}^{(\delta)}$ and $D_{\alpha,\beta;\gamma}^{(\delta)}$, respectively, under the Hadamard type operation as defined in Eqn. (3.2); (since $D_{\alpha,\alpha}^{(1)} = D_\alpha$ and $D_{\alpha,\alpha;\gamma}^{(1)} = D_{\alpha,\gamma}$) are under our investigation.

5.3 Finally, on making an appeal of the Theorem 4.2, this work may be helpful in the applications of Mittag - Leffler function given in the Eqn. (1.1) and its generalizations given in Eqns. (1.2), (5.1) and (5.2) in the theory of isomorphic Cayley graphs [8].

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