ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 2, p.151-166

The total graph of a module with respect to multiplicative-prime subsets

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Abstract

Let M be a module over a commutative ring R and U a nonempty proper subset of M. In this paper, a generalization of the total graph $T(\Gamma(M))$, denoted by $T(\Gamma_U(M))$ is presented, where U is a multiplicative-prime subset of M. It is the graph with all elements of M as vertices, and for two distinct elements $m, n \in M$, the vertices m and n are adjacent if and only if $m + n \in U$. The main purpose of this paper is to extend the definitions and properties given in [1] and [10] to a more general case.

Key words: Total graph; prime submodule; multiplicative-prime subset.
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1 Introduction

Throughout of this paper R is a commutative ring with nonzero identity and M is a unitary R-module. The concept of the graph of zero-divisors of R was first introduced in [8] and [2]. Recently, there has been considerable attention in associating graphs with algebraic structures (see [3],[4],[6],[7], [9] and [11]). Anderson and Badawi in [5] defined the notion of a *multiplicative-prime* subset of a commutative ring R. It is a nonempty proper H of R which satisfies the following two properties: (i) $ab \in H$ for every $a \in H$ and $b \in R$; (ii) if $rs \in H$ for some $r, s \in R$, then either $r \in H$ or $s \in H$. For any *multiplicative-prime* subset H of R, they introduced the notion of a generalized total graph $GT_H(R)$ with vertices in R and for any two vertices $x, y \in R$, they are adjacent if and only if $x + y \in H$. Let R be a commutative ring and U be a nonempty subset of an R-module M. The subset $\{r \in R : rM \subseteq U\}$ will be denoted by $(U :_R M)$ or $(U : M)$. It is clear that if U is a submodule of M, then $(U : M)$ is an ideal of R. We say that a nonempty subset U of M is a *multiplicative-prime* subset of M if the following two conditions hold: (i) $rm \in U$ for every $r \in R$ and $m \in U$; (ii) if $sx \in U$ for some $s \in R$ and $x \in M$, then $x \in U$ or $s \in (U : M)$. Note that if U is a submodule of M, then U is necessarily a prime submodule of M.

In the present paper, we introduce and investigate the generalized total graph of M, denoted by $GT_U(M)$, as a (undirected) graph with all elements of M as vertices, and for two distinct elements $m, n \in M$, the vertices m and n are adjacent if and only if $m + n \in U$ where U is a *multiplicative-prime* subset of M. Let $GT_U(U)$ be the (induced) subgraph of $GT_U(M)$ with vertex set U, and let $GT_U(M \setminus U)$ be the (induced) subgraph $GT_U(M)$ with vertices consisting of $M \setminus U$. The study of $GT_U(M)$ breaks naturally into two cases depending on whether or not U is a submodule of M. In the second section, we obtain some properties concerning U. In the third section, we consider the case when U is a submodule of M; in the forth section, we do the case when U is not a submodule of M. For every case, we characterize the girths and diameters of $GT_U(M)$, $GT_U(U)$ and $GT_U(M \setminus U)$.

We begin with some notation and definitions. For a graph Γ, by $E(\Gamma)$ and $V(\Gamma)$, we mean the set of all edges and vertices, respectively. We recall that a graph is connected if there exists a path connecting any two of it's distinct vertices. At the other extreme, we say that a graph is totally disconnected if no two vertices of this graph are adjacent. The distance between two distinct vertices a and b, denoted by $\mathrm{d}(a, b)$, is the length of a shortest path connecting them (if such a path does not exist, then $d(a, b) = \infty$. We also define $d(a, a) = 0$. The diameter of a graph Γ, denoted by $\mathrm{diam}(\Gamma)$, is equal to $\sup\{d(a, b) : a, b \in V(\Gamma)\}$. A graph is complete if it is connected with diameter less than or equal to one. The girth of a graph Γ, denoted $\mathrm{gr}(\Gamma)$, is the length of a shortest cycle in Γ, provided Γ contains a cycle; otherwise; $\mathrm{gr}(\Gamma) = \infty$. We denote the complete graph on n vertices by K^n and the complete bipartite graph on m and n vertices by $K^{m,n}$ (we allow m and n to be infinite cardinals). We will sometimes call a $K^{1,m}$ a star graph. For a graph Γ, the degree of a vertex v in Γ, denoted $deg(v)$, is the number of edges of Γ incident with v. For a nonnegative integer k, a graph is called k-regular if every vertex has degree k. We say that two (induced) subgraphs Γ_1 and Γ_2 of Γ are disjoint if Γ_1 and Γ_2 have no common vertices and no vertices of Γ_1 is adjacent(in Γ) to some vertex of Γ_2.

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 2, p.151-166

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 2, p.151-166

2 Multiplicative-prime subsets of a module

We devote this section to the *multiplicative-prime* subsets of an R-module M. Throughout this paper, we assume that every multiplicatively closed proper subset S of R contains 1, but does not contain 0. We first begin with the following lemma.

Lemma 2.1 *Let U be a proper multiplicative-prime subset of M. Then the following hold:*
(1) $(U : M)$ is a multiplicative-prime subset of R.
(2) $S = R \setminus (U : M)$ is a multiplicatively closed subset of R.

Proof. Let $r \in R$ and $s \in (U : M)$. Then $rsM \subseteq sM \subseteq U$. So $rs \in (U : M)$. Now, suppose that $ab \in (U : M)$ and $a \notin (U : M)$ for some $a, b \in R$. It suffices to show that $b \in (U : M)$. There exists $m \in M \setminus U$ such that $am \notin U$. Since $abm \in U$ and U is a multiplicative-prime subset of M, thus $b \in (U : M)$.
(2) Since U is a proper subset of M so it is clear that $1 \in S$. Let $r, s \in S$. Then $r \notin (U : M)$ and $s \notin (U : M)$. So $rs \notin (U : M)$ since $(U : M)$ is a multiplicative-prime subset of R. □
Now, we have the following Definition from [13, Definition 1].

Definition 2.2 *Let S be a multiplicatively closed subset of a ring R and M an R-module.*
(1) A non-empty subset S^* of M is said to be S-closed if $sx \in S^*$ for every $s \in S$ and $x \in S^*$.
(2) An S-closed subset S^* is said to be saturated if whenever $rm \in S^*$ for some $r \in R$ and $m \in M$, then $r \in S$ and $m \in S^*$.

Proposition 2.3 *Let U be a proper multiplicative-prime subset of M and $S = R \setminus (U : M)$. Then $S^* = M \setminus U$ is a saturated S-closed subset of M.*

Proof. First suppose that $s \in S$ and $x \in S^*$. It is clear that $sx \notin U$, since U is a multiplicative-prime of M. Then S^* is S-closed. Now suppose that $rm \in S^*$ for some $r \in R$ and $m \in M$, then $rm \notin U$. Since U is a multiplicative-prime subset of M, if $m \in U$, then $rm \in U$ which is a contradiction. So $m \in S^* = M \setminus U$. Now, suppose that $r \in (U : M)$. So $rm \in rM \subseteq U$ which is a contradiction. Thus $r \in S = R \setminus (U : M)$. So S^* is a saturated S-closed subset of M. □

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 2, p.151-166

Proposition 2.4 *Let M be a cyclic R-module and U be a proper multiplicative-prime subset of M. Then U is a union of prime submodules N_i, $i \in I$, of M and $(U : M)$ is a union of prime ideals $P_i = (N_i : M)$ for each $i \in I$.*

Proof. The proof is clear by Proposition 2.3 and [13, Theorem 4.8]. □

Proposition 2.5 *Let U be a proper multiplicative-prime subset of M and $S = R \setminus (U : M)$. Then*
(1) $S^{-1}(U :_R M) = (S^{-1}U :_{S^{-1}R} S^{-1}M)$.
(2) $S^{-1}U$ is a multiplicative-prime subset of $S^{-1}M$.

Proof. (1) It suffices to show that $(S^{-1}U :_{S^{-1}R} S^{-1}M) \subseteq S^{-1}(U :_R M)$. Let $r/s \in (S^{-1}U :_{S^{-1}R} S^{-1}M)$ such that $r \in R$ and $s \in S$ and let $m \in M$. Then $(r/s)(m/1) \in S^{-1}U$. There exist $u \in U$ and $t \in S$ such that $rm/s = u/t$. Then $t'trm = t'su$ for some $t' \in S$. It follows that $rm \in U$, since $t't \in S = R \setminus (U : M)$. Thus $r \in (U :_R M)$ and so $r/s \in S^{-1}(U :_R M)$.
(2) It is clear that $(r/s)(m/t) = rs/tm \in S^{-1}U$ for every $r/s \in S^{-1}R$ and $m/t \in S^{-1}U$. Now, let that $(a/s)(x/t) \in S^{-1}U$ for some $a/s \in S^{-1}R$ and $x/t \in S^{-1}M$. Then $ax/st = u/s'$ for some $s' \in S$ and $u \in U$. So $s''s'ax = s''stu$ for some $s'' \in S$. Hence $ax \in U$ since S is a multiplicatively closed subset of R and U is a multiplicative-prime subset of M. So either $a \in (U :_R M)$ or $u \in U$, then the result is clear by part (1) above. □

3 The case when U is a submodule of M

In this section, we study the case when U is a (prime) submodule of M. If $U = M$, then it is clear that $GT_U(M)$ is a complete graph and $GT_U(M)$ is a disconnected graph when $U = 0$ and $|M| \geq 2$. So we may assume that $U \neq 0$ and $U \neq M$.

Theorem 3.1 *Let M be a module over a commutative ring R and U be a prime submodule of M. Then $GT_U(U)$ is a complete subgraph of $GT_U(M)$ and is disjoint from $GT_U(M \setminus U)$. In particular, $GT_U(U)$ is connected and $GT_U(R)$ is disconnected.*

Proof. It is clear by the definitions. □

Theorem 3.2 *Let M be a module over a commutative ring R and U be a prime submodule of M. Then the following hold:*
(1) Suppose that G is an induced subgraph of $GT_U(M \setminus U)$ and let m and m' be distinct vertices of G that are connected by a path in G. Then there exists

a path in G of length 2 between m and m'. In particular, if $GT_U(M \setminus U)$ is connected, then $diam(GT_U(M \setminus U)) \leq 2$.
(2) Let m and m' be distinct elements of $GT_U(M \setminus U)$ that are connected by a path. If $m + m' \notin U$ then $m - (-m) - m'$ and $m - (-m') - m'$ are paths of length 2 between m and m' in $GT_U(M \setminus U)$.

Proof. (1) Let m_1, m_2, m_3 and m_4 are distinct vertices of G. It suffices to show that if there is a path $m_1 - m_2 - m_3 - m_4$ from m_1 to m_4, then m_1 and m_4 are adjacent. Now, $m_1 + m_2, m_2 + m_3, m_3 + m_4 \in U$ gives $m_1 + m_4 = (m_1 + m_2) - (m_2 + m_3) - (m_3 + m_4) \in U$. Thus m_1 and m_4 are adjacent. It's clear that if $GT_U(M \setminus U)$ is connected, then $diam(GT_U(M \setminus U)) \leq 2$.
(2) Since $m, m' \notin U$ and $m+m' \notin U$, there exists $w \in GT_U(M \setminus U)$ such that $m - w - m'$ is a path of length 2 by part (1) above. Thus $w + m, w + m' \in U$ and hence $m - m' = (m + w) - (w + m') \in U$. Also, since $m, m' \notin U$, we must have $m \neq -m'$ and $m' \neq -m'$. Thus $m - (-m') - m'$ is a path from m to m' in $GT_U(M \setminus U)$. □

Theorem 3.3 *Let M be a module over a commutative ring R and U be a prime submodule of M.Then the following statement are equivalent:*
(1) $GT_U(M \setminus U)$ is connected.
(2) Either $m + m' \in U$ or $m - m' \in U$ (but not both) for all $m, m' \in M \setminus U$.
(3) Either $m + m' \in U$ or $m + 2m' \in U$ for all $m, m' \in M \setminus U$.
In particular, if (3) is satisfied, then either $2m \in U$ or $3m \in U$ (but not both) for all $m \in M \setminus U$.

Proof. (1)$\Longrightarrow$ (2) Let $m, m' \in M \setminus U$ be such that $m + m' \notin U$. If $m = m'$, then $m - m' \in U$. Otherwise $m - (-m') - m'$ is a path from m to m' by Theorem 3.2 (2). Then $m - m' \in U$.
(2)$\Longrightarrow$ (3) Let $m, m' \in M \setminus U$ be distinct elements of M such that $m + m' \notin U$. Thus $(m + m') + m' \in U$ or $(m + m') - m' \in U$ by assumption. If $(m + m') - m' \in U$, then $m \in U$, that is a contradiction. Therefore, $(m + m') + m' = m + 2m' \in U$. In particular, $m + m = 2m \in U$ or $m + 2m = 3m \in U$ for all $m \in M \setminus U$. Both $2m$ and $3m$ can't be in U, since $m = 3m - 2m \in U$ is a contradiction.
(3) $\Longrightarrow$ (1) Let $m, m' \in M \setminus U$ be distinct elements of M such that $m + m' \notin U$. By hypothesis $m + 2m' \in U$ and we get $2m' \notin U$. Thus $3m' \in U$ by assumption. Moreover, since $m + m' \notin U$ and $3m' \in U$, hence $m \neq 2m'$. Therefore $m - (2m') - m'$ is a path from m to m' in $GT_U(M \setminus U)$. Thus $GT_U(M \setminus U)$ is connected. □

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 2, p.151-166

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 2, p.151-166

Example 3.4 *Let $R = Z_4$ denote the ring of integers modulo 4 and let $M = Z_8$ as an R-module. Let $U = 2Z_8 = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$. It is clear that $2 \in (U :_R M)$. Now $\bar{5} + \bar{2}, \bar{5} - \bar{2} \notin U$. So $GT_U(M \setminus U)$ is not connected by Theorem 3.3.*

Now, we give the main theorem of this section. Since $GT_U(U)$ is a complete subgraph of $GT_U(M)$ by Theorem 3.1, the next theorem gives a complete description of $GT_U(M)$. We allow α and β to be infinite, then of course $\beta - 1 = \frac{\beta-1}{2} = \beta$.

Theorem 3.5 *Let M be a module over a commutative ring R and U be a prime submodule of M and let $\alpha = |U|$ and $|M/U| = \beta$.*
(1) If $2 \in (U :_R M)$, then $GT_U(M \setminus U)$ is the union of $\beta - 1$ disjoint k^{α}'s.
(2) If $2 \notin (U :_R M)$, then $GT_U(M \setminus U)$ is the union of $\frac{\beta-1}{2}$ disjoint $k^{\alpha,\alpha}$'s.

Proof. (1) We first note that $m + U \subseteq M \setminus U$ for all $m \notin U$. Now, let $2 \in (U :_R M)$ and $m + n_1, m + n_2 \in m + U$ for some $n_1, n_2 \in U$. Then $(m+n_1)+(m+n_2) = 2m+(n_1+n_2) \in U$, since U is a submodule of M and $2 \in (U :_R M)$. So each coset $m+U$ induces a complete subgraph of $GT_U(M \setminus U)$. Moreover, distinct cosets form disjoint subgraphs of $GT_U(M \setminus U)$, since if $m + n$ and $m' + n'$ are adjacent for some $m, m' \in M \setminus U$ and $n, n' \in U$, then $m+m' = (m+n)+(m'+n')-(n+n') \in U$. Then $m-m' = (m+m')-2m' \in U$ that gives $m + U = m' + U$. Thus $GT_U(M \setminus U)$ is the union of $\beta - 1$ disjoint (induced) subgraphs $m + U$, each of which is k^{α} where $\alpha = |U| = |m + U|$.
(2) Let $m \in M \setminus U$ and $2 \notin (U :_R M)$. We claim that no two distinct elements in $m+U$ are adjacent. Suppose not. Let $m+m_1, m+m_2 \in m+U$ are adjacent for some $m_1, m_2 \in U$. Then $(m+m_1)+(m+m_2) = 2m+(m_1+m_2) \in U$. This implies $2m \in U$, since U is a prime submodule of M, we have $2 \in (U :_R M)$ which is a contradiction. Thus $(m + U) \cup (-m + U)$ is a complete bipartite (induced) subgraph of $GT_U(M \setminus U)$.
Moreover, if $m + x_1$ is adjacent to $m' + x_2$ for some $m, m' \in M \setminus U$ and $x_1, x_2 \in U$, then $m + x_1 + m' + x_2 \in U$, and hence $m + m' = m + x_1 + m' + x_2 - (x_1 + x_2) \in U$. Therefore $m + U = -m' + U$. Thus $GT_U(M \setminus U)$ is the union of $\frac{\beta-1}{2}$ disjoint subgraph $(m + U) \bigcup (-m + U)$, each of which is a $k^{\alpha,\alpha}$, Where $\alpha = |U| = |m + U|$. □

Example 3.6 *Let $R = Z_{12}$ and $M = Z_6$ as an R-module.*
(1) If $U = \{\bar{0}, \bar{2}, \bar{4}\}$, then it is clear that $2 \in (U :_R M)$. So $GT_U(M \setminus U)$ is the complete graph K^3 ($\alpha = 3, \beta = 2$).
(2) If $U = \{\bar{0}, \bar{3}\}$, then it is clear that $2 \notin (U :_R M)$. Thus $GT_U(M \setminus U)$ is the complete bipartite graph $K^{2,2}$ ($\alpha = 2, \beta = 3$).

Example 3.7 *Let $R = \mathbb{Z}$, $M = \mathbb{Z} \times \mathbb{Z}$.*
(a) If $U = 2Z \times 4Z$, then it is clear that $2 \in (U :_R M)$, so $GT_U(M \setminus U)$ is a union of complete graphs.
(b) If $U = 5Z \times 10Z$, then $2 \notin (U :_R M)$, then $GT_U(M \setminus U)$ is a union of complete bipartite graphs.

By the following theorem, we determine when $GT_U(M \setminus U)$ is either complete or connected.

Theorem 3.8 *Let M be a module over a commutative ring R and U be a prime submodule of M. Then*
(1) $GT_U(M \setminus U)$ is complete if and only if either $|M/U| = 2$ or $|M/U| = |M| = 3$.
(2) $GT_U(M \setminus U)$ is connected if and only if either $|M/U| = 2$ or $|M/U| = 3$.
(3) $GT_U(M \setminus U)$ (and hence $GT_U(U)$ and $GT_U(M)$) is totally disconnected if and only if $U = \{0\}$ and $2 \in (U :_R M)$.

Proof. (1) Let $GT_U(M \setminus U)$ be a complete subgraph of $GT_U(M)$. Then by Theorem 3.5, $GT_U(M \setminus U)$ is a single k^α or $k^{1,1}$. If $GT_U(M \setminus U)$ is k^α, then $\beta - 1 = 1$. Hence $\beta = 2$ and therefore $|M/U| = 2$. If $GT_U(M \setminus U)$ is $k^{1,1}$, then $\frac{\beta-1}{2} = 1$ and $\alpha = 1$. Thus $\beta = 3$ and $\alpha = 1$, therefore $|M/U| = 3$ and $U = \{0\}$, hence $|M/U| = |M| = 3$.
Conversely, let $|M/U| = 2$ and $M/U = \{U, x + U\}$ where $x \notin U$. Then $x + U = -x + U \in M/U$ gives $2x \in U$. Thus $2 \in (U :_R M)$. Now, we show that $GT_U(M \setminus U)$ is complete. Let $m, m' \in M \setminus U$. Then $m + m' = (m + x) + (m' + x) - 2x \in U$. Therefore $GT_U(M \setminus U)$ is complete. Now , let $|M/U| = |M| = 3$. In this case, we show that $2 \notin (U :_R M)$. Suppose not. Then $2m \in U$ for all $m \in M$. Thus $2(m + U) = 0_{M/U}$ for all $m \in M$ which is a contradiction, since M/U is a cyclic group with order 3. Thus $2 \notin (U :_R M)$ and hence $GT_U(M \setminus U)$ is complete. Then, every case leads to $GT_U(M \setminus U)$ is complete.
(2) Let $GT_U(M \setminus U)$ be connected. Then by Theorem 3.5, $GT_U(M \setminus U)$ is a single k^α or $k^{\alpha,\alpha}$. Thus by Theorem 3.5, if $2 \in (U :_R M)$, then $\beta - 1 = 1$ and so $|M/U| = 2$ and if $2 \notin (U :_R M)$, then $\frac{\beta-1}{2} = 1$ and hence $|M/U| = 3$.
Conversely, by part(1) above we may assume that $|M/U| = 3$. We claim that $2 \notin (U :_R M)$. Otherwise $2M \subseteq U$. Suppose that $M/U = \{U, x + U, y + U\}$ where $x, y \notin U$. Since M/U is a cyclic group with order 3, we conclude that $x + y \in U$ and hence x, y are adjacent that is a contradiction since $GT_U(M \setminus U)$ is union $3 - 1 = 2$ disjoint subgraph $x + U$ and $y + U$. Therefore $2 \notin (U :_R M)$. So by Theorem 3.5, $GT_U(M \setminus U)$ is a single $K^{\alpha,\alpha}$. Hence

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 2, p.151-166

is connected.
(3) $GT_U(M \setminus U)$ is totally disconnected if and only if it is a disjoint union of K^1's. So by Theorem 3.5, $GT_U(M \setminus U)$ is totally disconnected if and only if $2 \in (U :_R M)$, $|U| = 1$ and $|M/U| = 1$. □

By Theorem 3.8, the next theorem gives a more explicit description of the diameter of $GT_U(M \setminus U)$.

Theorem 3.9 *Let M be a module over a commutative ring R such that U is a prime submodule of M. Then $diam(GT_U(M \setminus U)) = 0, 1, 2, \infty$. In particular, if $GT_U(M \setminus U)$ is connected, then $diam(GT_U(M \setminus U)) \leq 2$.*

Proof. Suppose that $GT_U(M \setminus U)$ is connected. Then $GT_U(M \setminus U)$ is a singleton, a complete graph or a complete bipartite graph by Theorem 3.5. Hence $diam(GT_U(M \setminus U)) \leq 2$. □

Theorem 3.10 *Let M be a module over a commutative ring R such that U is a prime submodule of M.*
(1) $diam(GT_U(M \setminus U)) = 0$ if and only if $U = \{0\}$ and $|M| = 2$.
(2) $diam(GT_U(M \setminus U)) = 1$ if and only if either $U \neq \{0\}$ and $|M/U| = 2$ or $U = \{0\}$ and $|M| = 3$.
(3) $diam(GT_U(M \setminus U)) = 2$ if and only if $U \neq \{0\}$ and $|M/U| = 3$.
(4) Otherwise, $diam(GT_U(M \setminus U)) = \infty$.

Proof. These results follow from Theorem 3.5 and Theorem 3.8. □

Theorem 3.11 *Let M be a module over a commutative ring R such that U is a prime submodule of M. Then $gr(GT_U(M \setminus U)) = 3, 4$ or ∞. In particular, $gr(GT_U(M \setminus U)) \leq 4$ if $GT_U(M \setminus U)$ contains a cycle.*

Proof. Let $GT_U(M \setminus U)$ contains a cycle. Then since $GT_U(M \setminus U)$ is disjoint union of either complete or complete bipartite graphs by Theorem 3.8, it must contain either a 3-cycle or 4-cycle. Thus $gr(GT_U(M \setminus U)) \leq 4$. □

Theorem 3.12 *Let M be a module over a commutative ring R such that U is a prime submodule of M.*
(1) (a) $gr(GT_U(M \setminus U)) = 3$ if and only if $2 \in (U :_R M)$ and $|U| \geq 3$.
(b) $gr(GT_U(M \setminus U)) = 4$ if and only if $2 \notin (U :_R M)$ and $|U| \geq 2$.
(c) Otherwise, $gr(GT_U(M \setminus U)) = \infty$.
(2) (a) $gr(GT_U(M)) = 3$ if and only if $|U| \geq 3$.
(b) $gr(GT_U(M)) = 4$ if and only if $2 \notin (U :_R M)$ and $|U| = 2$.
(c) Otherwise, $gr(GT_U(M)) = \infty$.

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 2, p.151-166

Proof. Apply Theorem 3.1, Theorem 3.5 and Theorem 3.11. □

Example 3.13 *Let $R = Z_6$ and $M = R$ as an R-module. Let $U = \{\bar{0}, \bar{2}\}$. Then $|U| = 2$ and $2 \in (U : M)$. It is clear that $diam(GT_U(M)) = gr(GT_U(M)) = \infty$.*

4 The case when U is not a submodule of M

In this section we study $GT_U(M)$ when the multiplicative-prime subset U is not a submodule of M. Since U is always closed under multiplication by elements of R, this just means that $0 \in U$ and there are distinct $x, y \in U$ such that $x + y \in M \setminus U$. We first begin with the following theorem.

Theorem 4.1 *Let M be a module over a commutative ring R such that U is a multiplicative-prime subset of M that is not a submodule of M. Then the following hold:*
(1) $GT_U(U)$ is connected with $diam(GT_U(U)) = 2$.
(2) Some vertex of $GT_U(U)$ is adjacent to a vertex of $GT_U(M \setminus U)$. In particular, the subgraphs $GT_U(U)$ and $GT_U(M \setminus U)$ are not disjoint.
(3) If $GT_U(M \setminus U)$ is connected, then $GT_U(M)$ is connected.

Proof. (1) Let $m \in U^* = U \setminus \{0\}$. Then m is adjacent to 0. Thus $m - 0 - n$ is a path in $GT_U(U)$ of length two between any two distinct $m, n \in U^*$. Moreover, there exist nonadjacent $m, n \in U^*$ since U is not a submodule of M; thus $diam(GT_U(U)) = 2$.
(2) There exist distinct $m, n \in U^*$ such that $m + n \notin U$. Then $-m \in U$ and $m + n \in U$ are adjacent vertices in $GT_U(M)$. Finally, the"in particular" statement is clear.
(3) Since $GT_U(U)$ and $GT_U(M \setminus U)$ are connected and there is an edge between $GT_U(U)$ and $GT_U(M \setminus U)$, so $GT_U(M)$ is connected. □
We determine when $GT_U(M)$ is connected and compute $diam(GT_U(M))$ with the following theorem.

Theorem 4.2 *Let M be a module over a commutative ring R such that U is a multiplicative-prime subset of M that is not a submodule of M. Then $GT_U(M)$ is connected if only if $M =< U >$ (that is, $M =< a_1, a_2, \ldots, a_k >$ for some $a_1, a_2, \ldots, a_k \in U$).*

Proof. Suppose that $GT_U(M)$ is connected, and $m \in M$. Then there exist a path $0 - m_1 - m_2 - \ldots - m_n - m$ from 0 to m in $GT_U(M)$. Thus

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 2, p.151-166

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 2, p.151-166

$m_1, m_1+m_2, ..., m_n+m \in U$. Hence $m \in< m_1, m_1+m_2, ..., m_{n-1}+m_n, m_n+m >\subseteq< U >$; thus $M =< U >$. Conversely, suppose that $M =< U >$. We show that for each $0 \neq m \in M$, there exist a path in $GT_U(M)$ from 0 to m. By assumption, there are elements $m_1, m_2, ..., m_n \in U$ such that $m = m_1 + m_2 + ... + m_n$. Set $x_0 = 0$ and $x_k = (-1)^{n+k}(m_1 + m_2 + ... + m_k)$ for each integer k with $1 \leq k \leq n$. Then $x_k + x_{k+1} = (-1)^{n+k+1} m_{k+1} \in U$ for each integer k with $0 \leq k \leq n-1$, and thus $0-x_1-x_2-...-x_{n-1}-x_n = m$ is a path from 0 to m in $GT_U(M)$ of length at most n. Now let $u, w \in M$. Then by the preceding argument, there are paths from u to 0 and 0 to w in $GT_U(M)$; hence there is a path from u to w in $GT_U(M)$. Thus, $GT_U(M)$ is connected. □

Theorem 4.3 *Let M be a module over a commutative ring R such that U is a multiplicative-prime subset of M that is not a submodule of M, and let $M =< U >$ (that is , $GT_U(M)$ is connected). Let $n \geq 2$ be the least integer such that $M =< m_1, m_2, ..., m_n >$ for some $m_1, m_2, ..., m_n \in U$. Then $diam(GT_U(M)) \leq n$. In particular, if M is a cyclic R-module, then $diam(GT_U(M)) = n$.*

Proof. Let m and m' be distinct elements in M. We show that there exist a path from m to m' in $GT_U(M)$ with length at most n. By hypothesis, we can write $m = \sum_{i=1}^{n} r_i m_i$ and $m' = \sum_{i=1}^{n} s_i m_i$ for some $r_i, s_i \in R$. Define $x_0 = m$ and $x_k = (-1)^k(\sum_{i=k+1}^{n} r_i m_i + \sum_{i=1}^{k} s_i m_i)$, so $x_k + x_{k+1} = (-1)^k m_{k+1}(r_{k+1} - s_{k+1}) \in U$ for each integer k with $1 \leq k \leq n-1$. If we define $x_n = m'$, then $m - x_1 - x_2 - ... - x_{n-1} - m'$ is a path from m to m' in $GT_U(M)$ with length at most n.
Finally, assume that $M =< w >$. Let $0-y_1-y_2-...-y_{m-1}-w$ be a path from 0 to w in $GT_U(M)$ with length m. Thus $y_1, y_1 + y_2, ..., y_{m-1} + w \in< U >$, and hence $w \in< y_1, y_1 + y_2, ..., y_{m-1} + w >\subseteq U$. Thus $m \geq n$, as required. □

Theorem 4.4 *Let M be a module over a commutative ring R such that U is a multiplicative-prime subset of M that is not a submodule of M. Let $n \geq 2$ be the least integer such that $M =< m_1, m_2, ..., m_n >$ for some $m_1, m_2, ..., m_n \in U$.*
(1) If M is a cyclic module with generator m, then $diam(GT_U(M)) = d(0, m)$.
(2) If $diam(GT_U(M)) = n$ and M is a cyclic R-module with generator m, then $diam(GT_U(M \setminus U)) \geq n-2$.

Proof. (1) This follows from Theorem 4.3.
(2) Since $diam(GT_U(M)) = d(0, m) = n$, by part (1) above, let $0 - m_1 - ... - m_{n-1} - m$ be a shortest path from 0 to m in $GT_U(M)$. Clearly, $m_1 \in U$. If $m_i \in U$ for some i with $2 \leq i \leq n-1$, then $0 - m_i - ... - m_{n-1} - m$ is a path from 0 to m of length less than n in $GT_U(M)$, which is a contradiction. Thus $m_i \in GT_U(M \setminus U)$ for each integer i with $2 \leq i \leq n-1$. Therefore, $m_2 - m_3 - ... - m_{n-1} - m$ is a shortest path from m_2 to m in $GT_U(M \setminus U)$, and it has length $n-2$. Thus $diam(GT_U(M \setminus U)) \geq n-2$. □

Let M be a module over a commutative ring R such that U is a *multiplicative-prime* subset of M. Recall that two submodules L and K of M are called co-maximal if $M = L + K$. Note that if a proper subset U of M contains two co-maximal submodules of M, then U is not a submodule of M.

Theorem 4.5 *Let M be a finitely generated R-module and $n \geq 2$ be the least integer such that $M =< m_1, m_2, ..., m_n >$ for some $m_1, ..., m_n \in M$. Let U be a multiplicative-prime subset of M such that U contains two co-maximal submodules of M. Then $GT_U(M)$ is connected with $diam(GT_U(M)) \leq 2n$.*

Proof. Let $L, K \subseteq U$ be co-maximal submodules of M. Then $M = L + K$; so $m_i = x_i + y_i$ for some $x_i \in L$ and $y_i \in K$ for every $i = 1, 2, ..., n$. Hence $M =< x_1, ..., x_n, y_1, ..., y_n >$. Thus $GT_U(M)$ is connected with $diam(GT_U(M)) \leq 2n$ by Theorem 4.3 and Theorem 4.2. □

Theorem 4.6 *Let M be a cyclic R-module and let U be a multiplicative-prime subset of M that is not a submodule of M. If $S = R \setminus (U :_R M)$, then $GT_{S^{-1}U}(S^{-1}M)$ is connected with $diam(GT_{S^{-1}U}(S^{-1}M)) \leq 2$.*

Proof. Let $M = Rm$. There exist $u, w \in U$ such that $u + w \notin U$, since U is not a submodule of M. By Proposition 2.4, U is a union of prime submodules, so there are prime submodules N and L of M contained in U with $u \in N \setminus L$ and $w \in L \setminus N$. Then $u = rm$ and $w = sm$ for some $r, s \in R$. So $(r+s)m = u + w \notin U$; then $r + s \notin (U :_R M)$. Thus $r + s \in S$. This implies that $m/1 = (r+s)m/(r+s) = (u/(r+s)) + (w/(r+s)) \in S^{-1}L + S^{-1}N$. Thus the prime submodules $S^{-1}L$ and $S^{-1}N$ are co-maximal in $S^{-1}M$; so the result follows from Theorem 4.5. □

Now, by the following theorem we provide a proof for the converse of [1. Theorem 4.5 (4)] when M is a cyclic R-module.

Theorem 4.7 *Let M be a module over a commutative ring R such that U is a multiplicative-prime subset of M that is not a submodule of M.*
(1) Either $gr(GT_U(U)) = 3$ or $gr(GT_U(U)) = \infty$.
(2) $gr(GT_U(M)) = 3$ if and only if $gr(GT_U(U)) = 3$.
(3) If $gr(GT_U(M)) = 4$, then $gr(GT_U(U)) = \infty$.
(4) If M is a cyclic R-module and $gr(GT_U(U)) = \infty$, then $gr(GT_U(M)) = 4$.
(5) If $Nil(R) \neq 0$ and $2 \in (0 :_R M)$, then $gr(GT_U(M \setminus U)) = 3, 4 or \infty$.
(6) If $2 \notin (U :_R M)$, then $gr(GT_U(M \setminus U)) = 3, 4 or \infty$.

Proof. (1) If $m + m' \in U$ for some distinct $m, m' \in U^*$, then $0 - m - m' - 0$ is a 3-cyclic in $gr(GT_U(U))$; so $gr(GT_U(U)) = 3$. Otherwise, $m + m' \in M \setminus U$ for all distinct $m, m' \in U$. Therefore, in this case, each $m \in U^*$ is adjacent to 0, and no two distinct $m, m' \in U^*$ are adjacent. Thus $gr(GT_U(U))$ is a star graph with center 0; hence $gr(GT_U(U)) = \infty$.
(2) It suffices to show that $gr(GT_U(U)) = 3$ when $gr(GT_U(M)) = 3$. If $2m \neq 0$ for some $u \in U^*$, then $0 - u - (-u) - 0$ is a 3-cycle in U. Thus we may assume that $2m = 0$ for some $m \in U$. Let $m - m_1 - m_2 - m$ be a 3-cycle in $GT_U(M)$. Then $m + m_1, m_1 + m_2, m_2 + m \in U$. One can see that $m + m_1 \neq 0$ and $m + m_2 \neq 0$. So $0 - m + m_1 - m + m_2 - 0$ is a 3-cycle in $GT_U(U)$.
(3) If $gr(GT_U(M)) = 4$, then $gr(GT_U(M)) \neq 3$ by part (2) above. So $gr(GT_U(M)) = \infty$ by part (1) above .
(4) Since U is not a submodule of M, so $U \neq M$. Then $U = \bigcup_{i \in I} N_i$, where each N_i is a submodule of M by Proposition 2.4, then $|I| \geq 2$. If $gr(GT_U(U)) = \infty$, then $x + y \in M \setminus U$ for all distinct elements $x, y \in U^*$. So $|N_i| = 2$ for every $i \in I$. Hence the intersection of any two distinct N_i's is $\{0\}$ and so $|I| = 2$. So $U = N_1 \cup N_2$ for prime submodules N_1 and N_2 of M with $N_1 \cap N_2 = 0$ and $|N_1| = |N_2| = 2$. Thus we may assume that $N_1 = \{0, x\}$ and $N_2 = \{0, y\}$ where $2x = 2y = 0$. So $|U| = 3$ and $x + y \notin U$. Thus $0 - x - (x + y) - y - 0$ is a 4-cycle in $GT_U(M)$. Then $gr(GT_U(M)) \leq 4$. Hence $gr(GT_U(M)) = 4$ by part (2) above.
(5) Let $0 \neq r \in Nil(R)$. Assume that $GT_U(M \setminus U)$ contains a cycle, so there is a path $x - y - z$ in $GT_U(M \setminus U)$. If x and z are adjacent vertices in $GT_U(M \setminus U)$, then we are done. So we may assume that x and z are not adjacent in $GT_U(M \setminus U)$. Since $(U :_R M)$ is a multiplicative-prime subset of R, so $(U :_R M) = \bigcup_{i \in I} P_i$ for distinct prime ideals P_i of R by Proposition 2.4 and [12, Theorem 2]. So $0 \neq r \in Nil(R) \subseteq \bigcap_{i \in I} P_i$. Thus $r \in (U :_R M)$. So $rx, ry, rz \in U$ and $rx + x, ry + y$ and $rz + z$ are distinct elements of $M \setminus U$. Clearly $2m = 0$ for every $m \in M$ by assumption. We have split the proof

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 2, p.151-166

into four cases:
Case 1. $ry + y \neq z$ and $rz + z \neq y$. If $ry + y + z \in U$, then $(ry + y) - y - z - (ry + y)$ is a 3-cycle in $GT_U(M \setminus U)$. If $rz + z + y \in U$, then $(rz + z) - z - y - (rz + z)$ is a 3-cycle in $GT_U(M \setminus U)$. So we may assume that $ry + y + z, rz + z + y \notin U$. Then $(ry + y) - y - z - (rz + z) - (ry + y)$ is a 4-cycle in $GT_U(M \setminus U)$.
Case 2. $ry + y = z$ and $rz + z \neq y$. Since $rz + z + y = r(z + y) \in U$, so $(rz + z) - z - y - (rz + z)$ is a 3-cycle in $GT_U(M \setminus U)$.
Case 3. $ry + y \neq z$ and $rz + z = y$. By an argument like that the Case 2, $(ry + y) - y - z - (ry + y)$ is a 3-cycle in $GT_U(M \setminus U)$.
Case 4. $ry + y = z$ and $rz + z = y$. If $rx + x + y \in U$, then $(rx + x) - x - y - (rx + x)$ is a 3-cycle in $GT_U(M \setminus U)$. If $ry + y + x \in U$, then $(ry + y) - y - x - (ry + y)$ is a 3-cycle in $GT_U(M \setminus U)$. So we may assume that $ry + y + x, rx + x + y \notin U$. Thus $(rx + x) - x - y - (ry + y) - (rx + x)$ is a 4-cycle in $GT_U(M \setminus U)$.
(6) Assume that $GT_U(M \setminus U)$ contains a cycle, so there is a path $m - m_1 - m_2$ in $GT_U(M \setminus U)$. We may assume that $m + m_2 \notin U$. Since $m \neq m_2$, so either $m + m_1 \neq 0$ or $m_1 + m_2 \neq 0$. Assume that $m + m_1 \neq 0$. If $2m = 0$, then $m \in U$, since $2 \notin (U :_R M)$ and U is a multiplicative-prime subset of M. Thus $m - m_1 - (-m_1) - (-m) - m$ is a 4-cycle in $GT_U(M \setminus U)$. □

Example 4.8 *(1) Let $R = Z_6$ and $M = R$ as an R-module. Let $U = \{\bar{0}, \bar{2}, \bar{3}, \bar{4}\}$. Then $U = \langle\bar{2}\rangle \cup \langle\bar{3}\rangle$. So, diam$(GT_U(M)) = 2$ and gr$(GT_U(M)) = 3$.*
(2) Let $R = Z_{60}$ and $M = R$ as an R-module. Let $U = \langle\bar{2}\rangle \cup \langle\bar{3}\rangle \cup \langle\bar{5}\rangle$. It is clear that diam$(GT_U(M)) = 2$ and gr$(GT_U(M)) = 3$.

Theorem 4.9 *Let M be a cyclic R-module and U be a proper multiplicative-prime subset of M which is not a submodule of M. Let $U = \bigcup_{i \in I} N_i$ for prime submodule N_i of M. Suppose that $a - b - c$ is a path of length two in $GT_U(M \setminus U)$ for distinct vertices $a, b, c \in M \setminus U$.*
(1) If $2k \in U$ for some $k \in \{a, b, c\}$ and $\bigcap_i N_i \neq \{0\}$, then gr$(GT_U(M \setminus U)) = 3$.
(2) If $2k = 0$ for some $k \in \{a, b, c\}$ and $2 \notin (0 :_R M)$ then gr$(GT_U(M \setminus U)) = 3$.
(3) If $2k \notin U$ for every $k \in \{a, b, c\}$, then gr$(GT_U(M \setminus U)) \leq 4$.

Proof. (1) Suppose that $2k \in U$ for some $k \in \{a, b, c\}$ and there is a $0 \neq h \in \bigcap_i N_i$. Assume $2a \in U$. If $b \neq a + h$, then $a - b - (a + h) - a$ is a cycle of length three in $GT_U(M \setminus U)$. Hence, assume that $b = a + h$. Since

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 2, p.151-166

$(a+h)+c = b+c \in U$ and $h \in \bigcap_i N_i$, we have $a+c \in U$. Thus $a-b-c-a$ is a cycle of length three in $GT_U(M \setminus U)$. Assume $2b \in U$. If $c \neq b+h$, then $b-c-(b+h)-b$ is a cycle of length three in $GT_U(M \setminus U)$. So, let $c = b+h$. Then $a-b-(b+h)-a$ is a cycle of length three in $GT_U(M \setminus U)$. Assume $2c \in U$. If $b \neq c+h$, then $b-c-(c+h)-b$ is a cycle of length three in $GT_U(M \setminus U)$. Thus, let $b = c+h$. Since $a+(c+h) = a+b \in U$ and $h \in \bigcap_i N_i$, we have $a+c \in U$. Hence $a-b-c-a$ is a cycle of length three in $GT_U(M \setminus U)$. Thus $gr(GT_U(M \setminus U)) = 3$.
(2) Suppose that $2k = 0$ for some $k \in \{a,b,c\}$ and $2 \notin (0 :_R M)$. Thus $2 \neq 0$. Since $k \notin N_i$ for every $i \in I$, so $2 \in (N_i :_R M)$. Hence $0 \neq 2M \subseteq \bigcap_{i \in I} N_i$. Therefor $gr(GT_U(M \setminus U)) = 3$ by part(1) above.
(3) Suppose $2k \notin U$ for every $k \in \{a,b,c\}$. Then $z \neq -z$ for every $z \in \{a,b,c\}$. Hence there are distinct $x,y \in \{a,b,c\}$ such that $y \neq -x$. Thus $x-y-(-y)-(-x)-x$ is a 4 cycle in $GT_U(M \setminus U)$; So $gr(GT_U(M \setminus U)) \leq 4$.
□

Theorem 4.10 *Let M be a module over a commutative ring R such that U is a multiplicative-prime subset of M that is not a submodule of M and $H = (U :_R M)$. If $GT_H(R)$ is connected, then $GT_U(M)$ is connected. Moreover if $diam(GT_H(R)) = n$, then $diam(GT_U(M)) \leq 2n$.*

Proof. Let $m \in M$ and $GT_H(R)$ be connected. Then $diam(GT_H(R)) = d(0,1) = n$ by [5, Corollary 3.5]. Then there exists a path $0 - r_1 - r_2 - ... - r_{n-1} - 1$ from 0 to 1 of length n such that $r_{i-1} + r_i \in H$ for each $i = 2, ..., n-1$. So $(r_{i-1} + r_i)M \subseteq U$ for each $i = 2, ..., n-1$. Thus $0 - r_1 m - r_2 m - ... - r_{n-1} m - m$ is a path from 0 to m of length at most n in $GT_U(M)$. The "moreover" statement follows directly from the above arguments.
□

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References

[1] A. Abbasi and Sh. Habibi, *The total graph of a module over a commutative ring with respect to proper submodules*, Journal of Algebra and It's Applications. **11(3)** (2012), 1250048.

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 2, p.151-166

[2] D. D. Anderson and M. Naseer, *Beck's coloring of a commutative ring*, J. Algebra, **159** (1993), 500-514.

[3] D. F. Anderson, M. C. Axtell and J. A. Stickles, Jr., *Zero-divisor graphs in commutative rings*, commutative Algebra, Noetherian and Non-Noetherian Perspectives, eds. M. Fontana, S. E. Kabbaj, B. Olberding and I. Swanson (Springer-Verlag, New York, 2011), pp. 23-45.

[4] D. F. Anderson and A. Badawi, *The total graph of a commutative ring*, J. Algebra, **320** (2008), 2706-2719.

[5] D. F. Anderson and A. Badawi, *The generalized total graph of a commutative ring*, J. Algebra Appl., **12** (2013), 1250212.

[6] D. F. Anderson and P. F. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra, **217** (1999), 437-447.

[7] D. F. Anderson and S. B. Mulay, *On the diameter and girth of a zero-divisor graph*, J. Pure Appl. Algebra, **210** (2007), 543-550.

[8] I. Beck, *Coloring of a commutative ring*, J. Algebra, **116** (1988), 208-226.

[9] S. Ebrahimi Atani and F. Esmaeili Khalil Saraei, *The total graph of a commutative semiring*, An. St. Univ. Ovidius Constanta, **21(2)** (2013), 21-33.

[10] S. Ebrahimi Atani and S. Habibi, *The total torsion element graph of a module over a commutative ring*, An. St. Univ. Ovidius Constant, **19(1)** (2011), 23-34.

[11] F. Esmaeili Khalil Saraei, *The total torsion element graph without the zero element of modules over commutative rings*, J. Korean Math. Soc., **51(4)** (2014), 721-734.

[12] I. Kaplansky, *Commutative Rings*, rev. ed., University of Chicago Press, Chicago, 1974.

[13] Ch. Lu, *Union of prime submodules*, Huston Journal of Mathematics, **23(2)** (1997), 203-213.

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ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 2, p.151-166

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ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 2, p.151-166