CERTAIN CURVES ON α-PARA KENMOTSU MANIFOLDS

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Abstract. The aim of the present paper is to study biharmonic almost contact curves on three dimensional α-para Kenmotsu manifolds. Slant curves have been studied. We also consider locally ϕ-symmetric almost contact curves. Illustrative example is given.

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1. Introduction

A beautiful notion of classical differential geometry of curves is that of curves of constant slope, also called cylindrical helix. This is a curve in the Euclidean space E^3 for which the tangent vector field has a constant angle with a fixed direction called the axis. Analogous to the above concept there are notions of slant curves and in particular Legendre curves in contact structure geometry. In the study of contact manifolds, Legendre curves play an important role, e.g., a diffeomorphism of a contact manifold is a contact transformation if and only if it maps Legendre curves to Legendre curves. Legendre curves on contact manifolds have been studied by C. Baikoussis and D.E. Blair in the paper [1]. Recently the present authors have studied curves on three-dimensional quasi-Sasakian manifolds [27] and curves on some classes of Kenmotsu manifolds [28]. The Legendre property has been extended to almost contact metric manifolds [30]. Legendre curves on almost contact metric manifolds are called almost contact curves[15].

The theory of biharmonic functions is an old and rich subject. Biharmonic functions have been studied since 1862 by Maxwell and Airy to describe a mathematical model of elasticity. The theory of polyharmonic functions was developed later on. There are a few results on biharmonic curves in arbitrary Riemannian manifolds. Biharmonic curves on a surface was studied by R. Caddeo, et al. in the paer [5]. Later S. Montaldo and C. Oniciuc studied biharmonic maps between Riemannian manifolds. In the paper [12] D. Fetcu studied Biharmonic Legendre curves in Sasakian space forms. Legendre curves or almost contact curves have been studied by many authors [21], [30]. J. Welyczko studied slant curves in 3-dimensional normal almost paracontact metric manifolds in the paper [31]. The first author has also studied curves on different classes of almost contact manifolds [24], [25], [26]. Again for slant curves we refer [6],[9], [10], [16].

P. Majhi studied α-para Kenmotsu manifolds in the paper [19]. K. Srivastava and S. K. Srivastava studied a class of α-para Kenmotsu manifolds in the paper [29]. In this paper we like to study biharmonic almost contact curves, slant curves and locally ϕ-symmetric curves on three-dimensional α-para kenmotsu manifolds. The present paper is organized as follows:

After the introduction, we give some preliminaries in Section 2. In Section 3, we study biharmonic almost contact curves on three-dimensional α-para Kenmotsu manifolds. Then slant curves on three-dimensional α-para Kenmotsu manifolds have been studied in Section 4. Section 5 contains the study of locally ϕ-symmetric almost contact curves on α-para Kenmotsu manifolds of dimension three. The last section contains example.

2. Preliminaries

Let M be a $2n+1$ dimensional differentiable manifold. Let ϕ be an 1-1 tensor field, ξ a vector field and η a 1-form on M. Then (ϕ, ξ, η) is called an almost para contact structure on M if

$$\phi^2 X = X - \eta(X)\xi, \qquad \eta(\xi) = 1, \qquad \phi\xi = 0, \qquad \eta(\phi) = 0. \tag{2.1}$$

The tensor field ϕ induces an almost paracomplex structure on the distribution $\mathcal{D} = \ker\eta$, that is, the eigen distributions $\mathcal{D}^+$, $\mathcal{D}^-$ corresponding to the eigen values 1, -1 of ϕ, respectively, have equal dimension n.

M is said to be almost paracontact manifold if it is endowed with an almost paracontact structure [**?**], [11], [18], [32].

Let M be an almost paracontact manifold. M will be called an almost paracontact metric manifold if it is additionally endowed with a pseudo-Riemannian metric g of signature $(n+1, n)$ and such that

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \xi) = \eta(X) \tag{2.3}$$

for all $X, Y \in \chi(M)$.

Moreover, we can define a skew-symmetric 2-form Φ by $\Phi(X, Y) = g(X, \phi Y)$, which is called the fundamental form corresponding to the structure. Note that $\eta \wedge \Phi^n$ is up to a constant factor the Riemannian volume element of M.

On an almost paracontact manifold, one defines the (2,1)- tensor field $N^{(1)}$ by

$$N^{(1)}(X, Y) = [\phi, \phi](X, Y) - 2d\eta(X, Y)\xi,$$

where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ given by

$$[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y].$$

If $N^{(1)}$ vanishes identically, then the almost paracontact manifold(structure) is said to be normal [**?**], [18], [32]. The normality condition says that the almost paracomplex structure J defined on $M \times R$ by

$$J(X, \lambda\frac{d}{dt}) = (\phi X + \lambda\xi, \eta(X)\frac{d}{dt})$$

is integrable (paracomplex).

In the sequel, we are interested in dimension 3.

In a 3-dimensional α-para Kenmotsu manifold the following results hold [29]

$$\begin{aligned} R(X, Y)Z &= (\frac{r}{2} + 2\alpha^2)[g(Y, Z)X - g(X, Z)Y] \\ &- (\frac{r}{2} + 3\alpha^2)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]\xi \\ &+ (\frac{r}{2} + 3\alpha^2)[\eta(X)Y - \eta(Y)X]\eta(Z), \end{aligned} \tag{2.4}$$

where R is the curvature tensor of the manifold.

$$(\nabla_X \eta)Y = \alpha[g(X, Y) - \eta(X)\eta(Y)], \tag{2.5}$$

$$(\nabla_X \phi)Y = \alpha[g(\phi X, Y)\xi - \eta(Y)\phi X], \tag{2.6}$$

$$\nabla_X \xi = \alpha(X - \eta(X)\xi) \tag{2.7}$$

for all vector fields $X, Y, Z \in \chi(M)$.

A curve γ on M is called Frenet curve with respect to Levi-Civita connection on M if

$$\nabla_T T = kN, \tag{2.8}$$

$$\nabla_T N = -kT + \tau B, \tag{2.9}$$

$$\nabla_T B = -\tau N, \tag{2.10}$$

where k, τ are the curvature and torsion of the curve with respect to Levi-Civita connection and {T,N,B} is an orthonormal Frenet frame and $T = \dot{\gamma}$.

A Frenet curve γ in an almost contact metric manifold is said to be almost contact curve if it is an integral curve of the distribution $\mathcal{D} = \ker\eta$. Formally,it is also said that a Frenet curve γ in an almost contact metric manifold is an almost contact curve if and only if $\eta(\dot{\gamma}) = 0$ and $g(\dot{\gamma}, \dot{\gamma}) = 1$. For further details we refer [1], [15], [30]. It is to be mentioned that in the paper [15], curves satisfying the above properties on almost contact manifolds have been termed as almost contact curve, while Welyczko [30] has termed such curves on almost contact manifolds as Legendre curves. Henceforth by Legendre curves on almost contact manifolds we shall mean almost contact curves.

A Frenet curve γ is called a slant curve if it makes a constant angle with the Reeb vector field ξ [10]. If a unit speed curve γ on an almost contact metric manifold is slant curve, then $\eta(\dot{\gamma}) = \cos\theta$, where θ is a constant and is called slant angle. In particular, if the angle is $\frac{\pi}{2}$, the curve becomes almost contact curve. A slant curve γ is called proper if it is neither parallel nor perpendicular to the Reeb vector field ξ.

3. Biharmonic almost contact curves on three-dimensional α-para Kenmotsu manifolds with respect to Levi-Civita connection

Definition 3.1. An almost contact curve γ on a three-dimensional α-para Kenmotsu manifold is called biharmonic with respect to Levi-Civita connection ∇ if it satisfies [15]

$$\nabla_T^3 T + R(\nabla_T T, T)T = 0, \tag{3.1}$$

where $\dot{\gamma} = T$, and $\tilde{R}$ is the curvature of the manifold with respect to Levi-Civita connections.

Here we prove the following:

Theorem 3.1. *The curvature of a non-geodesic biharmonic almost contact curve on a three-dimensional α-para Kenmotsu manifold with respect to Levi-Civita connections is a non-zero constant and the torsion of the curve is given by $\tau = \pm\sqrt{\frac{r}{2} + 2\alpha^2 - k^2}$.*

Proof. Let us consider a biharmonic almost contact curve.

From Serret-Frenet formula (2.8) to (2.10), we get

$$\nabla_T^3 T = -3kk'T + (k'' - k^3 - k\tau^2)N + (2\tau k' + k\tau')B, \tag{3.2}$$

where k and τ are the curvature and torsion of the Frenet curve.

Using Serret-Frenet formula in (3.1) we get

$$\nabla_T^3 T + kR(N, T)T = 0. \tag{3.3}$$

Since we have considered Frenet frame as T, ϕT, ξ where $\phi T = -N$, so for an almost contact curve we get $\eta(T) = 0$, $\eta(N) = 0$. Using this fact and putting $X = N$, $Y = T$, $Z = T$ in (2.4) we get,

$$R(N,T)T = (\frac{r}{2} + 2\alpha^2)N. \tag{3.4}$$

From (3.2), (3.3) and (3.4) we get,
$(-3kk')T + (k'' - k^3 - k\tau^2 + k\frac{r}{2} + 2k\alpha^2)N + (2\tau k' + k\tau')B = 0.$
So we have,

$$-3kk' = 0. \tag{3.5}$$

From above we get k=constant, provided $k \neq 0$
Again,

$$k'' - k^3 - k\tau^2 + k\frac{r}{2} + 2k\alpha^2 = 0. \tag{3.6}$$

$$2\tau k' + k\tau' = 0. \tag{3.7}$$

For k=constant, k''=0.
So from (3.6) we have,

$$k(k^2 + \tau^2 - \frac{r}{2} - 2\alpha^2) = 0.$$

Since $k \neq 0$, $k^2 + \tau^2 - \frac{r}{2} - 2\alpha^2 = 0$
i.e., $\tau = \pm\sqrt{\frac{r}{2} + 2\alpha^2 - k^2}$.
This completes the proof. □

4. Slant curves on three-dimensional α-para Kenmotsu manifolds with respect to Levi-Civita connection

In this section, we study slant curves on three dimensional α-para Kenmotsu manifolds with respect to Levi-Civita connection and prove the following:

Theorem 4.1. *A proper slant curve γ on α-para Kenmotsu manifolds with respect to Levi-Civita connection is a geodesic if and only if $\alpha = 0$.*

Proof. Let us consider a proper slant curve γ on a α-para Kenmotsu manifold with respect to Levi-Civita connection.
Here $\dot{\gamma}(s) = T(s)$ is given by

$$\cos\theta(s) = g(T(s), \xi), \tag{4.1}$$

where θ is the constant slant angle.
By covariant differentiation with respect to ∇ we get from (4.1)

$$-\sin\theta.\theta^{'} = -g(\nabla_T T, \xi) - g(T, \nabla_T \xi). \tag{4.2}$$

Now using (2.7), (2.8) in (4.2) we get,

$$\begin{aligned} -\sin\theta.\theta^{'} &= -g(kN,\xi) - g(T, \alpha(T - \eta(T)\xi)) \\ &= -k\eta(N) - \alpha + (\eta(T))^2\alpha \\ &= -k\eta(N) - \alpha + \cos^2\theta.\alpha \\ &= -k\eta(N) - \alpha\sin^2\theta. \end{aligned} \tag{4.3}$$

If θ=constant, then from (4.3) we get, $k\eta(N) = -\alpha \sin^2\theta$.
So k=0 if and only if $\alpha = 0$.
This completes the proof. □

5. Locally ϕ-symmetric almost contact curves on three-dimensional α-para Kenmotsu manifolds with respect to Levi-Civita connection

Definition 5.1. With respect to Levi-Civita connection an almost contact curve γ on a three-dimensional α-para Kenmotsu manifold is called locally ϕ-symmetric if it satisfies [24]

$$\phi^2(\nabla_T R)(\nabla_T T, T)T = 0, \tag{5.1}$$

where $T = \dot{\gamma}$.

Here we shall establish the following:

Theorem 5.1. *A locally ϕ-symmetric curve on a three-dimensional α-para Kenmotsu manifold with constant structure function α and constant scalar curvature r is not necessarily a geodesic with respect to Levi-Civita connection.*

Proof. Now putting $X = \nabla_T T, \quad Y = Z = T$ in (2.4) and using Serret-Frenet formula, we get,

$$R(\nabla_T T, T)T = k(\frac{r}{2} + 2\alpha^2)N. \tag{5.2}$$

Again putting $X = B$, $Y = Z = T$ in (2.4) we get,

$$R(B, T)T = (\frac{r}{2} + 2\alpha^2)B. \tag{5.3}$$

By definition of covariant differentiation of R with respect to Levi-Civita connection and using Serret-Frenet formula, we get,

$$\begin{aligned}(\nabla_T R)(\nabla_T T, T)T &= \nabla_T R(\nabla_T T, T)T - k\tau R(B, T)T \\ &- k' R(N, T)T - k^2 R(N, T)N.\end{aligned} \tag{5.4}$$

Now from (5.2) and using Serret-Frenet formula we get,

$$\begin{aligned}\nabla_T R(\nabla_T T, T)T &= -[\frac{r}{2}k^2 + 2\alpha^2 k^2]T + [2\alpha^2 k' + \frac{r}{2}k' + \frac{r'}{2}k + 4k\alpha\alpha']N \\ &+ [\frac{r}{2}k\tau + 2k\tau\alpha^2]B.\end{aligned} \tag{5.5}$$

Again putting $X = N$, $Y = T$, $Z = N$ in (2.4) we get,

$$R(N, T)N = -(\frac{r}{2} + 2\alpha^2)T. \tag{5.6}$$

Now using (3.4), (5.3), (5.5) and (5.6) in (5.4) we get,

$$(\nabla_T R)(\nabla_T T, T)T = (\frac{r'}{2}k + 4k\alpha\alpha')N. \tag{5.7}$$

Applying ϕ^2 on both sides of (5.7), we get,

$$\phi^2(\nabla_T R)(\nabla_T T, T)T = (\frac{r'}{2}k + 4k\alpha\alpha')N. \tag{5.8}$$

If the curves are locally ϕ-symmetric, then we have

$k(4\alpha\alpha' + \frac{r'}{2}) = 0.$
The above equation is true if α and r are constants and $k \neq 0$.
Hence the theorem follows. $\square$

6. Example

In this section we give example of an almost contact curve and slant curve on α-para Kenmotsu manifold. To give the examples, we have followed the paper [31]
Let $M = \mathbb{R}^2 \times \mathbb{R}$. Then a normal almost para contact structure (ϕ, ξ, η) can be defined as follows: $\phi(e_1) = e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0, \quad \xi = e_3, \quad \eta = dz.$
Let the vector fields be

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the pseudo-Riemannian metric defined by

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0, \qquad g(e_1, e_1) = -2z, \qquad g(e_2, e_2) = 2z, \qquad g(e_3, e_3) = 1.$$

The quadraruple (ϕ, ξ, η, g) becomes a normal almost para contact metric structure on M.
For the Levi-CIvita connection, By Koszul formula, we have

$$\nabla_{e_1} e_3 = \frac{1}{2z} e_1, \qquad \nabla_{e_1} e_2 = 0, \qquad \nabla_{e_1} e_1 = e_3$$

$$\nabla_{e_2} e_3 = \frac{1}{2z} e_2, \qquad \nabla_{e_2} e_2 = -e_3, \qquad \nabla_{e_2} e_1 = 0$$

$$\nabla_{e_3} e_3 = 0, \qquad \nabla_{e_3} e_2 = \frac{1}{2z} e_2, \qquad \nabla_{e_3} e_1 = \frac{1}{2z} e_1.$$

Using the above and (2.7), we get $\alpha = \frac{1}{2z}$. Hence the manifold is a α-para Kenmotsu manifold.
Consider a curve $\gamma : I \longrightarrow M$ defined by $\gamma(s) = (\cosh s, \sinh s, 1)$.
This curve is an almost contact curve [31].
Again we consider a curve $\gamma : I \longrightarrow M$ defined by $\gamma_1(s) = (\cosh s, \sinh s, s)$.
This curve is a slant curve.

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