

# EDGE-TO-VERTEX DETOUR MONOPHONIC NUMBER OF A GRAPH

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ABSTRACT. For a connected graph  $G = (V, E)$  of order at least three, the monophonic distance  $d_m(u, v)$  is the length of a longest  $u - v$  monophonic path in  $G$ . For subsets  $A$  and  $B$  of  $V$ , the monophonic distance  $d_m(A, B)$  is defined as  $d_m(A, B) = \min\{d_m(x, y) : x \in A, y \in B\}$ . A  $u - v$  path of length  $d_m(A, B)$  is called an  $A - B$  detour monophonic path joining the sets  $A, B \subseteq V$ , where  $u \in A$  and  $v \in B$ . A set  $S \subseteq E$  is called an *edge-to-vertex detour monophonic set* of  $G$  if every vertex of  $G$  is incident with an edge of  $S$  or lies on a detour monophonic joining a pair of edges of  $S$ . The *edge-to-vertex detour monophonic number*  $dm_{ev}(G)$  of  $G$  is the minimum order of its edge-to-vertex detour monophonic sets and any edge-to-vertex detour monophonic set of order  $dm_{ev}(G)$  is an *edge-to-vertex detour monophonic basis* of  $G$ . Certain general properties of these concepts are studied. It is shown that for each pair of integers  $k$  and  $q$  with  $2 \leq k \leq q$ , there exists a connected graph  $G$  of order  $q + 1$  and size  $q$  with  $dm_{ev}(G) = k$ .

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**Key words:** monophonic distance, detour monophonic path, edge-to-vertex detour monophonic set, edge-to-vertex detour monophonic basis, edge-to-vertex detour monophonic number.

## 1. INTRODUCTION

By a graph  $G = (V, E)$  we mean a finite undirected connected graph without loops or multiple edges. The order and size of  $G$  are denoted by  $p$  and  $q$ , respectively. For basic graph theoretic terminology we refer to Harary [1, 5]. For vertices  $x$  and  $y$  in a connected graph  $G$ , the *distance*  $d(x, y)$  is the length of a shortest  $x - y$  path in  $G$ . An  $x - y$  path of length  $d(x, y)$  is called an  $x - y$  *geodesic*. The *neighborhood* of a vertex  $v$  is the set  $N(v)$  consisting of all vertices  $u$  which are adjacent with  $v$ . A vertex  $v$  is an *extreme vertex* if the subgraph induced by its neighbors is complete.

The *detour distance*  $D(u, v)$  between two vertices  $u$  and  $v$  in  $G$  is the length of a longest  $u - v$  path in  $G$ . An  $u - v$  path of length  $D(u, v)$  is called an  $u - v$  *detour*. It is known that  $D$  is a metric on the vertex set  $V$  of  $G$ . The closed detour interval  $I_D[x, y]$  consists of  $x, y$ , and all the vertices in some  $x - y$  detour of  $G$ . For  $S \subseteq V$ ,  $I_D[S]$  is the union of the sets  $I_D[x, y]$  for all  $x, y \in S$ . A set  $S$  of vertices is a *detour set* if  $I_D[S] = V$ , and the minimum cardinality of a detour set is the *detour number*  $dn(G)$ . The concept of detour number was introduced in [2, 3] and further studied in [3, 4].

A *chord* of a path  $P$  is an edge joining two non-adjacent vertices of  $P$ . A path  $P$  is called *monophonic* if it is a chordless path. A longest  $x - y$  monophonic path is called an  $x - y$  *detour monophonic path*. A set  $S$  of vertices of a graph  $G$  is a *detour monophonic*

set if each vertex  $v$  of  $G$  lies on an  $x - y$  detour monophonic path for some  $x, y \in S$ . The minimum cardinality of a detour monophonic set of  $G$  is the *detour monophonic number* of  $G$  and is denoted by  $dm(G)$ . The detour monophonic number of a graph was introduced in [9] and further studied in [10].

An *edge detour monophonic set* of  $G$  is a set  $S$  of vertices such that every edge of  $G$  lies on a detour monophonic path joining some pair of vertices in  $S$ . The *edge detour monophonic number* of  $G$  is the minimum cardinality of its edge detour monophonic sets and is denoted by  $edm(G)$ . An edge detour monophonic set of cardinality  $edm(G)$  is an *edm-set* of  $G$ . The edge detour monophonic number of a graph was introduced and studied in [8].

For any two vertices  $u$  and  $v$  in a connected graph  $G$ , the *monophonic distance*  $d_m(u, v)$  from  $u$  to  $v$  is defined as the length of a longest  $u - v$  monophonic path in  $G$ . The *monophonic eccentricity*  $e_m(v)$  of a vertex  $v$  in  $G$  is  $e_m(v) = \max \{d_m(v, u) : u \in V(G)\}$ . The *monophonic radius*,  $rad_m G$  of  $G$  is  $rad_m(G) = \min \{e_m(v) : v \in V(G)\}$  and the *monophonic diameter*,  $diam_m G$  of  $G$  is  $diam_m(G) = \max \{e_m(v) : v \in V(G)\}$ . A vertex  $u$  in  $G$  is a *monophonic eccentric vertex* of a vertex  $v$  in  $G$  if  $e_m(v) = d_m(u, v)$ . The monophonic distance was introduced in [6] and further studied in [7].

Throughout this paper  $G$  denotes a connected graph with at least three vertices.

## 2. EDGE-TO-VERTEX DETOUR MONOPHONIC NUMBER

**Definition 2.1.** Let  $G = (V, E)$  be a connected graph with at least three vertices. For subsets  $A$  and  $B$  of  $V$ , the *monophonic distance*  $d_m(A, B)$  is defined as  $d_m(A, B) = \min\{d_m(x, y) : x \in A, y \in B\}$ . A  $u - v$  detour monophonic path of length  $d_m(A, B)$  is called an  *$A - B$  detour monophonic path* joining the sets  $A$  and  $B$ , where  $u \in A$  and  $v \in B$ . For  $A = \{u, v\}$  and  $B = \{z, w\}$  with  $uv$  and  $zw$  edges, we write an  $A - B$  detour monophonic path as  $uv - zw$  detour monophonic path, and  $d_m(A, B)$  as  $d_m(uv, zw)$ .

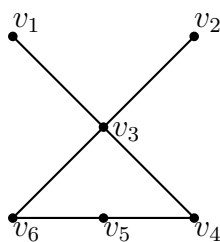


Figure 2.1:  $G$

**Example 2.2.** For the graph  $G$  given in Figure 2.1, with  $A = \{v_1, v_2\}$  and  $B = \{v_4, v_5\}$ ,  $P : v_1, v_3, v_4$  is the only  $v_1 - v_4$  detour monophonic path;  $Q : v_1, v_3, v_4, v_5$  and  $R : v_1, v_3, v_6, v_5$  are the only  $v_1 - v_5$  detour monophonic paths;  $P' : v_2, v_3, v_4$  is the only  $v_2 - v_4$  detour monophonic path,  $Q' : v_2, v_3, v_4, v_5$  and  $R' : v_2, v_3, v_6, v_5$  are the only  $v_2 - v_5$  detour monophonic paths. Hence  $d_m(A, B) = 2$  and  $P : v_1, v_3, v_4$  and  $P' : v_2, v_3, v_4$  are the only two  $A - B$  detour monophonic paths.

**Definition 2.3.** Let  $G = (V, E)$  be a connected graph with at least three vertices. A set  $S \subseteq E$  is called an *edge-to-vertex detour monophonic set* of  $G$  if every vertex of  $G$  is incident with an edge of  $S$  or lies on a detour monophonic path joining a pair of edges of  $S$ . The *edge-to-vertex detour monophonic number*  $dm_{ev}(G)$  of  $G$  is the minimum cardinality of its edge-to-vertex detour monophonic sets and any edge-to-vertex detour monophonic set of cardinality  $dm_{ev}(G)$  is an *edge-to-vertex detour monophonic basis* of  $G$ .

**Example 2.4.** For the graph  $G$  given in Figure 2.2, the four  $v_1v_2-v_4v_5$  detour monophonic paths are  $P_1 : v_1, v_2, v_3, v_4$ ,  $P_2 : v_1, v_6, v_5, v_4$ ,  $Q_1 : v_2, v_3, v_4, v_5$  and  $Q_2 : v_2, v_1, v_6, v_5$ , each of length 3 so that  $d_m(v_1v_2, v_4v_5) = 3$ . Since the vertices  $v_3$  and  $v_6$  lie on the  $v_1v_2 - v_4v_5$  detours monophonic paths  $P_1$  and  $P_2$  respectively,  $S_1 = \{v_1v_2, v_4v_5\}$  is an edge-to-vertex detour monophonic basis of  $G$  so that  $dm_{ev}(G) = 2$ . Also  $S_2 = \{v_2v_3, v_5v_6\}$  and  $S_3 = \{v_3v_4, v_1v_6\}$  are edge-to-vertex detour monophonic bases of  $G$ . Thus there can be more than one edge-to-vertex detour monophonic basis for a graph.

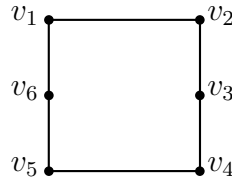


Figure 2.2:  $G$

It is clear that an edge-to-vertex detour monophonic set needs at least two edges, and the set of all edges of  $G$  is an edge-to-vertex detour monophonic set of  $G$ . Hence the following proposition is trivial.

**Proposition 2.5.** For any connected graph  $G$  of size  $q \geq 2$ ,  $2 \leq dm_{ev}(G) \leq q$ .

For the star  $K_{1,q}(q \geq 2)$ , it is clear that the set of all edges is the unique edge-to-vertex detour monophonic set so that  $dm_{ev}(K_{1,q}) = q$ . The set of two end-edges of a path  $P_n(n \geq 3)$  is its unique edge-to-vertex detour monophonic basis so that  $dm_{ev}(P_n) = 2$ . Thus the bounds in Proposition 2.5 are sharp.

**Definition 2.6.** An edge  $e$  in a graph  $G$  is an *edge-to-vertex detour monophonic edge* in  $G$  if  $e$  belongs to every edge-to-vertex detour monophonic basis of  $G$ . If  $G$  has a unique edge-to-vertex detour monophonic basis  $S$ , then every edge in  $S$  is an edge-to-vertex detour monophonic edge of  $G$ .

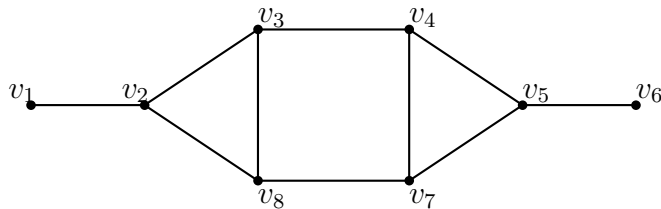


Figure 2.3:  $G$

**Example 2.7.** For the graph  $G$  given in Figure 2.3,  $S = \{v_1v_2, v_5v_6\}$  is the unique edge-to-vertex detour monophonic basis of  $G$  so that both the edges in  $S$  are edge-to-vertex detour monophonic edge of  $G$ . For the graph  $G$  given in Figure 2.1, it is easily verified that no two element subset of  $E$  is an edge-to-vertex detour monophonic set of  $G$ . Also, it is clear that  $S_1 = \{v_1v_3, v_2v_3, v_4v_5\}$  and  $S_2 = \{v_1v_3, v_2v_3, v_5v_6\}$  are the only edge-to-vertex detour monophonic bases of  $G$  so that the edges  $v_1v_3, v_2v_3$  are the edge-to-vertex detour monophonic edges of  $G$ .

An edge of a connected graph  $G$  is called an *extreme edge* of  $G$  if one of its ends is an extreme vertex of  $G$ .

**Theorem 2.8.** *If  $v$  is an extreme vertex of a non-complete connected graph  $G$ , then every edge-to-vertex detour monophonic set of  $G$  contains at least one extreme edge that is incident with  $v$ .*

*Proof.* Let  $v$  be an extreme vertex of  $G$ . Let  $e_1, e_2, \dots, e_k$  be the edges incident with  $v$ . Let  $S$  be any edge-to-vertex detour monophonic set of  $G$ . We claim that  $e_i \in S$  for some  $i(1 \leq i \leq k)$ . Otherwise,  $e_i \notin S$  for any  $i(1 \leq i \leq k)$ . Since  $S$  is an edge-to-vertex detour monophonic set and the vertex  $v$  is not incident with any element of  $S$ ,  $v$  lies on a detour monophonic path joining two elements say  $x, y \in S$ . Let  $x = v_1v_2$  and  $y = v_l v_m$ . Then  $v \neq v_1, v_2, v_l, v_m$  and since  $G$  is non-complete,  $d_m(x, y) \geq 2$ . Let  $u$  and  $w$  be the neighbors of  $v$  on  $P$ . Then  $u$  and  $w$  are not adjacent and so  $v$  is not an extreme vertex, which is a contradiction. Therefore,  $e_i \in S$  for some  $i(1 \leq i \leq k)$ .  $\square$

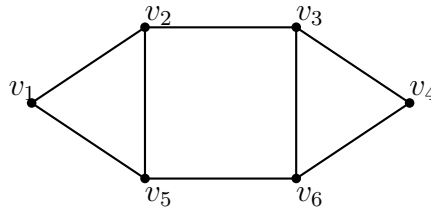


Figure 2.4:  $G$

**Remark 2.9.** For the graph  $G$  given in Figure 2.4,  $S = \{v_1v_5, v_3v_4\}$  is an edge-to-vertex detour monophonic set of  $G$ , which does not contain the extreme edge  $v_1v_2$ . Thus all the extreme edges of a graph need not belong to an edge-to-vertex detour monophonic set of  $G$ .

In the following theorem we show that there are certain edges in a connected graph  $G$  that are edge-to-vertex detour monophonic edges of  $G$ .

**Corollary 2.10.** *Every end-edge of a connected graph  $G$  belongs to every edge-to-vertex detour monophonic set of  $G$ . Also if the set  $S$  of all end-edges of  $G$  is an edge-to-vertex detour monophonic set, then  $S$  is the unique edge-to-vertex detour monophonic basis for  $G$ .*

*Proof.* This follows from Theorem 2.8. If  $S$  is the set of all end-edges of  $G$ , then by the first part of this corollary  $dm_{ev}(G) \geq |S|$ . Since  $S$  is an edge-to-vertex detour monophonic

set of  $G$ ,  $dm_{ev}(G) \leq |S|$ . Hence  $dm_{ev}(G) = |S|$  and  $S$  is the unique edge-to-vertex detour monophonic basis for  $G$ .  $\square$

**Corollary 2.11.** *If  $T$  is a tree with  $k$  end-edges, then  $dm_{ev}(T) = k$ .*

**Corollary 2.12.** *For any connected graph  $G$  with  $k$  end-edges,  $\max\{2, k\} \leq dm_{ev}(G) \leq q$ .*

*Proof.* This follows from Proposition 2.5 and Corollary 2.10.  $\square$

For a cutvertex  $v$  in a connected graph  $G$  and a component  $H$  of  $G - v$ , the subgraph  $H$  and the vertex  $v$  together with all edges joining  $v$  and  $V(H)$  is called a *branch* of  $G$  at  $v$ .

**Theorem 2.13.** *Let  $G$  be a connected graph with cutvertices and  $S$  an edge-to-vertex detour monophonic set of  $G$ . Then every branch of  $G$  contains an element of  $S$ .*

*Proof.* Assume that there is a branch  $B$  of  $G$  at a cutvertex  $v$  such that  $B$  contains no element of  $S$ . Then by Corollary 2.10,  $B$  does not contain any end-edge of  $G$ . Hence it follows that no vertex of  $B$  is an endvertex of  $G$ . Let  $u$  be any vertex of  $B$  (note that  $|V(B)| \geq 2$ ). Then  $u$  is not incident with any end-edge of  $G$  and so  $u$  lies on a  $e - f$  detour monophonic path  $P : u_1, u_2, \dots, u, \dots, u_t$  where  $u_1$  is an end of  $e$ ,  $u_t$  is an end of  $f$  and  $e, f \in S$ . Since  $v$  is a cutvertex of  $G$ , the  $u_1 - u$  and  $u - u_t$  subpaths of  $P$  both contain  $v$  and so  $P$  is not a path, which is a contradiction. Hence every branch of  $G$  contains an element of  $S$ .  $\square$

**Corollary 2.14.** *Let  $G$  be a connected graph with cut-edges and  $S$  an edge-to-vertex detour monophonic set of  $G$ . Then every branch of  $G$  contains an element of  $S$ .*

**Corollary 2.15.** *Let  $G$  be a connected graph with cut-edges and  $S$  an edge-to-vertex detour monophonic set of  $G$ . Then for any cut-edge  $e$  of  $G$ , which is not an end-edge, each component of  $G - e$  contains an element of  $S$ .*

*Proof.* Let  $e = uv$ . Let  $G_1$  and  $G_2$  be the two components of  $G - e$  such that  $u \in V(G_1)$  and  $v \in V(G_2)$ . Since  $u$  and  $v$  are cutvertices of  $G$ , the result follows from Theorem 2.13.  $\square$

**Corollary 2.16.** *If  $G$  is a connected graph with  $k \geq 2$  endblocks, then  $dm_{ev}(G) \geq k$ .*

**Corollary 2.17.** *If  $G$  is a connected graph with a cutvertex  $v$  and the number of components of  $G - v$  is  $r$ , then  $dm_{ev}(G) \geq r$ .*

**Remark 2.18.** By Corollary 2.16, if  $S$  is an edge-to-vertex detour monophonic set of a graph  $G$ , then every endblock of  $G$  must contain at least one element of  $S$ . However, it is possible that some blocks of  $G$  that are not endblocks must contain an element of  $S$  as well. For example, consider the graph  $G$  given in Figure 2.5, where the cycle  $C_5 : x, y, t, w, s, x$  is a block of  $G$  that is not an endblock. By Corollary 2.10, every edge-to-vertex detour monophonic set of  $G$  must contain  $us$  and  $tv$ . Since the  $us - tv$  detour monophonic path does not contain the vertex  $w$ , it follows that  $\{us, tv\}$  is not an edge-to-vertex detour monophonic set of  $G$ . Thus every edge-to-vertex detour monophonic set of  $G$  must contain at least one edge from the block  $C_5$ .

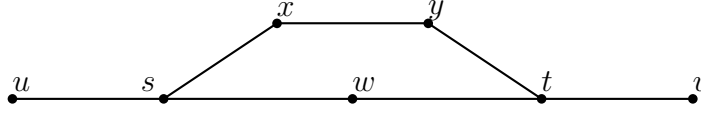


Figure 2.5:  $G$

**Theorem 2.19.** *Let  $G$  be a connected graph with cut-edges. Then no cut-edge which is not an end-edge in  $G$  belongs to any edge-to-vertex detour monophonic basis of  $G$ .*

*Proof.* Suppose that  $S$  is an edge-to-vertex detour monophonic basis that contains a cut-edge  $e = uv$  which is not an end-edge of  $G$ . Let  $G_1, G_2$  be the two components of  $G - e$  such that  $u \in G_1$  and  $v \in G_2$ . Then by Corollary 2.15, each of  $G_1$  and  $G_2$  contains an element of  $S$ . Let  $S' = S - \{uv\}$ . We show that  $S'$  is an edge-to-vertex detour monophonic set of  $G$ . Since  $S$  is an edge-to-vertex detour monophonic set of  $G$  and  $uv \in S$ , let  $s$  be any vertex of  $G$  that lies on a detour monophonic path  $P$  joining the edges, say  $xy$  and  $uv$  of  $S$ . We may assume that  $xy \in E(G_1)$  and so  $V(P) \subseteq V(G_1)$ . Let  $P_1$  be the  $xy - uv$  detour monophonic path that contains the vertex  $s$  and  $P_2$  be any  $uv - wz$  detour monophonic path in  $G$ , where  $wz \in E(G_2) \cap S$ . Then, since  $uv$  is a cut-edge of  $G$ , the detour monophonic path  $P_1$  followed by the edge  $uv$  and the detour monophonic path  $P_2$  is an  $xy - wz$  detour monophonic path which contains the vertex  $s$ . Thus we have shown that a vertex that lies on a detour monophonic path joining a pair of edges  $xy$  and  $uv$  of  $S$  also lies on a detour monophonic path joining a pair of edges  $xy$  and  $wz$  of  $S'$ . Hence it follows that  $S'$  is an edge-to-vertex detour monophonic set of  $G$ . Since  $|S'| = |S| - 1$ , this contradicts that  $S$  is an edge-to-vertex detour monophonic basis of  $G$ . Thus the result is proved.  $\square$

### 3. EDGE-TO-VERTEX DETOUR MONOPHONIC NUMBERS OF SOME STANDARD GRAPHS

**Theorem 3.1.** *For  $p$  even, a set  $S$  of edges of  $G = K_p$  ( $p \geq 4$ ) is an edge-to-vertex detour monophonic basis of  $K_p$  if and only if  $S$  consists of  $p/2$  independent edges.*

*Proof.* Let  $S$  be any set of  $p/2$  independent edges of  $K_p$ . Since each vertex of  $K_p$  is incident with an edge of  $S$ , it follows that  $dm_{ev}(G) \leq p/2$ . If  $dm_{ev}(G) < p/2$ , then there exists an edge-to-vertex detour monophonic set  $S'$  of  $K_p$  such that  $|S'| < p/2$ . Therefore, there exists at least one vertex  $v$  of  $K_p$  such that  $v$  is not incident with any edge of  $S'$ . Since  $d_m(e, f) = 1$  if  $e$  and  $f$  are independent edges, it follows that  $v$  is neither incident with any edge of  $S'$  nor lies on a detour monophonic path joining a pair of edges of  $S'$  and so  $S'$  is not an edge-to-vertex detour monophonic set of  $G$ , which is a contradiction. Thus  $S$  is an edge-to-vertex detour monophonic basis of  $K_p$ .

Conversely, let  $S$  be an edge-to-vertex detour monophonic basis of  $K_p$ . Let  $S'$  be any set of  $p/2$  independent edges of  $K_p$ . Then as in the first part of this theorem,  $S'$  is an edge-to-vertex detour monophonic basis of  $K_p$ . Therefore,  $|S| = p/2$ . If  $S$  is not independent, then there exists a vertex  $v$  of  $K_p$  such that  $v$  is not incident with any edge

of  $S$  and it follows that  $S$  is not an edge-to-vertex detour monophonic set of  $G$ , which is a contradiction. Therefore,  $S$  consists of  $p/2$  independent edges.  $\square$

**Corollary 3.2.** *For the complete graph  $K_p(p \geq 4)$  with  $p$  even,  $dm_{ev}(K_p) = p/2$ .*

For any real  $x$ ,  $\lceil x \rceil$  denotes the smallest integer greater than or equal to  $x$ .

**Theorem 3.3.** *For the complete graph  $G = K_p(p \geq 3)$  with  $p$  odd,  $dm_{ev}(G) = \frac{p+1}{2}$ .*

*Proof.* Let  $S$  be any set of  $\frac{p-1}{2}$  independent edges of  $G$ . Then there exists a unique vertex  $v$  which is not incident with an edge of  $S$ . Let  $S_1$  be the union of  $S$  and an edge incident with  $v$ . Then  $S_1$  is an edge-to-vertex detour monophonic set of  $G$  so that  $dm_{ev}(G) \leq \frac{p-1}{2} + 1$ . Now, if  $dm_{ev}(G) \leq \frac{p-1}{2}$ , then let  $S_2$  be an edge-to-vertex detour monophonic set of  $G$  such that  $|S_2| \leq \frac{p-1}{2}$ . Then there exists a vertex  $u$  such that  $u$  is not incident with any edge of  $S_2$ . Obviously,  $u$  does not lie on a detour monophonic path joining a pair of edges of  $S_2$ , which is a contradiction to  $S_2$  an edge-to-vertex detour monophonic set of  $G$ . Hence  $dm_{ev}(G) = \frac{p-1}{2} + 1 = \frac{p+1}{2}$ .  $\square$

**Corollary 3.4.** *For the complete graph  $K_p(p \geq 3)$ ,  $dm_{ev}(K_p) = \lceil \frac{p}{2} \rceil$ .*

Two vertices  $u$  and  $v$  of  $G$  are called *antipodal* if  $d(u, v) = \text{diam } G$ , where  $\text{diam } G$  is the usual diameter of the graph  $G$ .

**Theorem 3.5.** *For the cycle  $C_p(p \geq 3)$ ,  $dm_{ev}(C_p) = \begin{cases} 2 & \text{if } p \neq 5 \\ 3 & \text{if } p = 5. \end{cases}$*

*Proof.* For  $p = 3$ ,  $C_p = K_3$  and any set of two edges is an edge-to-vertex detour monophonic basis and so  $dm_{ev}(G) = 2$ .

Let  $p \geq 4$  and  $p \neq 5$ . Let  $C_p : v_1, v_2, v_3, \dots, v_k, v_{k+1}, v_{k+2}, \dots, v_p, v_1$  be the cycle of order  $p$  such that  $v_{k+1}$  is the unique antipodal vertex of  $v_1$  if  $p$  is even; and  $v_{k+1}$  and  $v_{k+2}$  are the antipodal vertices of  $v_1$  if  $p$  is odd. Then it is easily checked that  $S = \{v_1v_2, v_{k+1}v_{k+2}\}$  is an edge-to-vertex detour monophonic set of  $C_p$  so that  $dm_{ev}(C_p) = 2$ .

For  $p = 5$ , it is easily seen that no 2-element subset of edges of  $C_5$  is an edge-to-vertex detour monophonic set of  $C_5$  since  $d_m(e, f) = 1$  if  $e$  and  $f$  are two independent edges in  $C_5$ . Also, since  $S = \{v_1v_2, v_2v_3, v_4v_5\}$  is an edge-to-vertex detour monophonic set of  $C_5$ , it follows that  $dm_{ev}(C_5) = 3$ .  $\square$

#### 4. MONOPHONIC DIAMETER AND EDGE-TO-VERTEX DETOUR MONOPHONIC NUMBER

**Theorem 4.1.** *For each pair of integers  $k$  and  $q$  with  $2 \leq k \leq q$ , there exists a connected graph  $G$  of order  $q + 1$  and size  $q$  with  $dm_{ev}(G) = k$ .*

*Proof.* For  $2 \leq k \leq q$ , let  $P$  be a path of order  $q - k + 3$ . Then the graph  $G$  obtained from  $P$  by adding  $k - 2$  new vertices to  $P$  and joining them to any cutvertex of  $P$  is a tree of order  $q + 1$  and size  $q$  with  $k$  end-edges and so by Corollary 2.11,  $dm_{ev}(G) = k$ .  $\square$

Proposition 2.5 shows that if  $G$  is a connected graph of size  $q \geq 2$ , then  $2 \leq dm_{ev}(G) \leq q$ . Indeed, by Theorem 4.1, for each pair  $k, q$  of integers with  $2 \leq k \leq q$ , there is a tree of size  $q$  with edge-to-vertex detour monophonic number  $k$ . An improved upper bound for the edge-to-vertex detour monophonic number of a graph can be given in terms of its size  $q$  and detour monophonic diameter. For convenience, we denote the detour monophonic diameter  $diam_m(G)$  by  $d_m$  itself.

**Theorem 4.2.** *If  $G$  is a connected graph of size  $q$  and monophonic diameter  $d_m$ , then  $dm_{ev}(G) \leq q - d_m + 2$ .*

*Proof.* Let  $u$  and  $v$  be vertices of  $G$  such that  $d_m(u, v) = d_m$  and let  $P : u = v_0, v_1, v_2, \dots, v_{d_m-1}, v_{d_m} = v$  be a  $u - v$  detour monophonic path of length  $d_m$ . Let  $S = (E(G) - E(P)) \cup \{uv_1, v_{d_m-1}v\}$ . Then it is clear that  $S$  is an edge-to-vertex detour monophonic set of  $G$  so that  $dm_{ev}(G) \leq |S| = q - d_m + 2$ .  $\square$

We give below a characterization theorem for trees.

**Theorem 4.3.** *For any tree  $T$  of size  $q \geq 2$  and monophonic diameter  $d_m$ ,  $dm_{ev}(T) = q - d_m + 2$  if and only if  $T$  is a caterpillar.*

*Proof.* Let  $T$  be any tree of size  $q \geq 2$  and  $P : v_0, v_1, \dots, v_{d_m-1}, v_{d_m}$  be a monophonic diametral path of  $T$ . Let  $e_1, e_2, \dots, e_{d_m-1}, e_{d_m}$  be the edges of  $P$ , where  $e_i = v_{i-1}v_i$  ( $1 \leq i \leq d_m$ ),  $k$  the number of end-edges of  $T$  and  $l$  the number of internal edges of  $T$  other than  $e_2, \dots, e_{d_m-1}$ . Then  $k + l + d_m - 2 = q$ . By Corollary 2.11,  $dm_{ev}(T) = k = q - d_m - l + 2$ . Hence  $dm_{ev}(T) = k = q - d_m + 2$  if and only if  $l = 0$ , if and only if all the internal edges of  $T$  lie on the monophonic diametral path  $P$ , if and only if  $T$  is a caterpillar.  $\square$

**Corollary 4.4.** *For a wounded spider  $T$  of size  $q \geq 2$ ,  $dm_{ev}(T) = q - d_m + 2$  if and only if  $T$  is obtained from  $K_{1,t}$  ( $t \geq 2$ ) by subdividing at most two of its edges.*

*Proof.* Since a wounded spider  $T$  is a caterpillar if and only if  $T$  is obtained from  $K_{1,t}$  ( $t \geq 2$ ) by subdividing at most two of its edges, the result follows from Theorem 4.3.  $\square$

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