ON DOUBLE ALMOST LACUNARY SUMMABLE SEQUENCES OF ORDER θ DEFINED VIA ORLICZ FUNCTION

RABIA SAVAS

Abstract. The primary aim of this article is to present the double almost lacunary strong P-convergence of order θ via Orlicz function and study some characteristics of the resulting sequence spaces.

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1. First Section

Kuttner [3] examined spaces of strongly summable sequences and later on it was discussed by Maddox [5], and others. Also Maddox [4] studied the set of sequences which are strongly Cesàro summable with respect to a modulus as a generalization of the notion of strongly Cesàro summable sequences. Further, Connor [1] considered the strong A-summability with regard to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and examined some connections between strong A-summability and strong A-summability with respect to a modulus.

We recall [7] in that $y = (y_{r,s})$ is said to be convergent in Pringsheim sense to some complex number ϖ if for every $\epsilon > 0$ there exists $n_0 = n_0 \in \mathbf{N}$ such that

$$|y_{r,s} - \varpi| < \epsilon \text{ if } min\{r, s\} > n_0.$$

We shall describe such an y more shortly as "**P-convergent**".

Furthermore, Moricz and Rhoades [6] presented P-almost convergent sequences as below:

Definition 1.1. *A double sequence $y = (y_{r,s})$ of real numbers is called almost P-convergent to a limit ϖ if*

$$P - \lim_{p,q\to\infty} \sup_{t,z\geq 0} \frac{1}{pq} \sum_{r=t}^{t+p-1} \sum_{s=z}^{z+q-1} |y_{r,s} - \varpi| = 0$$

that is the average value of $(y_{r,s})$ taken over any rectangle $\{(r, s) : t \leq r \leq t + p - 1, z \leq s \leq z + q - 1\}$ tends to ϖ as both p and q tend to ∞, and this P-convergences is uniform in t and z. We denote the set of sequence with this property by $[\hat{c}^2]$.

Later on the following definition was given by Savaş and Patterson [8].

Definition 1.2. *The double sequence $\Phi_{\xi,\eta} = \{(r_\xi, s_\eta)\}$ is called* **double lacunary** *if there exist two increasing of integers such that*

$$r_0 = 0, \gamma_\xi = r_\xi - r_{\xi-1} \to \infty \quad as \quad \xi \to \infty$$

and

$$s_0 = 0, \overline{\gamma}_\eta = s_\eta - s_{\eta-1} \to \infty \quad as \quad \eta \to \infty.$$

Also, $r_{\xi,\eta} = r_\xi s_\eta$, $\gamma_{\xi,\eta} = \gamma_\xi \overline{\gamma}_\eta$, $\Phi_{\xi,\eta}$ is determine by $J_{\xi,\eta} = \{(r,s) : r_{\xi-1} < r \leq r_\xi \ \& \ s_{\eta-1} < s \leq s_\eta\}$, $\zeta_\xi = \frac{r_\xi}{r_{\xi-1}}$, $\overline{\zeta}_\eta = \frac{s_\eta}{s_{\eta-1}}$, and $\zeta_{\xi,\eta} = \zeta_\xi \overline{\zeta}_\eta$. We will denote the set of all double lacunary sequences by $\mathbf{N}_{\Phi_{\xi,\eta}}$.
Additionally, Savas [9] presented some results by using double sequence and Orlicz functions.

Recall in [2] that an Orlicz function $\mathbf{F}$ is continuous, convex, nondecreasing function such that $\mathbf{F}(0) = 0$ and $\mathbf{F}(y) > 0$ for $y > 0$.

2. Some New Definitions and Notations

In this section, we will present some new definitions and notations that will be needed in main result.

Definition 2.1. *Let $\mathbf{F}$ be an Orlicz function and $\theta \in (0,1]$ be any real number and $\tau = (\tau_{r,s})$ be any factorable double sequence of strictly positive real numbers, we consider the following sequence space:*

$$[\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}, \tau]^\theta = \{y = (y_{r,s}) : P - \lim_{\xi,\eta} \frac{1}{\gamma^\theta_{\xi,\eta}} \sum_{(r,s)\in J_{\xi,\eta}} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z} - \varpi|}{\rho}\right)\right]^{\tau_{r,s}} = 0,$$
$$\textit{uniformly in } t \textit{ and } z \textit{ for some } \varpi \textit{ and } \rho > 0\}.$$

If y is in $[\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}, \tau]^\theta$, we say that y is almost lacunary strongly P-convergent of order θ with regard to the Orlicz function $\mathbf{F}$. If we take $\varpi = 0$, we have $[\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}, \tau]^\theta = [\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}, \tau]^\theta_0$.

Also note if $\mathbf{F}(y) = y$, $\tau_{r,s} = 1$ for all r and s, then $[\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}, \tau]^\theta = [\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}]^\theta$ which is presented as follows:

$$[\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}] = \{y : \text{ for some } \varpi, P - \lim_{\xi,\eta} \frac{1}{\gamma^\theta_{\xi,\eta}} \sum_{(r,s)\in J_{\xi,\eta}} |y_{r+t,s+z} - \varpi| = 0,$$
$$\text{uniformly in } t \text{ and } z\}.$$

If $\tau_{r,s} = 1$ for all r and s, then $[\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}, \tau]^\theta = [\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}]^\theta$ which is presented as follows:

$$[\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}]^\theta = \{y = (y_{r,s}) : P - \lim_{\xi,\eta} \frac{1}{\gamma^\theta_{\xi,\eta}} \sum_{(r,s)\in J_{\xi,\eta}} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z} - \varpi|}{\rho}\right)\right] = 0,$$
$$\text{uniformly in } t \text{ and } z \text{ for some } \varpi \text{ and } \rho > 0\}.$$

Note that if $\tau_{r,s} = 1$, $\theta = 1$ for all r and s, then $[\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}, \tau]^\theta = [\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}]$ which is given as follows:

$$[\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}] = \{y = (y_{r,s}) : P - \lim_{\xi,\eta} \frac{1}{\gamma_{\xi,\eta}} \sum_{(r,s)\in J_{\xi,\eta}} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z} - \varpi|}{\rho}\right)\right] = 0,$$
$$\text{uniformly in } t \text{ and } z \text{ for some } \varpi \text{ and } \rho > 0\}.$$

Almost P-convergent of order θ double sequences to Orlicz function is considered as follows:

Definition 2.2. *The double sequence $y = (y_{r,s})$ of real numbers is called almost P-convergent of order θ to a limit ϖ with regard to the Orlicz function $\mathbf{F}$ if*

$$P - \lim_{u,w} \frac{1}{(uw)^\theta} \sum_{r,s=1,1}^{u,w} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z} - \varpi|}{\rho}\right)\right]^{\tau_{r,s}} = 0,$$
$$\textit{uniformly in } t \textit{ and } z \textit{ for some } \varpi \textit{ and } \rho > 0.$$

Almost P-convergent of order θ double sequences with regard to Orlicz function can be shown by a standard argument that $[\hat{c}^2, \mathbf{F}, \tau]^\theta$. An Orlicz function $\mathbf{F}$ is said to fulfill Δ_2-condition for all values of $\widetilde{u}$, if there exists a constant $\widehat{K} > 0$ such that $\mathbf{F}(2\widetilde{u}) \leq \widehat{K}\mathbf{F}(\widetilde{u})$, $\widetilde{u} \geq 0$.

3. Main Results

We first present the following lemma for the next theorem.

Lemma 3.1. *Let $\mathbf{F}$ be an Orlicz function which satisfies Δ_2- condition and let $0 < \widetilde{\delta} < 1$. Then for each $y \geq \widetilde{\delta}$ we have $\mathbf{F}(y) < \widehat{K}\widetilde{\delta}^{-1}\mathbf{F}(2)$ for some constant $\widehat{K} > 0$.*

Theorem 3.2. *For any Orlicz function $\mathbf{F}$ which satisfies Δ_2 condition, we have $[\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}]^\theta \subseteq [\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}]^\theta$.*

Proof. Let $y \in \left[\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}\right]^\theta$ so that

$$A_{\xi,\eta} = \left\{ y: \text{ for some } \varpi, P - \frac{1}{\gamma^\theta_{\xi,\eta}} \lim_{\xi,\eta} \sum_{(r,s)\in J_{\xi,\eta}} |y_{r+t,s+z} - \varpi| = 0 \right\}.$$

Let $\epsilon > 0$ and choose $\widetilde{\delta}$ with $0 < \widetilde{\delta} < 1$ such that $\mathbf{F}(\varsigma) < \epsilon$ for every ς with $0 \leq \varsigma \leq \widetilde{\delta}$. We obtain the following

$$\begin{aligned}
& \frac{1}{\gamma^\theta_{\xi,\eta}} \sum_{(r,s)\in J_{\xi,\eta}} \mathbf{F}(|y_{r+t,s+z} - \varpi|) \\
= & \frac{1}{\gamma^\theta_{\xi,\eta}} \sum_{(r,s)\in J_{\xi,\eta} \& |y_{r+t,s+z}-\varpi| \leq \widetilde{\delta}} \mathbf{F}(|y_{r+t,s+z} - \varpi|) + \frac{1}{\gamma^\theta_{\xi,\eta}} \sum_{(r,s)\in J_{\xi,\eta} \& |y_{r+t,s+z}-\varpi| > \widetilde{\delta}} \mathbf{F}(|y_{r+t,s+z} - \varpi|) \\
\leq & \frac{1}{\gamma^\theta_{\xi,\eta}} \gamma^\theta_{\xi,\eta}\epsilon + \frac{1}{\gamma^\theta_{\xi,\eta}} \sum_{(r,s)\in J_{\xi,\eta} \& |y_{r+t,s+z}-\varpi| > \widetilde{\delta}} \mathbf{F}(|y_{r+t,s+z} - \varpi|) \\
< & \frac{1}{\gamma^\theta_{\xi,\eta}} (\gamma^\theta_{\xi,\eta}\varepsilon) + \frac{1}{\gamma^\theta_{\xi,\eta}} \widehat{K}\widetilde{\delta}^{-1}\mathbf{F}(2)\gamma_{\xi,\eta}A_{\xi,\eta}.
\end{aligned}$$

Therefore, as ξ and η go to infinity, for each t and z, it is obvious $y \in [\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}]^\theta$. □

In the next theorems we shall interest the connection between $[\hat{c}^2, \mathbf{F}, \tau]^\theta$ and $[\widehat{N}_{\Phi_{\xi,\eta}}, \mathbf{F}, \tau]^\theta$.

Theorem 3.3. *Let $\Phi_{\xi,\eta} = \{(r_\xi, s_\eta)\}$ be a double lacunary sequence, $\mathbf{F}$ is Orlicz function and $\theta \in (0, 1]$. In order for $[\hat{c}^2, \mathbf{F}, \tau]^\theta \subset [\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}, \tau]^\theta$ it is sufficient that $\liminf_\xi \zeta_\xi > 1$ and $\liminf_\eta \overline{\zeta}_\eta > 1$.*

Proof. It is sufficient to show that $[\hat{c}^2, \mathbf{F}, \tau]^\theta_0 \subset [\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}, \tau]^\theta_0$. The general inclusion follows by linearity. Suppose $\liminf_\xi \zeta_\xi > 1$ and $\liminf_\eta \overline{\zeta}_\eta > 1$, then there exists $\widetilde{\delta} > 0$ such that $\zeta_\xi > 1 + \widetilde{\delta}$ and $\overline{\zeta}_\eta > 1 + \widetilde{\delta}$.

This implies $\frac{\gamma_\xi^\theta}{r_\xi^\theta} \geq \frac{\widetilde{\delta}^\theta}{(1+\widetilde{\delta})^\theta}$ and $\frac{\overline{\gamma}_\eta^\theta}{s_\eta^\theta} \geq \frac{\widetilde{\delta}^\theta}{(1+\widetilde{\delta})^\theta}$. Then for $y \in [\hat{c}^2, \mathbf{F}, \tau]_0^\theta$, we can write for each t and z

$$\begin{aligned}
A_{\xi,\eta} &= \frac{1}{\gamma_{\xi,\eta}^\theta} \sum_{(r,s)\in J_{\xi,\eta}} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}} \\
&= \frac{1}{\gamma_{\xi,\eta}^\theta} \sum_{r=1}^{r_\xi}\sum_{s=1}^{s_\eta} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}} \\
&- \frac{1}{\gamma_{\xi,\eta}^\theta} \sum_{r=1}^{r_{\xi-1}}\sum_{s=1}^{s_{\eta-1}} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}} \\
&- \frac{1}{\gamma_{\xi,\eta}^\theta} \sum_{r=r_{\xi-1}+1}^{r_\xi}\sum_{s=1}^{s_{\eta-1}} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}} \\
&- \frac{1}{\gamma_{\xi,\eta}^\theta} \sum_{s=s_{\eta-1}+1}^{s_\eta}\sum_{r=1}^{r_{\xi-1}} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}} \\
&= \frac{r_\xi^\theta s_\eta^\theta}{\gamma_{\xi,\eta}^\theta}\left(\frac{1}{r_\xi s_\eta} \sum_{r=1}^{r_\xi}\sum_{s=1}^{s_\eta} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}}\right) \\
&- \frac{r_{\xi-1}^\theta s_{\eta-1}^\theta}{\gamma_{\xi,\eta}^\theta}\left(\frac{1}{r_{\xi-1}^\theta s_{\eta-1}^\theta} \sum_{r=1}^{r_{\xi-1}}\sum_{s=1}^{s_{\eta-1}} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}}\right) \\
&- \frac{1}{\gamma_\xi^\theta} \sum_{r=r_{\xi-1}+1}^{r_\xi} \frac{s_\eta^\theta - 1}{\overline{\gamma}_\eta^\theta} \frac{1}{s_\eta^\theta - 1} \sum_{s=1}^{s_\eta - 1} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}} \\
&- \frac{1}{\overline{\gamma}_\eta^\theta} \sum_{s=s_{\eta-1}+1}^{s_\eta} \frac{r_{\xi-1}^\theta}{\gamma_\xi^\theta} \frac{1}{r_{\xi-1}^\theta} \sum_{r=1}^{r_{\xi-1}} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}}.
\end{aligned}$$

Since $y \in [\hat{c}^2, \mathbf{F}, \tau]^\theta$ the last two terms tend to zero uniformly in t and z in the Pringsheim sense, thus for each t and z

$$\begin{aligned}
A_{\xi,\eta} &= \frac{r_\xi^\theta s_\eta^\theta}{\gamma_{\xi,\eta}^\theta}\left(\frac{1}{r_\xi^\theta s_\eta^\theta} \sum_{r=1}^{r_\xi}\sum_{s=1}^{s_\eta} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}}\right) \\
&- \frac{r_{\xi-1}^\theta s_{\eta-1}^\theta}{\gamma_{\xi,\eta}^\theta}\left(\frac{1}{r_{\xi-1}^\theta s_{\eta-1}^\theta} \sum_{r=1}^{r_{\xi-1}}\sum_{s=1}^{s_{\eta-1}} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}}\right) + o(1).
\end{aligned}$$

Since $\gamma_{\xi,\eta} = r_\xi s_\eta - r_{\xi-1}s_{\eta-1}$ we are granted for each t and z the following:

$$\frac{r_\xi^\theta s_\eta^\theta}{\gamma_{\xi,\eta}^\theta} \leq \frac{(1+\widetilde{\delta})^\theta}{(\widetilde{\delta})^\theta} \text{ and } \frac{r_{\xi-1}^\theta s_{\eta-1}^\theta}{\gamma_{\xi,\eta}^\theta} \leq \frac{1}{(\delta)^\theta}.$$

The terms

$$\frac{1}{r_\xi^\theta s_\eta^\theta} \sum_{r=1}^{r_\xi}\sum_{s=1}^{s_\eta} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}}$$

and

$$\frac{1}{r_{\xi-1}^\theta s_{\eta-1}^\theta} \sum_{r=1}^{r_{\xi-1}}\sum_{s=1}^{s_{\eta-1}} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}}$$

are both Pringsheim null sequences. Thus, $A_{\xi,\eta}$ is a Pringsheim null sequence for each t and z. Consequently, y is in $[\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}, \tau]_0^\theta$. □

Theorem 3.4. *Let $\Phi_{\xi,\eta} = \{(r_\xi, s_\eta)\}$ be a double lacunary sequence, $\mathbf{F}$ is Orlicz function and $\theta \in (0,1]$. In order for $[\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}, \tau]^\theta \subset [\hat{c}^2, \mathbf{F}, \tau]^\theta$ it is sufficient that $\limsup_\xi \frac{r_\xi}{r_{\xi-1}^\theta} < \infty$ and $\limsup_\eta \frac{s_\eta}{s_{\eta-1}^\theta} < \infty$.*

Proof. Since $\limsup_\xi \frac{r_\xi}{r_{\xi-1}^\theta} < \infty$ and $\limsup_\eta \frac{s_\eta}{s_{\eta-1}^\theta} < \infty$ there exists $\overline{H} > 0$ such that $\frac{r_\xi}{r_{\xi-1}^\theta} < \overline{H}$ and $\frac{s_\eta}{s_{\eta-1}^\theta} < \overline{H}$ for all ξ and η. Let $y \in [\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}, \tau]^\theta$ and $\epsilon > 0$. Also there exist $\xi_0 > 0$ and $\eta_0 > 0$ such that for every $\overline{u} \geq \xi_0$ and $\overline{v} \geq s_0$

$$A_{\overline{u},\overline{v}} = \frac{1}{\gamma_{\xi,\eta}^\theta} \sum_{(r,s)\in J_{\xi,\eta}} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}} < \epsilon.$$

Let $M = \max\{A_{\overline{u},\overline{v}} : 1 \leq \xi \leq \xi_0 \text{ and } 1 \leq \eta \leq \eta_0\}$, and u and w be such that $r_{\xi-1} < u \leq r_\xi$ and $s_{\eta-1} < w \leq s_\eta$. Thus we obtain the following:

$$\begin{aligned}
& \frac{1}{(uw)^\theta} \sum_{r,s=1,1}^{u,w} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}} \\
\leq \; & \frac{1}{r_{\xi-1}^\theta s_{\eta-1}^\theta} \sum_{r,s=1,1}^{r_\xi s_\eta} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}} \\
\leq \; & \frac{1}{r_{\xi-1}^\theta s_{\eta-1}^\theta} \sum_{\overline{u},\overline{v}=1,1}^{\xi,\eta} \left(\sum_{(r,s)\in I_{\overline{u},\overline{v}}} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}} \right) \\
= \; & \frac{1}{r_{\xi-1}^\theta s_{\eta-1}^\theta} \sum_{\overline{u},\overline{v}=1,1}^{\xi_0,\eta_0} \gamma_{\overline{u},\overline{v}} A_{\overline{u},\overline{v}} + \frac{1}{r_{\xi-1}^\theta s_{\eta-1}^\theta} \sum_{(\xi_0<\overline{u}\leq\xi)\cup(\eta_0<\overline{v}\leq\eta)} \gamma_{\overline{u},\overline{v}} A_{\overline{u},\overline{v}} \\
\leq \; & \frac{M}{r_{\xi-1}^\theta s_{\eta-1}^\theta} \sum_{\overline{u},\overline{v}=1,1}^{\xi_0,\eta_0} \gamma_{\overline{u},\overline{v}} + \frac{1}{r_{\xi-1}^\theta s_{\eta-1}^\theta} \sum_{(\xi_0<\overline{u}\leq\xi)\cup(\eta_0<\overline{v}\leq\eta)} \gamma_{\overline{u},\overline{v}} A_{\overline{u},\overline{v}} \\
\leq \; & \frac{M\ r_{\xi_0} s_{\eta_0} \xi_0 \eta_0}{r_{\xi-1}^\theta s_{\eta-1}^\theta} + \frac{1}{r_{\xi-1}^\theta s_{\eta-1}^\theta} \sum_{(\xi_0<\overline{u}\leq\xi)\cup(\eta_0<\overline{v}\leq\eta)} \gamma_{\overline{u},\overline{v}} A_{\overline{u},\overline{v}} \\
\leq \; & \frac{M\ r_{\xi_0} s_{\eta_0} \xi_0 \eta_0}{r_{\xi-1}^\theta s_{\eta-1}^\theta} + \left(\sup_{\overline{u}\geq\xi_0\cup\overline{v}\geq\eta_0} A_{\overline{u},\overline{v}} \right) \frac{1}{r_{\xi-1}^\theta s_{\eta-1}^\theta} \sum_{(\xi_0<\overline{u}\leq\xi)\cup(\eta_0<\overline{v}\leq\eta)} \gamma_{\overline{u},\overline{v}} \\
\leq \; & \frac{M\ r_{\xi_0} s_{\eta_0} \xi_0 \eta_0}{r_{\xi-1}^\theta s_{\eta-1}^\theta} + \epsilon \sum_{(\xi_0<\overline{u}\leq\xi)\cup(\eta_0<\overline{v}\leq\eta)} \gamma_{\overline{u},\overline{v}} \\
\leq \; & \frac{M\ r_{\xi_0} s_{\eta_0} \xi_0 \eta_0}{r_{\xi-1}^\theta s_{\eta-1}^\theta} + \epsilon \overline{H}^2.
\end{aligned}$$

Since r_ξ and s_η both approach infinity as both u and w approach infinity. Therefore

$$\frac{1}{(uw)^\theta} \sum_{r,s=1,1}^{u,w} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}} \to 0, \text{uniformly in } t \text{ and } z.$$

As a result, $y \in [\hat{c}^2, \mathbf{F}, \tau]^\theta$. □

The following theorem is a clear consequence of Theorem 3.3 and Theorem 3.4.

Theorem 3.5. *Let* $\Phi_{\xi,\eta} = \{(r_\xi, s_\eta)\}$ *be a double lacunary sequence with* $1 < \liminf_{\xi,\eta} \zeta_{\xi,\eta} \leq \limsup_{\xi,\eta} \zeta_{\xi,\eta} < \infty$*, then for any Orlicz function* $\mathbf{F}$*,* $[\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}, \tau]^\theta = [\hat{c}^2, \mathbf{F}, \tau]^\theta$.

References

[1] Connor, J., "On strong matrix summability with respect to a modulus and statistical convergence", Canadian Mathematical Bulletin, 32(2): 194-198, 1989.

[2] Krasnoselskii, M. A. and Rutisky, Y. B. "Convex function and Orlicz spaces", Groningen, Netherlands, 1961.

[3] Kuttner, B., "Note on strong summability", Journal of the London Mathematical Society, 21: 118-122, 1946.

[4] Maddox, I. J., "Sequence spaces defined by a modulus", Mathematical Proceedings of the Cambridge Philosophical Society, 100(1): 161-166, 1986.

[5] Maddox, I. J., "On strong almost convergence", Mathematical Proceedings of the Cambridge Philosophical Society, 85(2): 345-350, 1979.

[6] Moricz, F. and Rhoades, B. E. "Almost convergence of double sequences and strong regularity of Summability matrices", Mathematical Proceedings of the Cambridge Philosophical Society, 104: 283-293, 1988.

[7] Pringsheim, A. "Zur theorie der zweifach unendlichen Zahlenfolgen", Mathematische Annalen, 53: 289-321, 1900.

[8] Savaş, E. and Patterson, R. F., "Lacunary statistical convergence of multiple sequences", Applied Mathematics Letters. 19 (6): 527-534, 2006.

[9] Savaş, E. "Double almost lacunary statistical convergence of order alpha", Advances in Difference Equations, 2013:254, 2013.

Sakarya University, Department of Mathematics, Sakarya -TURKEY

Email address: rabiasavass@hotmail.com