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SOME RESULTS IN LOCAL COHOMOLOGY AND SERRE SUBCATEGORY

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ABSTRACT. Let R be a noetherain ring, let $\mathfrak{a}$ be an ideal of R, and let M be an R-module. Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules and let i be a non-negative integer. In this paper we find some conditions under which $H^i_{\mathfrak{a}}(M) \in \mathcal{S}$ or $\mathrm{Supp}_R(H^i_{\mathfrak{a}}(M)) \subseteq \mathrm{Supp}(\mathcal{S})$.

1. Introduction

Throughout this paper R is a commutative noetherian ring and $\mathcal{S}$ is a Serre subcategory of R-modules. The main aim of this paper is to investigate when local cohomology modules belong to the Serre subcategory $\mathcal{S}$.

Recall that a Serre subcategory of the category of R-modules is a full subcategory whenever it is closed under taking submodules, quotient modules and extension. Some examples of these subcategories are the subcategories of finite generated R-modules; coatomic R-modules [Zos1]; minimax R-modules [Zos2].

Recently some results have been proved concerning with the local cohomology modules $H^i_{\mathfrak{a}}(M)$ of a module M in some certain Serre subcategory of the category of modules (cf. [AM, AT1, AT2]).

M. Aghapournahr and L. Melkersson [AM] gave a condition on a Serre subcategory $\mathcal{S}$. To give more details, let $\mathfrak{a}$ be an ideal of R, let M be an R-module and let $(0 :_M \mathfrak{a}) = \{x \in M | \mathfrak{a}x = 0\}$. The R-module M is said to satisfy $C_{\mathfrak{a}}$ condition on $\mathcal{S}$ whenever the following condition holds:

$$\text{If } \Gamma_{\mathfrak{a}}(M) = M \text{ and } (0 :_M \mathfrak{a}) \in \mathcal{S}, \text{ then } M \in \mathcal{S}.$$

From [AM], $\mathcal{S}$ is said to satisfy $C_{\mathfrak{a}}$ condition whenever every R-module satisfies $C_{\mathfrak{a}}$ condition on $\mathcal{S}$.

In this paper, by dealing with this condition, we answer to the following question:

When do the local cohomology modules $H^i_{\mathfrak{a}}(M)$ belong to $\mathcal{S}$ for a non-negative integer i?

To be more precise, let M be a finitely generated R-module. We show that if $M \in \mathcal{S}$ or $R/\mathfrak{a} \in \mathcal{S}$, then $H^i_{\mathfrak{a}}(M) \in \mathcal{S}$ for all $i \geq 0$. We prove that if $D(\mathfrak{a}) \subseteq \mathrm{Supp}(\mathcal{S})$, then $H^i_{\mathfrak{a}}(M) \in \mathcal{S}$ for all $i > \mathcal{S} - \dim M$. Let n be a fixed non-negative integer. We show that if M is an R-module such that $\mathrm{Supp}_R(\mathrm{Ext}^j_R(R/\mathfrak{a}, M)) \subseteq \mathrm{Supp}(\mathcal{S})$ for all $j \leq n$, then $\mathrm{Supp}_R(H^j_{\mathfrak{a}}(M)) \subseteq \mathrm{Supp}(\mathcal{S})$ for all $j \leq n$.

2. The main results

We start this section by a definition on Serre subcategory of modules due to M. Aghapournahr and L. Melkersson [AM].

Definition 2.1. Let $\mathfrak{a}$ be an ideal of R, let M be an R-module and let $(0 :_M \mathfrak{a}) = \{x \in M | \mathfrak{a}x = 0\}$. We say that M satisfy $C_{\mathfrak{a}}$ condition on $\mathcal{S}$ whenever the following condition holds:

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If $\Gamma_{\mathfrak{a}}(M) = M$ *and* $(0 :_M \mathfrak{a}) \in \mathcal{S}$*, then* $M \in \mathcal{S}$*.*

From [AM], $\mathcal{S}$ is said to satisfy $C_{\mathfrak{a}}$ condition whenever every R-module satisfies $C_{\mathfrak{a}}$ condition on $\mathcal{S}$.

Example 2.2. The class of artinian modules satisfy the condition $C_{\mathfrak{a}}$ for every ideal $\mathfrak{a}$ of R. But the class of noetherian modules $\mathcal{N}$ over a non-artinian local ring $(R, \mathfrak{m})$ does not satisfy $C_{\mathfrak{m}}$ condition, because the injective envelope $E(R/\mathfrak{m})$ of $R/\mathfrak{m}$, does not satisfy $C_{\mathfrak{m}}$ condition on $\mathcal{N}$.

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Theorem 2.3. *Let* $\mathfrak{a}$ *be an ideal of* R*, let* $\mathcal{S}$ *be a Serre subcategory satisfying* $C_{\mathfrak{a}}$ *condition and let* M *be a finitely generated* R*-module such that* $M \in \mathcal{S}$*. Then* $H^i_{\mathfrak{a}}(M) \in \mathcal{S}$ *for all* $i \geq 0$*.*

Proof. We proceed the assertion by induction on i. Let $i = 0$. As $H^0_{\mathfrak{a}}(M) = \Gamma_{\mathfrak{a}}(M)$ is a submodule of $M \in \mathcal{S}$ and $\mathcal{S}$ is closed under taking submodules, we deduce that $\Gamma_{\mathfrak{a}}(M) \in \mathcal{S}$. Now suppose, inductively that $i > 0$ and the result has been proved for all values smaller than i. By the basic properties of local cohomology, for each $j > 0$, there is an isomorphism $H^j_{\mathfrak{a}}(M) \cong H^j_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M))$; moreover, since $\mathcal{S}$ is closed under quotient modules, $M/\Gamma_{\mathfrak{a}}(M) \in \mathcal{S}$. Thus by replacing M by $M/\Gamma_{\mathfrak{a}}(M)$, we may assume that $\Gamma_{\mathfrak{a}}(M) = 0$. In this case, we assert that $\mathfrak{a} \nsubseteq \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}(M)$; otherwise there exists $\mathfrak{q} \in \mathrm{Ass}(M)$ such that $\mathfrak{a} \subseteq q$. Since $\mathfrak{q} \in \mathrm{Ass}(M)$, there exists a non-zero element $m \in M$ such that $\mathfrak{q} = \mathrm{Ann}(m)$ and so $\mathfrak{a}m \subseteq \mathfrak{q}m = 0$. But this fact implies that $m \in \Gamma_{\mathfrak{a}}(M)$ and so $\Gamma_{\mathfrak{a}}(M) \neq 0$ which is a contradiction. Since M is a finitely generated, $\mathrm{Ass}(M)$ is finite; and hence, by the Prime Avoidance Theorem, $\mathfrak{a} \nsubseteq \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}$. Thus $\mathfrak{a}$ contains a non-zero divisor x on M which gives rise an exact sequence of modules $0 \to M \xrightarrow{x} M \to M/xM \to 0$. Applying the functor $H^i_{\mathfrak{a}}(.)$, there exists a long exact sequence of modules

$$\cdots \to H^{i-1}_{\mathfrak{a}}(M/xM) \to H^i_{\mathfrak{a}}(M) \xrightarrow{x} H^i_{\mathfrak{a}}(M) \to \dots$$

which yields an exact sequence of modules $H^{i-1}_{\mathfrak{a}}(M/xM) \to (0 :_{H^i_{\mathfrak{a}}(M)} x) \to 0$. We notice that since $M \in \mathcal{S}$ and $\mathcal{S}$ is closed under taking quotients modules, $M/xM \in \mathcal{S}$. Now, the induction hypothesis implies that $H^{i-1}_{\mathfrak{a}}(M/xM) \in \mathcal{S}$, and then the quotients module $(0 :_{H^i_{\mathfrak{a}}(M)} x)$ lies in $\mathcal{S}$. On the other hand $(0 :_{H^i_{\mathfrak{a}}(M)} \mathfrak{a}) \subset (0 :_{H^i_{\mathfrak{a}}(M)} x) \in \mathcal{S}$ and since $\mathcal{S}$ is closed under taking submodules, $(0 :_{H^i_{\mathfrak{a}}(M)} \mathfrak{a}) \in \mathcal{S}$. Furthermore, by the basic properties of local cohomology, we have $\Gamma_{\mathfrak{a}}(H^i_{\mathfrak{a}}(M)) = H^i_{\mathfrak{a}}(M)$. Now, since $H^i_{\mathfrak{a}}(M)$ satisfies $C_{\mathfrak{a}}$ condition on S, $H^i_{\mathfrak{a}}(M) \in \mathcal{S}$. □

Proposition 2.4. *Let* $(R, \mathfrak{m})$ *be a local ring, let* $\mathfrak{a}$ *be an ideal of* R *and let* $\mathcal{S}$ *be a Serre subcategory satisfying* $C_{\mathfrak{a}}$ *condition. If* n *is a fixed non-negative integer such that* $H^i_{\mathfrak{a}}(M) \in \mathcal{S}$ *for all* $i < n$*, then* $\Gamma_{\mathfrak{m}}(H^n_{\mathfrak{a}}(M)) \in \mathcal{S}$*.*

Proof. It follows from [AT2, Corollary 2.9] that $\mathrm{Hom}_R(R/\mathfrak{a}, H^n_{\mathfrak{a}}(M)) \in \mathcal{S}$. On the other hand, there is the following isomorphisms and equalities

$$(0 :_{\Gamma_{\mathfrak{m}}(H^n_{\mathfrak{a}}(M))} \mathfrak{a}) = \Gamma_{\mathfrak{m}}(0 :_{H^n_{\mathfrak{a}}(M)} \mathfrak{a}) \cong \Gamma_{\mathfrak{m}}(\mathrm{Hom}_R(R/\mathfrak{a}, H^n_{\mathfrak{a}}(M))).$$

As $\mathcal{S}$ is closed under taking submodules, $\Gamma_{\mathfrak{m}}(\mathrm{Hom}_R(R/\mathfrak{a}, H^n_{\mathfrak{a}}(M))) \in \mathcal{S}$. Therefore the preceding isomorphism implies that $(0 :_{\Gamma_{\mathfrak{m}}(H^n_{\mathfrak{a}}(M))} \mathfrak{a}) \in \mathcal{S}$. Moreover, it is clear to see that $\Gamma_{\mathfrak{a}}(\Gamma_{\mathfrak{m}}(H^n_{\mathfrak{a}}(M))) = \Gamma_{\mathfrak{m}}(H^n_{\mathfrak{a}}(M))$. Lastly, since $\Gamma_{\mathfrak{m}}(H^n_{\mathfrak{a}}(M))$ satisfies $C_{\mathfrak{a}}$ condition on $\mathcal{S}$, we conclude that $\Gamma_{\mathfrak{m}}(H^n_{\mathfrak{a}}(M)) \in \mathcal{S}$. □

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Following [BS], the *ideal transform functor with respect to an ideal* $\mathfrak{a}$ *of* R, denoted by $D_{\mathfrak{a}}(.) = \varinjlim_{n\in\mathbb{N}} \mathrm{Hom}_R(\mathfrak{a}^n, .)$ is a functor from the category of all R-modules and R-homomorphisms $\mathcal{C}(R)$ to itself.

We now have the following proposition.

Proposition 2.5. *Let* $\mathfrak{a}$ *be an ideal of* R*, let* $\mathcal{S}$ *be a Serre subcategory satisfying* $C_{\mathfrak{a}}$ *condition and let* M *be an* R*-module such that* $\mathrm{Ext}^i_R(R/\mathfrak{a}, M) \in \mathcal{S}$ *for* $i = 0, 1$*. Then* $H^i_{\mathfrak{a}}(M) \in \mathcal{S}$ *for* $i = 0, 1$.

Proof. It is clear to see that $(0 :_{\Gamma_{\mathfrak{a}}(M)} \mathfrak{a}) = (0 :_M \mathfrak{a}) \cong \mathrm{Hom}_R(R/\mathfrak{a}, M) \in \mathcal{S}$, and moreover $\Gamma_{\mathfrak{a}}(\Gamma_{\mathfrak{a}}(M) = \Gamma_{\mathfrak{a}}(M)$. Now, since by the assumption $\Gamma_{\mathfrak{a}}(M)$ satisfies $C_{\mathfrak{a}}$ condition on $\mathcal{S}$, $\Gamma_{\mathfrak{a}}(M) \in \mathcal{S}$. We now prove the assertion for $i = 1$. By the definition of Ext, for each $j \geq 0$, the module $\mathrm{Ext}^j_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M))$ is a quotient of some submodules of finite direct sums of $\Gamma_{\mathfrak{a}}(M)$ and since S is closed under taking submodules, quotients and extensions, $\mathrm{Ext}^j_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)) \in \mathcal{S}$ for all $j \geq 0$. From this, if we apply the functor $\mathrm{Hom}_R(R/\mathfrak{a}, .)$ to the exact sequence $0 \to \Gamma_{\mathfrak{a}}(M) \to M \to M/\Gamma_{\mathfrak{a}}(M) \to 0$, then we deduce that $\mathrm{Ext}^i_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M)) \in \mathcal{S}$ for $i = 0, 1$. On the other hand, there is an isomorphism $H^1_{\mathfrak{a}}(M) \cong H^1_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M))$ and so replacing M by $M/\Gamma_{\mathfrak{a}}(M)$, we may assume that $\Gamma_{\mathfrak{a}}(M) = 0$. Hence, by the basic properties of local cohomology, we have the following exact sequence

$$0 \to M \to D_{\mathfrak{a}}(M) \to H^1_{\mathfrak{a}}(M) \to 0.$$

Application of the functor $\mathrm{Hom}_R(R/\mathfrak{a}, -)$ induces the following exact sequence

$$\cdots \to \mathrm{Hom}_R(R/\mathfrak{a}, D_{\mathfrak{a}}(M)) \to \mathrm{Hom}_R(R/\mathfrak{a}, H^1_{\mathfrak{a}}(M)) \to \mathrm{Ext}^1_R(R/\mathfrak{a}, M) \to \ldots.$$

We note that $\mathrm{Hom}_R(R/\mathfrak{a}, D_{\mathfrak{a}}(M)) = 0$ and so the assumption implies that $\mathrm{Hom}_R(R/\mathfrak{a}, H^1_{\mathfrak{a}}(M)) \in \mathcal{S}$. Now since $\Gamma_{\mathfrak{a}}(H^1_{\mathfrak{a}}(M)) = H^1_{\mathfrak{a}}(M)$ and $H^1_{\mathfrak{a}}(M)$ satisfies $C_{\mathfrak{a}}$ condition on $\mathcal{S}$, $H^1_{\mathfrak{a}}(M) \in \mathcal{S}$. □

Corollary 2.6. *Let* $x \in R$ *and let* $\mathcal{S}$ *satisfies* C_{xR} *condition. If* $M \in \mathcal{S}$*, then* $M_x \in \mathcal{S}$.

Proof. It is clear to see that $\mathrm{Ext}^i_R(R/xR, M) \in \mathcal{S}$ and so in view of the above proposition $H^i_{xR}(M) \in \mathcal{S}$ for $i = 0, 1$. Now the result follows by the following exact sequence

$$0 \to \Gamma_{xR}(M) \to M \to M_x \to H^1_{xR}(M) \to 0.$$

□

Definition 2.7. For any Serre subcategory $\mathcal{S}$ we define the *support of* $\mathcal{S}$ as follows:

$$\mathrm{Supp}(\mathcal{S}) = \{\mathfrak{p} \in \mathrm{Spec}R | R/\mathfrak{p} \in \mathcal{S}\}.$$

For an R-module M we define $\mathrm{Supp}_R(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) | M_{\mathfrak{p}} \neq 0\}$. We also define $\mathcal{S}$*-Support of* M as follows:

$$\mathcal{S} - \mathrm{Supp}_R(M) = \mathrm{Supp}_R(M) \setminus \mathrm{Supp}(\mathcal{S}).$$

Following [AT1], we define *Krull dimension of* M *with respect to* $\mathcal{S}$, denoted by $\mathcal{S} - \dim(M)$ as:

$$\mathcal{S} - \dim(M) = \sup\{\mathrm{ht}_M(\mathfrak{p}) | \mathfrak{p} \in \mathcal{S} - \mathrm{Supp}_R(M)\}$$

where $\mathrm{ht}_M(\mathfrak{p}) = \sup\{n | \ \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{p}$ is a chain of prime ideals in $\mathrm{Supp}_R(M)\}$.

For an ideal $\mathfrak{a}$ of R, we define $D(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}(R) | \mathfrak{a} \not\subseteq \mathfrak{p}\}$. This set is an open subset of $\mathrm{Spec}(R)$ with respect to Zariski topology.

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Theorem 2.8. *Let $\mathfrak{a}$ be an ideal of R, let $\mathcal{S}$ be a Serre subcategory satisfying $C_{\mathfrak{a}}$ condition and that $D(\mathfrak{a}) \subseteq \mathrm{Supp}(\mathcal{S})$. If M is a finitely generated R-module, then $H^i_{\mathfrak{a}}(M) \in \mathcal{S}$ for all $i > \mathcal{S} - \dim M$.*

Proof. We proceed the claim by induction on $n = \mathcal{S} - \dim M$. Let $n = 0$. For every $\mathfrak{p} \in \mathcal{S} - \mathrm{Supp}_R(M)$, we have $\mathrm{ht}\mathfrak{p} = 0$ and so $\mathfrak{p} \in \mathrm{Ass}(M)$. In this case by Definition 2.7, $\mathfrak{p} \notin \mathrm{Supp}(\mathcal{S})$. Now, using the assumption $D(\mathfrak{a}) \subseteq \mathrm{Supp}(\mathcal{S})$, we deduce that $\mathfrak{p} \notin D(\mathfrak{a})$ and so $\mathfrak{a} \subseteq \mathfrak{p}$. As $\mathfrak{p} \in \mathrm{Ass}(M)$, there exists some $m \in M$ such that $\mathfrak{p} = \mathrm{Ann}(m)$ and so $\mathfrak{a}m = 0$ which forces $m \in \Gamma_{\mathfrak{a}}(M)$. Therefore $\mathfrak{p} \in \mathrm{Ass}(\Gamma_{\mathfrak{a}}(M))$. We note that $\mathrm{Ass}(M) = \mathrm{Ass}(\Gamma_{\mathfrak{a}}(M)) \cup \mathrm{Ass}(M/\Gamma_{\mathfrak{a}}(M))$ and $\mathrm{Ass}(\Gamma_{\mathfrak{a}}(M)) \cap \mathrm{Ass}(M/\Gamma_{\mathfrak{a}}(M)) = \emptyset$. Now we assert that $\mathcal{S} - \mathrm{Supp}_R(M/\Gamma_{\mathfrak{a}}(M)) = \emptyset$. Suppose on the contrary that $\mathcal{S} - \mathrm{Supp}_R(M/\Gamma_{\mathfrak{a}}(M)) \neq \emptyset$ and then there exists some $\mathfrak{q} \in \mathcal{S} - \mathrm{Supp}_R(M/\Gamma_{\mathfrak{a}}(M))$. As $M/\Gamma_{\mathfrak{a}}(M)$ is finitely generated, $(0 :_{M/\Gamma_{\mathfrak{a}}(M)} \mathfrak{a}) \subseteq \mathfrak{q}$. Hence there exists a minimal prime ideal $\mathfrak{p}_1$ of $(0 :_{M/\Gamma_{\mathfrak{a}}(M)} \mathfrak{a})$ such that $(0 :_{M/\Gamma_{\mathfrak{a}}(M)} \mathfrak{a}) \subseteq \mathfrak{p}_1 \subseteq \mathfrak{q}$. But, it is known that the minimal prime ideals of $(0 :_{M/\Gamma_{\mathfrak{a}}(M)} \mathfrak{a})$ are contained in $\mathrm{Ass}(M/\Gamma_{\mathfrak{a}}(M))$; and hence $\mathfrak{p}_1 \in \mathrm{Ass}(M/\Gamma_{\mathfrak{a}}(M))$. Now, since $\mathfrak{p}_1 \subseteq \mathfrak{q}$, there exists an epimorphism of R-modules $R/\mathfrak{p}_1 \twoheadrightarrow R/\mathfrak{q}$. Moreover, since $\mathfrak{q} \in \mathcal{S} - \mathrm{Supp}_R(M/\Gamma_{\mathfrak{a}}(M))$, by the definition $\mathfrak{q} \notin \mathrm{Supp}(\mathcal{S})$ and so $R/\mathfrak{q} \notin \mathcal{S}$. Since $\mathcal{S}$ is closed under taking quotient modules, $R/\mathfrak{p}_1 \notin \mathcal{S}$ and then $\mathfrak{p}_1 \notin \mathrm{Supp}(\mathcal{S})$. Therefore $\mathfrak{p}_1 \in \mathcal{S} - \mathrm{Supp}_R(M)$. But the first argument implies that $\mathfrak{p}_1 \in \mathrm{Ass}(\Gamma_{\mathfrak{a}}(M))$ and this contradicts that $\mathrm{Ass}(\Gamma_{\mathfrak{a}}(M)) \cap \mathrm{Ass}(M/\Gamma_{\mathfrak{a}}(M)) = \emptyset$. Thus $\mathcal{S} - \mathrm{Supp}_R(M/\Gamma_{\mathfrak{a}}(M)) = \emptyset$ and then $M/\Gamma_{\mathfrak{a}}(M) \in \mathcal{S}$. Since $M/\Gamma_{\mathfrak{a}}(M) \in \mathcal{S}$, by the basic properties of local cohomology and using Theorem 2.3, we have $H^i_{\mathfrak{a}}(M) \cong H^i_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M)) \in \mathcal{S}$ for all $i > 0$. Now, assume that $n \geq 1$ and the result has been proved for all values smaller than n. Without loss of generality we may assume that $\Gamma_{\mathfrak{a}}(M) = 0$ and so $\mathfrak{a}$ contains a non-zerodivisor x on M. Then $x \notin \mathfrak{p}$ for every $\mathfrak{p} \in \mathrm{Ass}(M)$. As $\mathcal{S} - \mathrm{Supp}_R(M/xM) \subseteq \mathcal{S} - \mathrm{Supp}_R(M)$, the choice of x implies that $\mathcal{S} - \dim M/xM \leq n-1$ and so using the induction hypothesis $H^i_{\mathfrak{a}}(M/xM) \in \mathcal{S}$ for all $i > n-1$. Now applying the functor $H^i_{\mathfrak{a}}(.)$ to the exact sequence

$$0 \to M \xrightarrow{x} M \to M/xM \to 0$$

we get a long exact sequence

$$\cdots \to H^{i-1}_{\mathfrak{a}}(M/xM) \to H^i_{\mathfrak{a}}(M) \xrightarrow{x} H^i_{\mathfrak{a}}(M) \to \dots$$

which yields an exact sequence $H^{i-1}_{\mathfrak{a}}(M/xM) \to (0 :_{H^i_{\mathfrak{a}}(M)} x) \to 0$. As $\mathcal{S}$ is closed under taking quotient modules, $(0 :_{H^i_{\mathfrak{a}}(M)} x) \in \mathcal{S}$ for all $i > n$. Moreover, since $\mathcal{S}$ is closed under taking submodules and $(0 :_{H^i_{\mathfrak{a}}(M)} \mathfrak{a}) \subseteq (0 :_{H^i_{\mathfrak{a}}(M)} x)$, we deduce that $(0 :_{H^i_{\mathfrak{a}}(M)} \mathfrak{a}) \in \mathcal{S}$ for each $i > n$. Furthermore, $\Gamma_{\mathfrak{a}}(H^i_{\mathfrak{a}}(M)) = H^i_{\mathfrak{a}}(M)$ for each $i > n$. Now, since $H^i_{\mathfrak{a}}(M)$ satisfies $C_{\mathfrak{a}}$ condition on $\mathcal{S}$, one can conclude that $H^i_{\mathfrak{a}}(M) \in \mathcal{S}$ for all $i > n$. □

Corollary 2.9. *Let $\mathfrak{a}$ be an ideal of R, let $\mathcal{S}$ be a Serre subcategory satisfying $C_{\mathfrak{a}}$ condition and $D(\mathfrak{a}) \subseteq \mathrm{Supp}(\mathcal{S})$, and let M be an R-module. Then $\mathrm{Supp}_R(H^i_{\mathfrak{a}}(M)) \subseteq \mathrm{Supp}(\mathcal{S})$ for every $i > \mathcal{S} - \dim(M)$ where $\mathrm{Supp}_R(H^i_{\mathfrak{a}}(M)) = \{\mathfrak{p} \in \mathrm{Spec}(R) | H^i_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0\}$.*

Proof. It is easy to see that for each finitely generated submodule N of M there is $\mathcal{S} - \dim(N) \leq \mathcal{S} - \dim(M)$. Then by virtue of the previous theorem, $H^i_{\mathfrak{a}}(N) \in \mathcal{S}$ for all $i > \mathcal{S} - \dim(M)$. We now show that $\mathrm{Supp}_R(H^i_{\mathfrak{a}}(N)) \subseteq \mathrm{Supp}(\mathcal{S})$ for all $i > \mathcal{S} - \dim(M)$. Let $\mathfrak{p} \in \mathrm{Supp}_R(H^i_{\mathfrak{a}}(N))$. Then there exists a non-zero element $x \in H^i_{\mathfrak{a}}(N)$ such that $\mathrm{Ann}(x) \subseteq \mathfrak{p}$ and so there exists a natural epimorphism $xR \twoheadrightarrow R/\mathfrak{p}$. Since $H^i_{\mathfrak{a}}(N) \in \mathcal{S}$ and $\mathcal{S}$ is closed under taking submodules, $xR \in \mathcal{S}$. Subsequently, since $\mathcal{S}$ is closed under taking quotient modules, $R/\mathfrak{p} \in \mathcal{S}$ and so $\mathfrak{p} \in \mathrm{Supp}(\mathcal{S})$. For every R-module M, we have $M = \varinjlim M_i$ where M_i is taken over finitely generated submodules of M. Then $\mathrm{Supp}_R(H^i_{\mathfrak{a}}(M)) \subseteq \mathrm{Supp}_R(\coprod H^i_{\mathfrak{a}}(M_i)) \subseteq \mathrm{Supp}(\mathcal{S})$ for every $i > \mathcal{S} - \dim(M)$. □

Theorem 2.10. *Let $\mathfrak{a}$ be an ideal of R, let $\mathcal{S}$ be a Serre subcategory satisfying $C_{\mathfrak{a}}$ condition, and let $R/\mathfrak{a} \in \mathcal{S}$. Then $H^i_{\mathfrak{a}}(M) \in \mathcal{S}$ for every $i \geq 0$ and every finitely generated R-module M.*

Proof. Let M be a finitely generated R-module. We proceed by induction on i. Let $i = 0$. For every $\mathfrak{p} \in V(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}(R) | \mathfrak{a} \subseteq \mathfrak{p}\}$, there is an epimorphism $R/\mathfrak{a} \twoheadrightarrow R/\mathfrak{p}$. As $R/\mathfrak{a} \in \mathcal{S}$ and $\mathcal{S}$ is closed under quotient modules, $R/\mathfrak{p} \in \mathcal{S}$, hence $\mathrm{Supp}_R(\Gamma_{\mathfrak{a}}(M)) \subseteq V(\mathfrak{a}) \subseteq \mathrm{Supp}(\mathcal{S})$. Since $\Gamma_{\mathfrak{a}}(M)$ is noetherian, there is a filtration $0 = N_0 \subseteq N_1 \subseteq N_2 \subseteq \ldots \subset N_n = \Gamma_{\mathfrak{a}}(M)$ such that $N_i/N_{i-1} \cong R/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \mathrm{Supp}_R(\Gamma_{\mathfrak{a}}(M))$. We show that by induction on j that each N_j lies $\mathcal{S}$. For $j = 1$, since $N_1 = R/\mathfrak{p}_1$ and $\mathfrak{p}_1 \in \mathrm{Supp}(\mathcal{S})$, the assertion is clear. Suppose that $j > 1$ and the result has been proved for $j - 1$. There exists an exact sequence of modules $0 \to N_{j-1} \to N_j \to R/\mathfrak{p}_j \to 0$. By the induction hypothesis, we have $N_{j-1} \in \mathcal{S}$ and by the previous argument $R/\mathfrak{p}_j \in \mathcal{S}$. Now since $\mathcal{S}$ is closed under taking extension, we deduce that $N_j \in \mathcal{S}$. Hence $N_n = \Gamma_{\mathfrak{a}}(M) \in \mathcal{S}$. Let $i > 0$ and assume that the result has been proved for all values smaller than i. By the same way mentioned in the previous theorem we may assume that $\Gamma_{\mathfrak{a}}(M) = 0$ and so $\mathfrak{a}$ contains a non-zerodivisor x on M. Applying the functor $H^i_{\mathfrak{a}}(.)$ to the exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$, we get an exact sequence $H^{i-1}_{\mathfrak{a}}(M/xM) \to (0 :_{H^i_{\mathfrak{a}}(M)} \mathfrak{a}) \to 0$. Now using the induction hypothesis, $H^{i-1}_{\mathfrak{a}}(M/xM) \in \mathcal{S}$. As $\mathcal{S}$ is closed under taking quotient modules, $(0 :_{H^i_{\mathfrak{a}}(M)} x) \in \mathcal{S}$ for all $i > n$. Moreover, since $\mathcal{S}$ is closed under taking submodules and $(0 :_{H^i_{\mathfrak{a}}(M)} \mathfrak{a}) \subseteq (0 :_{H^i_{\mathfrak{a}}(M)} x)$, we deduce that $(0 :_{H^i_{\mathfrak{a}}(M)} \mathfrak{a}) \in \mathcal{S}$ for each $i > n$ and furthermore $\Gamma_{\mathfrak{a}}(H^i_{\mathfrak{a}}(M)) = H^i_{\mathfrak{a}}(M)$. Lastly, since $H^i_{\mathfrak{a}}(M)$ satisfies $C_{\mathfrak{a}}$ condition on $\mathcal{S}$, $H^i_{\mathfrak{a}}(M) \in \mathcal{S}$.

□

Theorem 2.11. *Let $\mathfrak{a}$ be an ideal of R; let $\mathcal{S}$ be a Serre subcategory and let n be a fixed non-negative integer. Let M be an R-module such that $\mathrm{Supp}_R(\mathrm{Ext}^j_R(R/\mathfrak{a}, M)) \subseteq \mathrm{Supp}(\mathcal{S})$ for all $j \leq n$. Then $\mathrm{Supp}_R(H^j_{\mathfrak{a}}(M)) \subseteq \mathrm{Supp}(\mathcal{S})$ for all $j \leq n$.*

Proof. We proceed by induction on j. At first assume that $j = 0$. In order to prove the assertion in this case, we show by induction on i that $\mathrm{Supp}_R(\mathrm{Hom}_R(R/\mathfrak{a}^i, M)) \subseteq \mathrm{Supp}(\mathcal{S})$ for all $i \geq 1$, where by Definition 2.7, $\mathrm{Supp}_R(\mathrm{Hom}_R(R/\mathfrak{a}^i, M)) = \{\mathfrak{p} \in \mathrm{Spec}(R) | \mathrm{Hom}_R(R/\mathfrak{a}^i, M)_{\mathfrak{p}} \neq 0\}$. The case $i = 1$ is the assumption. Suppose, inductively, that $i > 1$ and that the result have been proved for i. There exists an exact sequence of R-modules

$$0 \to \mathfrak{a}^i/\mathfrak{a}^{i+1} \to R/\mathfrak{a}^{i+1} \to R/\mathfrak{a}^i \to 0 \quad (\dagger).$$

Applying the functor $\mathrm{Hom}_R(., M)$ to the exact sequence and using the induction hypothesis, it suffices to show that

$$\mathrm{Supp}_R(\mathrm{Hom}_R(\mathfrak{a}^i/\mathfrak{a}^{i+1}, M)) \subseteq \mathrm{Supp}(\mathcal{S}).$$

Since R is noetherian, $\mathfrak{a}^i/\mathfrak{a}^{i+1}$ is a finitely generated $R/\mathfrak{a}$-module and so there exist some elements $a_1 + \mathfrak{a}^{i+1}, \ldots, a_t + \mathfrak{a}^{i+1} \in \mathfrak{a}^i/\mathfrak{a}^{i+1}$ such that $\mathfrak{a}^i/\mathfrak{a}^{i+1} =< a_1 + \mathfrak{a}^{i+1}, \ldots, a_t + \mathfrak{a}^{i+1} > R/\mathfrak{a}$. We now define a homomorphism of $R/\mathfrak{a}$-module, $\varphi : (R/\mathfrak{a})^t \to \mathfrak{a}^i/\mathfrak{a}^{i+1}$ by $\varphi(r_1 + \mathfrak{a}, \ldots, r_t + \mathfrak{a}) = r_1 a_1 + \cdots + r_t a_t + \mathfrak{a}^{i+1}$ for every $(r_1 + \mathfrak{a}, \ldots, r_t + \mathfrak{a}) \in (R/\mathfrak{a})^t$. One can see at once that φ is

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an epimorphism and if we consider $X = \ker\varphi$, then we have the following exact sequence of $R/\mathfrak{a}$-modules

$$0 \to X \to (R/\mathfrak{a})^t \to \mathfrak{a}^i/\mathfrak{a}^{i+1} \to 0 \quad (\ddagger).$$

Applying the functor $\mathrm{Hom}_R(-, M)$ to $(\ddagger)$, we conclude that $\mathrm{Supp}_R(\mathrm{Hom}_R(\mathfrak{a}^i/\mathfrak{a}^{i+1}, M)) \subseteq \mathrm{Supp}_R(\mathrm{Hom}_R(R/\mathfrak{a}, M)) \subseteq \mathrm{Supp}(\mathcal{S})$. Let $j > 0$ and assume that the result has been proved for all values smaller than $j < n$. We prove again by induction on i that $\mathrm{Supp}_R(\mathrm{Ext}^j_R(R/\mathfrak{a}^i, M) \subseteq \mathrm{Supp}(\mathcal{S})$. The case $i = 1$ is clear by the assumption. Assume that the result is true for i and so we prove it for $i+1$. Applying the functor $\mathrm{Ext}^j_R(., M)$ to $(\dagger)$ and using the induction hypothesis on i, we have to prove that $\mathrm{Supp}_R(\mathrm{Ext}^j_R(\mathfrak{a}^i/\mathfrak{a}^{i+1}, M)) \subseteq \mathcal{S}$. Application of the same functor to $(\ddagger)$ gives rise the following exact sequence

$$\mathrm{Ext}^{j-1}_R(X, M) \to \mathrm{Ext}^j_R(\mathfrak{a}^i/\mathfrak{a}^{i+1}, M) \to (\mathrm{Ext}^j_R(R/\mathfrak{a}, M))^t.$$

Using the assumption on i, it suffices to prove that $\mathrm{Supp}_R(\mathrm{Ext}^{j-1}_R(X, M)) \subseteq \mathrm{Supp}(\mathcal{S})$. As R is noetherian, X is a finitely generated $R/\mathfrak{a}$-module and so by applying an analogous argument of $\mathfrak{a}^i/\mathfrak{a}^{i+1}$, we have a non-negative integer t_1 and an exact sequence of R-modules

$$0 \to X_1 \to (R/\mathfrak{a})^{t_1} \to X \to 0 \quad (\dagger_1).$$

Continuing this way for $i \geq 2$, we obtain $R/\mathfrak{a}$-module X_i, non-negative integer t_i and the exact sequence

$$0 \to X_i \to (R/\mathfrak{a})^{t_i} \to X_{i-1} \to 0 \quad (\dagger_i).$$

Application of the functor $\mathrm{Hom}_R(., M)$ to $(\ddagger_1)$ induces the following exact sequence of R-modules

$$\mathrm{Ext}^{j-2}_R(X_1, M) \to \mathrm{Ext}^{j-1}_R(X, M) \to \mathrm{Ext}^{j-1}_R(R/\mathfrak{a}^{t_1}, M).$$

Using the induction hypothesis on j, we get $\mathrm{Supp}(\mathrm{Ext}^{j-1}_R(R/\mathfrak{a}^{t_1}, M)) \subseteq \mathrm{Supp}(\mathcal{S})$ and so in order to prove that $\mathrm{Supp}_R(\mathrm{Ext}^{j-1}_R(X, M)) \subseteq \mathrm{Supp}(\mathcal{S})$, it suffices to show that $\mathrm{Supp}_R(\mathrm{Ext}^{j-2}_R(X_1, M)) \subseteq \mathrm{Supp}(\mathcal{S})$. Iterating this way on $\ddagger_2, \ldots \ddagger_{j-1}$, it suffices to show that $\mathrm{Supp}_R(\mathrm{Hom}_R(X_{j-1}, M)) \subseteq \mathrm{Supp}(\mathcal{S})$. Finally applying the functor $\mathrm{Hom}_R(., M)$ to $(\ddagger_j)$, we get the exact sequence $0 \to \mathrm{Hom}_R(X_{j-1}, M) \to \mathrm{Hom}_R(R/\mathfrak{a}, M)^{t_j}$ which follows the result.

□

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