ON THE HYPER-ORDER OF SOLUTIONS OF SOME HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS WITH ENTIRE COEFFICIENTS

MANSOURIA SAIDANI AND BENHARRAT BELAÏDI

Abstract. In this paper, we deal with the growth of solutions of certain type of higher order nonhomogeneous linear differential equations with entire coefficients having the same order. Under some conditions on the coefficients, we prove that every solution is of infinite order with bounded hyper order. Our results extend the previous results in references [4] and [11].

Mathematics Subject Classification (2010): 34M10, 30D35.
Key words: order of growth, hyper-order, differential equation, entire function.

Article history:
Received 27 January 2019
Received in revised form 27 June 2019
Accepted 09 July 2019

1. Introduction and main results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory (see [13], [15], [21]) . Let $\rho(f)$ denote the order of growth of a meromorphic function f and the hyper-order of f is defined by

$$\rho_2(f) := \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r},$$

where $T(r,f)$ is the Nevanlinna characteristic function of f(see [13], [19], [21]).

Definition 1.1 (see [15], [19]) Let f be a meromorphic function. Then, the convergence exponent of the zero-sequence of a meromorphic function f is defined by

$$\lambda(f) := \limsup_{r\to+\infty} \frac{\log N\left(r,\frac{1}{f}\right)}{\log r},$$

where $N\left(r,\frac{1}{f}\right)$ is the integrated counting function of zeros of f in $\{z : |z| \leq r\}$, and the exponent of convergence of the sequence of distinct zeros of f is defined by

$$\overline{\lambda}(f) := \limsup_{r\to+\infty} \frac{\log \overline{N}\left(r,\frac{1}{f}\right)}{\log r},$$

where $\overline{N}\left(r,\frac{1}{f}\right)$ is the integrated counting function of distinct zeros of f in $\{z : |z| \leq r\}$. The hyper convergence exponents of zero-sequence and distinct zeros of f are defined respectively by

$$\lambda_2(f) := \limsup_{r\to+\infty} \frac{\log\log N\left(r,\frac{1}{f}\right)}{\log r},\ \overline{\lambda}_2(f) := \limsup_{r\to+\infty} \frac{\log\log \overline{N}\left(r,\frac{1}{f}\right)}{\log r}.$$

Several authors [6] , [9], [14] have studied the growth of solutions of the second order linear differential equation

$$f'' + A_1(z)\, e^{P(z)} f' + A_2(z)\, e^{Q(z)} f = 0, \tag{1.1}$$

where $P(z)$, $Q(z)$ are nonconstant polynomials, $A_1(z)$, $A_2(z)$ $(\not\equiv 0)$ are entire functions such that $\rho(A_1) < \deg P(z)$, $\rho(A_2) < \deg Q(z)$. Gundersen showed in [9] that if $\deg P(z) \neq \deg Q(z)$, then every nonconstant solution of $(1,1)$ is of infinite order. If $\deg P(z) = \deg Q(z)$, then $(1,1)$ may have nonconstant solutions of finite order. For instance $f(z) = e^{iz} + i$ satisfies $f'' + e^{iz} f' - ie^{iz} f = 0$.

In [1], Belaïdi investigated the order and hyper-order of solutions of some higher order linear differential equations as follows.

Theorem A ([1]) *Let $k \geq 2$ be an integer and $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i$ $(j = 0, 1, \cdots, k-1)$ be nonconstant polynomials, where $a_{0,j}, \cdots, a_{n,j}$ $(j = 0, \cdots, k-1)$ are complex numbers such that $a_{n,j} \neq 0 (j = 0, \cdots, k-1)$. Let $A_j(z) (\not\equiv 0)$, $B_j(z)$ $(\not\equiv 0)$ $(j = 0, \cdots, k-1)$ be entire functions. Suppose that $\arg a_{n,j} \neq \arg a_{n,0}$ or $a_{n,j} = c_j a_{n,0} (0 < c_j < 1)(j = 1, \cdots, k-1)$ and $\rho(A_j) < n, \rho(B_j) < n$ $(j = 0, \cdots, k-1)$. Then every solution f $(\not\equiv 0)$ of the differential equation*

$$f^{(k)} + \left(A_{k-1}(z)\, e^{P_{k-1}(z)} + B_{k-1}(z)\right) f^{(k-1)} + \cdots + \left(A_0(z)\, e^{P_0(z)} + B_0(z)\right) f = 0 \tag{1.2}$$

is of infinite order and satisfies $\rho_2(f) = n$.

After that, Hamani in [12] determined the hyper-order of solutions of equation (1.2), under certain conditions and obtained the following result.

Theorem B ([12]) *Let $k \geq 2$ be an integer and $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i (j = 0, 1, \cdots, k-1)$ be nonconstant polynomials, where $a_{0,j}, \cdots, a_{n,j}$ $(j = 0, 1, \cdots, k-1)$ are complex numbers such that $a_{n,j} \neq 0$ $(j = 0, 1, \cdots, k-1)$. Let $A_j(z) (\not\equiv 0)$, $B_j(z)$ $(\not\equiv 0)$ $(j = 0, 1, \cdots, k-1)$ be entire functions with $\rho(A_j) < n$ and $\rho(B_j) < n$. Suppose that there exists $s \in \{1, \cdots, k-1\}$ such that $a_{n,j} = c_j a_{n,s} (0 < c_j < 1)(j \neq s)$. Then every transcendental solution f of (1.2) is of infinite order and satisfies $\rho_2(f) = n$.*

In [22], Zhan and Xiao have investigated the homogeneous and nonhomogeneous higher order differential equations and obtained the following results.

Theorem C ([22]) *Let $A_{ji}(z)$ $(\not\equiv 0)$ be entire functions with $\rho(A_{ji}) < n$, $n \geq 1$ is an integer, $j = 0, 1, \cdots, k-1; i = 1, 2$. Let $P_j(z) = a_{j,n} z^n + \cdots + a_{j,0}$ and $Q_j(z) = b_{j,n} z^n + \cdots + b_{j,0}$ be polynomials, where $a_{j,q}, b_{j,q}$ $(j = 0, 1, \cdots, k-1; q = 0, 1, \cdots, n)$ are complex numbers such that $a_{j,n} b_{j,n} \neq 0$, $a_{0,n} \neq b_{0,n}$ and $a_{j,n} = c_j a_{0,n}$, $b_{j,n} = c_j b_{0,n}$, $c_j > 1$, $j = 1, \cdots, k-1$ are distinct numbers. Then every solution $f\,(\not\equiv 0)$ of the equation*

$$f^{(k)} + \left(A_{k-1,1}(z)\, e^{P_{k-1}(z)} + A_{k-1,2}(z)\, e^{Q_{k-1}(z)}\right) f^{(k-1)}$$
$$+ \cdots + \left(A_{0,1}(z)\, e^{P_0(z)} + A_{0,2}(z)\, e^{Q_0(z)}\right) f = 0$$

is of finite order.

Theorem D ([22]) *Let $A_{ji}(z)$ $(\not\equiv 0)$ be entire functions with $\rho(A_{ji}) < n$, where $n \geq 1$ is an integer, $j = 0, 1, \cdots, k-1; i = 1, 2$. Let $P_j(z) = a_{j,n}z^n + \cdots + a_{j,0}$ and $Q_j(z) = b_{j,n}z^n + \cdots + b_{j,0}$ be polynomials, where $a_{j,q}, b_{j,q}$ $(j = 0, 1, \cdots, k-1; q = 0, 1, \cdots, n)$ are complex numbers such that $a_{j,n}b_{j,n} \neq 0$, $a_{0,n} \neq b_{0,n}$ and $a_{j,n} = c_j a_{0,n}$, $b_{j,n} = c_j b_{0,n}$, $c_j > 1$, $j = 1, \cdots, k-1$ are distinct numbers, $F(z)\,(\not\equiv 0)$ is an entire function of a finite order. Then the equation*

$$f^{(k)} + \left(A_{k-1,1}(z)\, e^{P_{k-1}(z)} + A_{k-1,2}(z)\, e^{Q_{k-1}(z)}\right) f^{(k-1)}$$

$$+ \cdots + \left(A_{0,1}(z)\, e^{P_0(z)} + A_{0,2}(z)\, e^{Q_0(z)}\right) f = F(z) \tag{1.3}$$

satisfies the following statements:
(i) *There exists at most one exceptional solution f_0 with finite order, and all other solutions satisfy $\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty$ and $\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) \leq \max\{n, \rho(F)\}$.*
(ii) *If there exists an f_0 of a finite order, then $\rho(f_0) \leq \max\left\{n, \overline{\lambda}(f_0), \rho(F)\right\}$.*
(iii) *If $F(z)$ is an entire function of order less than n and $\arg a_{0,n} \neq \arg b_{0,n}$, then each solution of (1.3) is of infinite order.*

Very recently, Habib and Belaïdi [11] have investigated the growth of solutions of non-homogeneous linear differential equations of type

$$f^{(k)} + A_{k-1}(z)\, e^{az} f^{(k-1)} + \cdots + A_0(z)\, e^{az} f = F_1(z)e^{az} + F_2(z)e^{bz} \tag{1.4}$$

and they obtained the following result.

Theorem E ([11]) *Let $A_j(z)(\not\equiv 0)$ $(j = 0, 1, \cdots, k-1)$, $F_i(z)$ $(i = 1, 2)$ be entire functions with $\rho(A_j) < 1$ $(j = 0, 1, \cdots, k-1)$ and $\rho(F_i) < 1$ $(i = 1, 2)$, a and b be non-zero complex numbers such that $b = ca$ $(0 < c < 1)$. Suppose the following:*
(i) *there is exactly one $s\,(0 \leq s \leq k-1)$ such that*

$$\max\{\rho(A_j) \quad (j = 0, 1, \cdots, k-1), j \neq s\} < \rho(A_s) = \rho,$$

(ii) *for any τ satisfying $0 < \tau < \rho$, there exists a subset $H \subset (1, +\infty)$ with infinite logarithmic measure, such that $|z| = r \in H$,*

$$\log|A_s(z)| > r^{\tau},$$

then every solution f of equation (1.4) is of infinite order.

In [4], they obtained the following result.

Suppose that

$$\begin{aligned} I &= \{0, 1, \cdots, k-1\}, \\ I_1 &= \{i \in I, c_i > 1\} \neq \emptyset, \\ I_2 &= \{i \in I, 0 < c_i < 1\} \neq \emptyset, \\ I_3 &= \{i \in I, c_i < 0\} \neq \emptyset, \\ I_4 &= \{i \in I, c_i = 1\} \neq \emptyset, \end{aligned}$$

where $I_1 \cup I_2 \cup I_3 \cup I_4 = I$ and c_i $(i \in I)$ are real numbers.

Theorem F ([4]) *Let $A_j(z)(\not\equiv 0)$ $(j \in I)$, $F(z) \not\equiv 0$ be entire functions with $\max\{\rho(A_j) \quad (j \in I), \rho(F)\} < 1$, $a \neq 0$ and $b_i \neq 0$ $(i \in I)$ be complex numbers such that $b_i = c_i a$ $(i \in I)$. Suppose that there is one*

$s\in I_1$ such that $c_s>c_j$ for all $j\in I_1\backslash\{s\}$, suppose that there is one $l\in I_3$ such that $c_l<c_j$ for all $j\in I_3\backslash\{l\}$ and suppose that $c_0\neq 1$ and $c_0\neq c_j$ for all $j\in I\backslash\{0\}$. Then, every solution f of the differential equation

$$f^{(k)}+A_{k-1}(z)e^{b_{k-1}z}f^{(k-1)}+\cdots+A_1(z)e^{b_1z}f+A_0(z)e^{b_0z}f=F(z)e^{az}$$

is of infinite order and the hyper-order satisfies $\rho_2(f)\leq 1$.

In this paper, we are concerned with the more general problem and we obtain the following results which extend and improve Theorem E and Theorem F. In fact we will prove the following results.

Theorem 1.1 *Let $P(z)=a_nz^n+\cdots+a_0$ and $Q(z)=b_nz^n+\cdots+b_0$, be polynomials $(n\geq 1)$, where a_q, b_q, $(q=0,1,\cdots,n)$ are complex numbers such that $a_nb_n\neq 0$ and $b_q=ca_q$, $(0<c<1)$, $q=0,\cdots,n$. Let $A_j(z)(\not\equiv 0)$ $(j=0,1,\cdots,k-1)$, $F_i(z)$ $(i=1,2)$ be entire functions of finite order with*

$$\max\{\rho(A_j)\ \ (j=0,1,\cdots,k-1),\rho(F_i)\,(i=1,2)\}<n.$$

Suppose there is exactly one s, $0\leq s\leq k-1$ satisfying

$$\max\{\rho(A_j)\ \ (j=0,1,\cdots,k-1),j\neq s\}<\rho(A_s)<+\infty.$$

Then every solution f of equation

$$f^{(k)}+A_{k-1}(z)e^{P(z)}f^{(k-1)}+\cdots+A_0(z)e^{P(z)}f=e^{P(z)}F_1(z)+e^{Q(z)}F_2(z)\tag{1.5}$$

is of infinite order and the hyper-order satisfies $\rho_2(f)\leq n$.

Theorem 1.2 *Under the hypotheses of Theorem 1.1, suppose further that $\varphi(z)\not\equiv 0$ is an entire function with finite order. Then every solution f of (1.5) satisfies*

$$\overline{\lambda}(f)=\lambda(f)=\overline{\lambda}(f-\varphi)=\lambda(f-\varphi)=\rho(f)=+\infty$$

and

$$\overline{\lambda}_2(f)=\lambda_2(f)=\overline{\lambda}_2(f-\varphi)=\lambda_2(f-\varphi)=\rho_2(f)\leq n.$$

Theorem 1.3 *Let $A_j(z)(\not\equiv 0)$, $(j=0,1,\cdots,k-1)$, $F(z)\not\equiv 0$ be entire functions of finite order with*

$$\max\{\rho(A_j)\ \ (j\in I),\rho(F)\}<n.$$

Let $P_j(z)=a_{j,n}z^n+\cdots+a_{j,0}$ and $Q(z)=b_nz^n+\cdots+b_0$ be polynomials, where $a_{j,q},b_q$ $(j\in I;q=0,1,\cdots,n)$ are complex numbers such that $a_{j,n}b_n\neq 0$ $(j\in I)$ and $a_{j,q}=c_jb_q$ $(j\in I,q=0,1,\cdots,n)$. Suppose that there is one $s\in I_1$ such that $c_s>c_j$ for all $j\in I_1\backslash\{s\}$, suppose that there is one $l\in I_3$ such that $c_l<c_j$ for all $j\in I_3\backslash\{l\}$ and suppose that $c_0\neq 1$ and $c_0\neq c_j$ for all $j\in I\backslash\{0\}$. Then every solution f of the equation

$$f^{(k)}+A_{k-1}(z)e^{P_{k-1}(z)}f^{(k-1)}+\cdots+A_1(z)e^{P_1(z)}f'+A_0(z)e^{P_0(z)}f=e^{Q(z)}F(z)\tag{1.6}$$

is of infinite order and the hyper-order satisfies $\rho_2(f)\leq n$.

Theorem 1.4 *Under the hypotheses of Theorem 1.3, suppose further that $\varphi(z)\not\equiv 0$ is an entire function with finite order. Then every solution f of (1.6) satisfies*

$$\overline{\lambda}(f)=\lambda(f)=\overline{\lambda}(f-\varphi)=\lambda(f-\varphi)=\rho(f)=+\infty$$

and

$$\overline{\lambda}_2(f)=\lambda_2(f)=\overline{\lambda}_2(f-\varphi)=\lambda_2(f-\varphi)=\rho_2(f)\leq n.$$

2. Lemmas for the proofs of the theorems

First, we recall the following definitions. The linear measure of a set $E \subset [0, +\infty)$ is defined as $m(E) = \int_0^{+\infty} \chi_E(t)\, dt$ and the logarithmic measure of a set $F \subset [1, +\infty)$ is defined by $lm(F) = \int_1^{+\infty} \frac{\chi_F(t)}{t} dt$, where $\chi_H(t)$ is the characteristic function of a set H. The upper density of a set $E \subset [0, +\infty)$ is defined by

$$\overline{dens} E = \limsup_{r \to +\infty} \frac{m(E \cap [0, r])}{r}$$

and the upper logarithmic density of a set $F \subset [1, +\infty)$ is defined by

$$\overline{\log dens} F = \limsup_{r \to +\infty} \frac{lm(F \cap [1, r])}{\log r}$$

For all $H \subset [1, +\infty)$ the following statements hold ([3]):
(i) If $lm(H) = \infty$, then $m(H) = \infty$;
(ii) If $\overline{dens} H > 0$, then $m(H) = \infty$;
(iii) If $\overline{\log dens} H > 0$, then $lm(H) = \infty$.

In order to prove our theorems, we need the following lemmas.

Lemma 2.1 ([8], [17]) *Let $P_1(z), P_2(z), \cdots, P_n(z)\, (n \geq 1)$ be non-constant polynomials with degree $d_1, d_2, \cdots, d_n$, respectively, such that $\deg(P_i(z) - P_j(z)) = \max\{d_i, d_j\}$ for $i \neq j$. Let $A(z) = \sum_{j=0}^{n} B_j(z) e^{P_j(z)}$, where $B_j(z) (\not\equiv 0)$ are entire functions with $\rho(B_j) < d_j$. Then $\rho(A) = \max_{0 \leq j \leq n}\{d_j\}$.*

Lemma 2.2 ([23]) *Let f be an entire function with $\rho(f) = \rho$, $(0 < \rho < \infty)$. Then for any given $\tau < \rho$, there exists a set $H \subset (1, +\infty)$ with positive upper logarithmic density such that for all $|z| = r \in H$, we have*

$$\log M(r, f) > r^{\tau},$$

where $M(r, f) = \max_{|z|=r} |f(z)|$.

Lemma 2.3 ([7]) *Suppose that $P(z) = (\alpha + i\beta) z^n + \cdots$ (α, β are real numbers, $|\alpha| + |\beta| \neq 0$) is a polynomial with degree $n \geq 1$, that $A(z)$ is an entire function with $\rho(A) < n$. Set*

$$f(z) = A(z) e^{P(z)},\ z = re^{i\theta},\ \delta(P, \theta) = \alpha \cos n\theta - \beta \sin n\theta.$$

Then for any given $\varepsilon > 0$, there is a set $E_1 \subset [0, 2\pi)$ that has linear measure zero, such that for any $\theta \in [0, 2\pi) \setminus (E_1 \cup E_2)$, there is $R > 0$, such that for $|z| = r > R$, we have
(i) *if $\delta(P, \theta) > 0$, then*

$$\exp\{(1 - \varepsilon)\delta(P, \theta) r^n\} \leq |f(re^{i\theta})| \leq \exp\{(1 + \varepsilon)\delta(P, \theta) r^n\},$$

(ii) *if $\delta(P, \theta) < 0$, then*

$$\exp\{(1 + \varepsilon)\delta(P, \theta) r^n\} \leq |f(re^{i\theta})| \leq \exp\{(1 - \varepsilon)\delta(P, \theta) r^n\},$$

where $E_2 = \{\theta \in [0; 2\pi) : \delta(P, \theta) = 0\}$ is a finite set.

Lemma 2.4 ([10]) *Let f be a transcendental meromorphic function of finite order ρ. Let $\varepsilon > 0$ be a constant, k and j be integers satisfying $k > j \geq 0$. Then the following two statements hold:*

(i) *There exists a set* $E_3 \subset [1,+\infty)$ *which has finite logarithmic measure, such that for all* z *satisfying* $|z| \notin E_3 \cup [0,1]$, *we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\rho-1+\varepsilon)}. \tag{2.1}$$

(ii) *There exists a set* $E_4 \subset [0,2\pi)$ *which has linear measure zero, such that if* $\theta \in [0,2\pi)\backslash E_4$, *then there is a constant* $R = R(\theta) > 0$ *such that* (2.1) *holds for all* z *satisfying* $\arg z = \theta$ *and* $|z| \geq R$.

Lemma 2.5 ([18]) *Let* f *be an entire function and suppose that*

$$G(z) := \frac{\log^+ \left|f^{(k)}(z)\right|}{|z|^\rho}$$

is unbounded on some ray $\arg z = \theta$ *with constant* $\rho > 0$. *Then, there exists an infinite sequence of points* $z_n = r_n e^{i\theta}\ (n = 1,2,\cdots)$, *where* $r_n \to +\infty$ *such that* $G(z_n) \to +\infty$ *and*

$$\left|\frac{f^{(j)}(z_n)}{f^{(k)}(z_n)}\right| \leq \frac{1}{(k-j)!}(1+o(1))\, r_n^{k-j},\ j < k$$

as $n \to +\infty$.

Lemma 2.6 ([18]) *Let* f *be an entire function with* $\rho(f) = \rho < +\infty$. *Suppose that there exists a set* $E_5 \subset [0,2\pi)$ *which has linear measure zero, such that* $\log^+ \left|f\left(re^{i\theta}\right)\right| \leq Mr^{\rho_1}$ *for any ray* $\arg z = \theta \in [0,2\pi)\backslash E_5$, *where* M *is a positive constant depending on* θ, *while* ρ_1 *is a positive constant independent of* θ. *Then* $\rho(f) \leq \rho_1$.

Lemma 2.7 ([2], [5]), *Let* $A_j(z)\ (j = 0,1,\cdots,k-1)$, $F(z) \not\equiv 0$ *be finite order meromorphic functions.*
(i) *If* f *is a meromorphic solution of the differential equation*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_0(z) f = F, \tag{2.2}$$

with $\rho(f) = +\infty$, *then* f *satisfies*

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty.$$

(ii) *If* f *is a meromorphic solution of equation* (2.2) *with* $\rho(f) = +\infty$, $\rho_2(f) = \rho$, *then* f *satisfies*

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty,\ \overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) = \rho.$$

Lemma 2.8 ([20], [21]) *Suppose that* $f_1(z), f_2(z), \cdots, f_n(z)\ (n \geq 2)$ *are meromorphic functions and* $g_1(z), g_2(z), \cdots, g_n(z)\ (n \geq 2)$ *are entire functions satisfying the following conditions:*
(i) $\sum\limits_{j=1}^{n} e^{g_i(z)} f_j(z) \equiv f_{n+1}$,
(ii) *If* $1 \leq j \leq n+1$ *and* $1 \leq k \leq n$, *then the order of* f_j *is less than the order of* $e^{g_k(z)}$. *If* $n \geq 2$, $1 \leq j \leq n+1$ *and* $1 \leq h < k \leq n$, *then the order of* f_j *is less than the order of* $e^{g_h - g_k}$. *Then* $f_j(z) \equiv 0, j = 1,2,\cdots,n+1$.

Lemma 2.9 ([4]) *Let* $B_1(z), B_2(z), \cdots, B_{k-1}(z)$ *be entire functions of finite order. If* f *is a solution of the equation*

$$f^{(k)} + B_{k-1}(z) f^{(k-1)} + \cdots + B_1(z) f' + B_0(z) f = H(z),$$

then $\rho_2(f) \leq \max\{\rho(B_j)\ \ (j = 0,1,\cdots,k-1), \rho(H)\}$.

3. Proof of Theorem 1.1

First, we prove that each solution f of equation (1.5) is transcendental of order $\rho(f) \geq n$. We assume that f is solution to equation (1.5) with $\rho(f) < n$. Rewrite (1.5) as

$$e^{Q(z)-P(z)}F_2(z) - e^{-P(z)}f^{(k)} = \sum_{j=0}^{k-1} A_j(z) f^{(j)} - F_1(z). \tag{3.1}$$

Since

$$\max\{\rho(A_j) \ \ (j = 0, 1, \cdots, k-1)\} < n$$

and $\rho(f) < n$, then $\rho\left(A_j f^{(j)}\right) < n$ $(j = 0, 1, \cdots, k-1)$ and $\rho\left(f^{(k)}\right) < n$. By $b_q = ca_q, (0 < c < 1), q = 0, \cdots, n$, we get

$$\deg(Q(z) - P(z)) = \deg((c-1)P(z)) = n.$$

So, by Lemma 2.1 and (3.1), we have

$$n = \rho\left(e^{Q(z)-P(z)}F_2(z) - e^{-P(z)}f^{(k)}\right) = \rho\left(\sum_{j=0}^{k-1} A_j(z) f^{(j)} - F_1(z)\right)$$

$$\leq \max\left\{\rho\left(A_j f^{(j)}\right) \ \ (j = 0, 1, \cdots, k-1), \rho(F_1) \ \right\} < n.$$

This is a contradiction. Consequently, every solution f of equation (1.5) is transcendental with order $\rho(f) \geq n$. Now, we prove that $\rho(f) = +\infty$. Suppose that $\rho(f) = \rho < +\infty$. Set

$$\beta = \max\{\rho(A_j) \ \ (j = 0, 1, \cdots, k-1), j \neq s\} < \tau < \rho(A_s) = \alpha,$$

$$\gamma = \max\{\rho(F_i) \ \ (i = 1, 2)\}.$$

It is clear that, $0 \leq \beta < \tau < \alpha < n$ and $0 \leq \gamma < n$. Then, from the definition of the order, for any given ε with $0 < \varepsilon < \min\{n - \alpha, n - \beta, n - \gamma\}$ and for sufficiently large r, we have

$$|A_j(z)| \leq \exp\left\{r^{\beta+\varepsilon}\right\}, \ \ (j = 0, 1, \cdots, k-1, j \neq s), \tag{3.2}$$

$$|A_s(z)| \leq \exp\left\{r^{\alpha+\varepsilon}\right\}, \ |F_i(z)| \leq \exp\left\{r^{\gamma+\varepsilon}\right\}, (i = 1, 2). \tag{3.3}$$

By Lemma 2.2, for $0 \leq \beta < \tau < \alpha < n$ there exists a subset $H \subset (1, +\infty)$ having positive upper logarithmic density (so with infinite logarithmic measure) such that

$$\log M(r, A_s) > r^{\tau} \tag{3.4}$$

for $|z| = r \in H$. By Lemma 2.3, there exists a set $E \subset [0, 2\pi)$ of linear measure zero, such that whenever $\theta \in [0, 2\pi) \backslash E$, then $\delta(P, \theta) \neq 0$. By Lemma 2.4, there exists a set $E_4 \subset [0, 2\pi)$ which has linear measure zero, such that if $\theta \in [0, 2\pi) \backslash E_4$, then there is a constant $R = R(\theta) > 1$ such that for all z satisfying $\arg z = \theta$ and $|z| \geq R$, we have

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leq |z|^{(j-i)(\rho-1+\varepsilon)} \leq |z|^{k\rho}, \ 0 \leq i < j \leq k. \tag{3.5}$$

Since $P(z) = a_n z^n + \cdots + a_0$ and $Q(z) = b_n z^n + \cdots + b_0$ with a_q, b_q, $(q = 0, 1, \cdots, n)$ are complex numbers such that $a_n b_n \neq 0$ and $b_q = ca_q, 0 < c < 1, q = 0, \cdots, n$, then

$$Q(z) - P(z) = (c-1)P(z).$$

For any fixed $\theta \in [0, 2\pi) \backslash (E \cup E_4)$, set

$$\delta_1 = \delta(-P, \theta), \ \delta_2 = \delta(Q - P, \theta).$$

We can obtain

$$\delta_2 = (1-c)\delta(-P, \theta) = (1-c)\delta_1,$$

then $\delta_1 \neq 0$, $\delta_2 \neq 0$. We now discuss two cases separately.

Case 1. Suppose that $\delta_1 > 0$, then $\delta_2 > 0$, so $0 < \delta_2 < \delta_1$. By Lemma 2.3, for any given ε with $0 < 2\varepsilon < \min\left\{\left(\frac{\delta_1-\delta_2}{\delta_1}\right), n-\alpha, n-\beta, n-\gamma\right\}$, we have for any $\theta \in [0, 2\pi)\backslash E$ and $|z| = r > R$ sufficiently large

$$\left|e^{Q(z)-P(z)}\right| \le \exp\{(1+\varepsilon)\,\delta_2 r^n\}, \tag{3.6}$$

$$\left|e^{-P(z)}\right| \ge \exp\{((1-\varepsilon))\,\delta_1 r^n\}. \tag{3.7}$$

We now prove that $\frac{\log^+|f^{(k)}(z)|}{|z|^{\gamma+\varepsilon}}$ is bounded on the ray $\arg z = \theta$. We suppose that $\frac{\log^+|f^{(k)}(z)|}{|z|^{\gamma+\varepsilon}}$ is unbounded on the ray $\arg z = \theta$. Then by Lemma 2.5, there is a sequence of points $z_m = r_m e^{i\theta}\,(m = 1, 2, \cdots)$, where $r_m \to +\infty$ such that

$$\frac{\log^+|f^{(k)}(z_m)|}{|z_m|^{\gamma+\varepsilon}} \to +\infty.$$

Thus, for any sufficiently large number $A > 1$, we have

$$|f^{(k)}(z_m)| > \exp\left\{A|z_m|^{\gamma+\varepsilon}\right\} \tag{3.8}$$

and

$$\left|\frac{f^{(j)}(z_m)}{f^{(k)}(z_m)}\right| \le \frac{1}{(k-j)!}(1+o(1))\,r_m^{k-j} \le 2r_m^{k-j},\quad (j = 0, 1, \cdots, k-1) \tag{3.9}$$

as $m \to +\infty$. From (3.3) and (3.8), for m sufficiently large, we get

$$\frac{|F_i(z_m)|}{\left|f^{(k)}(z_m)\right|} \le \frac{\exp\{r_m^{\gamma+\varepsilon}\}}{\exp\left\{Ar_m^{\gamma+\varepsilon}\right\}},\quad (i = 1, 2).$$

Since $A > 1$, then we have

$$\left|\frac{F_i(z_m)}{f^{(k)}(z_m)}\right| \to 0 \text{ as } m \to +\infty,\quad (i = 1, 2). \tag{3.10}$$

From (1.5), we obtain

$$\left|e^{-P(z)}\right| \le \left\{\left|e^{Q(z)-P(z)}\right|\left|\frac{F_2}{f^{(k)}}\right| + \left|\frac{F_1}{f^{(k)}}\right| + |A_s(z)|\left|\frac{f^{(s)}}{f^{(k)}}\right| + \sum_{\substack{j=0\\ j\neq s}}^{k-1}|A_j(z)|\left|\frac{f^{(j)}}{f^{(k)}}\right|\right\}. \tag{3.11}$$

Substituting $(3.2), (3.3), (3.6), (3.7), (3.9), (3.10)$ into (3.11), we have

$$\begin{aligned}\exp\{(1-\varepsilon)\,\delta_1 r_m^n\} \le \left|e^{-P(z_m)}\right| &\le \exp\{(1+\varepsilon)\,\delta_2 r_m^n\}\,o(1) + o(1)\\ &+2r_m^{k-s}\exp\left\{r_m^{\alpha+\varepsilon}\right\} + 2(k-1)\,r_m^k\exp\left\{r_m^{\beta+\varepsilon}\right\}\\ &\le 2(k+2)\,r_m^k\exp\left\{r_m^{\alpha+\varepsilon}\right\}\exp\{(1+\varepsilon)\,\delta_2 r_m^n\},\end{aligned}$$

it follows that

$$\exp\{(1-\varepsilon)\,\delta_1 r_m^n - (1+\varepsilon)\,\delta_2 r_m^n\} \le 2(k+2)\,r_m^k\exp\left\{r_m^{\alpha+\varepsilon}\right\}. \tag{3.12}$$

By $0 < \varepsilon < \dfrac{\delta_1-\delta_2}{2\delta_1}$, we have

$$\begin{aligned}&\exp\{(1-\varepsilon)\,\delta_1 r_m^n - (1+\varepsilon)\,\delta_2 r_m^n\} = \exp\{((\delta_1-\delta_2) - \varepsilon(\delta_1+\delta_2))\,r_m^n\}\\ &= \exp\left\{(\delta_1-\delta_2)\left(1-\varepsilon\frac{\delta_1+\delta_2}{\delta_1-\delta_2}\right)r_m^n\right\} > \exp\left\{(\delta_1-\delta_2)\left(1-\frac{\delta_1-\delta_2}{2\delta_1}\cdot\frac{\delta_1+\delta_2}{\delta_1-\delta_2}\right)r_m^n\right\}\end{aligned}$$

$$= \exp\left\{\frac{(\delta_1-\delta_2)^2}{2\delta_1}r_m^n\right\}. \tag{3.13}$$

By (3.12) and (3.13), we obtain

$$\exp\left\{\frac{(\delta_1-\delta_2)^2}{2\delta_1}r_m^n\right\}\leq 2(k+2)\,r_m^k\exp\left\{r_m^{\alpha+\varepsilon}\right\},$$

which is a contradiction because $\alpha+\varepsilon<n$. Therefore, $\frac{\log^+|f^{(k)}(z)|}{|z|^{\gamma+\varepsilon}}$ is bounded on the ray $\arg z=\theta$, hence there is $M>0$, such that

$$|f^{(k)}(z)|\leq\exp\left\{M|z|^{\gamma+\varepsilon}\right\}. \tag{3.14}$$

Using Lemma 2.5, for $j=0<k$, we get

$$|f(z)|\leq\frac{(1+o(1))}{k!}\left|f^{(k)}(z)\right|r^k. \tag{3.15}$$

By (3.14) and (3.15), we have

$$|f(z)|\leq\frac{(1+o(1))}{k!}r^k\exp\left\{M|z|^{\gamma+\varepsilon}\right\}\leq C\exp\left\{|z|^{\gamma+2\varepsilon}\right\}$$

with $C>0$, on the ray $\arg z=\theta$.

Case 2. Suppose that $\delta_1<0$, then $\delta_2<0$. By Lemma 2.3, for any given

$$0<2\varepsilon<\min\left\{\tau-\beta,n-\alpha,n-\beta,n-\gamma\right\},$$

we have for any $\theta\in[0,2\pi)\backslash E$ and $|z|=r>R$ sufficiently large

$$\left|e^{Q(z)-P(z)}\right|\leq\exp\left\{(1-\varepsilon)\,\delta_2r^n\right\}<1, \tag{3.16}$$

$$\left|e^{-P(z)}\right|\leq\exp\left\{((1-\varepsilon))\,\delta_1r^n\right\}<1. \tag{3.17}$$

We now prove that $\frac{\log^+|f^{(s)}(z)|}{|z|^{\gamma+\varepsilon}}$ is bounded on the ray $\arg z=\theta$. We suppose the contrary. Then by Lemma 2.5, there is a sequence of points $z_m=r_me^{i\theta}\,(m=1,2,\cdots)$, where $r_m\to+\infty$ such that

$$\frac{\log^+|f^{(s)}(z_m)|}{|z_m|^{\gamma+\varepsilon}}\to+\infty.$$

Thus, for any sufficiently large number $A>1$, we have

$$|f^{(s)}(z_m)|>\exp\left\{A|z_m|^{\gamma+\varepsilon}\right\} \tag{3.18}$$

and

$$\left|\frac{f^{(j)}(z_m)}{f^{(s)}(z_m)}\right|\leq\frac{1}{(s-j)!}(1+o(1))\,r_m^{s-j}\leq 2r_m^{s-j},\quad(j=0,1,\cdots,s-1) \tag{3.19}$$

as $m\to+\infty$. From (3.3) and (3.18), for m sufficiently large, we get

$$\frac{|F_i(z_m)|}{\left|f^{(s)}(z_m)\right|}\leq\frac{\exp\left\{r_m^{\gamma+\varepsilon}\right\}}{\exp\left\{Ar_m^{\gamma+\varepsilon}\right\}},\quad(i=1,2).$$

Since $A>1$, then we have

$$\left|\frac{F_i(z_m)}{f^{(s)}(z_m)}\right|\to 0\text{ as }\ m\to+\infty,\quad(i=1,2). \tag{3.20}$$

From (1.5), we obtain

$$1\leq\left|e^{Q(z)-P(z)}\right|\frac{1}{|A_s(z)|}\left|\frac{F_2}{f^{(s)}}\right|+\frac{1}{|A_s(z)|}\left|\frac{F_1}{f^{(s)}}\right|$$

$$+\frac{\left|e^{-P(z)}\right|}{|A_s(z)|}\left|\frac{f^{(k)}}{f^{(s)}}\right|+\sum_{j=0}^{s-1}\frac{|A_j(z)|}{|A_s(z)|}\left|\frac{f^{(j)}}{f^{(s)}}\right|+\sum_{j=s+1}^{k-1}\frac{|A_j(z)|}{|A_s(z)|}\left|\frac{f^{(j)}}{f^{(s)}}\right|. \tag{3.21}$$

By (3.4) for all z_m, $|z_m| = r_m \in H$ satisfying $|A_s(z_m)| = M(r_m, f)$, we have

$$\frac{1}{|A_s(z_m)|} < \exp\{-r_m^\tau\}, \tag{3.22}$$

so by (3.2), we obtain

$$\frac{|A_j(z_m)|}{|A_s(z_m)|} < \exp\left\{r_m^{\beta+\varepsilon} - r_m^\tau\right\} \quad (j = 0, 1, \cdots, k-1, j \neq s) \tag{3.23}$$

for m is large enough. Substituting (3.5), (3.16), (3.17), (3.19), (3.20), (3.22), (3.23) into (3.21), we have

$$\begin{aligned} 1 \le\ & o(1)\exp\{-r_m^\tau\}\exp\{(1-\varepsilon)\delta_2 r_m^n\} + o(1)\exp\{-r_m^\tau\} \\ & + r_m^{k\rho}\exp\{-r_m^\tau\}\exp\{(1-\varepsilon)\delta_1 r_m^n\} + \sum_{j=0}^{s-1} 2r_m^{s-j}\exp\left\{r_m^{\beta+\varepsilon} - r_m^\tau\right\} \\ & + \sum_{j=s+1}^{k-1} \exp\left\{r_m^{\beta+\varepsilon} - r_m^\tau\right\} r_m^{k\rho} \le o(1)\exp(-r_m^\tau)\exp\{(1-\varepsilon)\delta_2 r_m^n\} \\ & + o(1)\exp\{-r_m^\tau\} + r_m^{k\rho}\exp\{-r_m^\tau\}\exp\{(1-\varepsilon)\delta_1 r_m^n\} \\ & + 2sr_m^s\exp\left\{r_m^{\beta+\varepsilon} - r_m^\tau\right\} + (k-1-s)\, r_m^{k\rho}\exp\left\{r_m^{\beta+\varepsilon} - r_m^\tau\right\}. \end{aligned} \tag{3.24}$$

Since each term in the right side of (3.24) tends to zero as $r_m \to +\infty$ because $\beta + \varepsilon < \tau$, then from (3.24) we get $1 \le 0$ as $r_m \to +\infty$. which is a contradiction. Therefore, $\frac{\log^+ |f^{(s)}(z)|}{|z|^{\gamma+\varepsilon}}$ is bounded on the ray $\arg z = \theta$, hence there is $M > 0$, such that

$$|f^{(s)}(z)| \le \exp\left\{M|z|^{\gamma+\varepsilon}\right\}.$$

This implies, as in Case 1, that

$$|f(z)| \le C\exp\left\{|z|^{\gamma+2\varepsilon}\right\}$$

with $C > 0$, on the ray $\arg z = \theta$. Then by Lemma 2.6, we have $\rho(f) \le \gamma + 2\varepsilon < n$, which is a contradiction. Hence, every transcendental solution f of (1.5) must be of infinite order. Since

$$\deg(Q(z) - P(z)) = \deg((c-1)P(z)) = n,$$

then by Lemma 2.1, we have

$$\max\left\{\rho\left(e^{P(z)}F_1(z) + e^{Q(z)}F_2(z)\right)\right\} = n.$$

By using Lemma 2.9 we obtain $\rho_2(f) \le n$.

4. Proof of Theorem 1.2

Suppose that f is a solution of equation (1.5). Then, by Theorem 1.1 we have $\rho(f) = +\infty$ and $\rho_2(f) \le n$. Set $g(z) = f(z) - \varphi(z)$. Then $g(z)$ is an entire function with $\rho(f) = \rho(g) = +\infty$ and $\rho_2(g) = \rho_2(f) \le n$. Substituting $f = g + \varphi$ into (1.5), we have

$$\begin{aligned} & g^{(k)} + A_{k-1}(z)e^{P(z)}g^{(k-1)} + \cdots + A_1(z)e^{P(z)}g' \\ & + A_0(z)e^{P(z)}g = D(z), \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} D(z) =\ & e^{P(z)}F_1(z) + e^{Q(z)}F_2(z) \\ & - \left[\varphi^{(k)} + A_{k-1}(z)e^{P(z)}\varphi^{(k-1)} + \cdots + A_1(z)e^{P(z)}\varphi' + A_0(z)e^{P(z)}\varphi\right]. \end{aligned}$$

We prove that $D \not\equiv 0$. In fact, if $D \equiv 0$, then

$$\varphi^{(k)} + A_{k-1}(z)e^{P(z)}\varphi^{(k-1)} + \cdots + A_1(z)e^{P(z)}\varphi'$$

$$+A_0(z)e^{P(z)}\varphi = e^{P(z)}F_1(z) + e^{Q(z)}F_2(z).$$

Hence φ is a solution of equation (1.5), then $\rho(\varphi) = +\infty$, which is a contradiction. Therefore $D \not\equiv 0$. We know that the functions $A_j(z)e^{P(z)}(j = 0, 1, \cdots, k-1), D$ are of finite order. By Lemma 2.7 and (4.1), we have

$$\overline{\lambda}(g) = \lambda(g) = \rho(g) = \rho(f) = +\infty,\ \overline{\lambda}_2(g) = \lambda_2(g) = \rho_2(g) = \rho_2(f) \leq n.$$

Then, by f is infinite order solution of equation (1.5) and Lemma 2.7 we obtain

$$\begin{aligned}\overline{\lambda}(f) &= \lambda(f) = \overline{\lambda}(f-\varphi) = \lambda(f-\varphi) = \rho(f) = +\infty,\\ \overline{\lambda}_2(f) &= \lambda_2(f) = \overline{\lambda}_2(f-\varphi) = \lambda_2(f-\varphi) = \rho_2(f) \leq n\end{aligned}$$

which completes the proof.

5. Proof of Theorem 1.3

We can rewrite (1.6) as

$$e^{-Q(z)}f^{(k)} + A_{k-1}(z)e^{P_{k-1}(z)-Q(z)}f^{(k-1)} + \cdots + A_1(z)e^{P_1(z)-Q(z)}f'$$

$$+A_0(z)e^{P_0(z)-Q(z)}f = F(z). \tag{5.1}$$

Since $a_{j,q} = c_j b_q\ (j \in I, q = 0, 1, \cdots, n)$, then $P_j(z) - Q(z) = (c_j - 1)Q(z)$, and by (5.1), we get

$$e^{-Q(z)}f^{(k)} + \sum_{j\in I_1} A_j(z)e^{(c_j-1)Q(z)}f^{(j)} + \sum_{j\in I_2} A_j(z)e^{(c_j-1)Q(z)}f^{(j)}$$

$$+\sum_{j\in I_3} A_j(z)e^{(c_j-1)Q(z)}f^{(j)} + \sum_{j\in I_4} A_j(z)f^{(j)} = F(z). \tag{5.2}$$

First we prove that every solution f of (5.2) is transcendental of order $\rho(f) \geq n$. We assume that f is solution to equation (5.2) with $\rho(f) < n$. It is clear that $f \not\equiv 0$. We can rewrite (5.2) in the form

$$e^{-Q(z)}f^{(k)} + \sum_{j\in\Gamma} B_j(z)e^{(c_j-1)Q(z)} + A_0(z)e^{(c_0-1)Q(z)}f = F(z) - \sum_{j\in I_4} A_j(z)f^{(j)} \tag{5.3}$$

where $\Gamma = I\backslash(I_4 \cup \{0\})$ and $B_j(z) = A_j(z)f^{(j)}$. Obviously, $\rho\left(f^{(k)}\right) < n$ and $\rho\left(A_j(z)f^{(j)}\right) < n\ (j \in I)$. We can see that $(c_0 - 1)b_q, (c_j - 1)b_q, -b_q (q = 0, \cdots, n,\ j \in \Gamma)$ are distinct numbers. Then by (5.3) and the Lemma 2.8, we have $A_0(z)f \equiv 0$. This is a contradiction. Hence, every solution f of equation (1.6) is transcendental with order $\rho(f) \geq n$.

Now, we prove that $\rho(f) = +\infty$. Suppose that $\rho(f) = \rho < +\infty$. Set

$$\alpha = \max\{\rho(A_j)\ \ (j = 0, 1, \cdots, k-1), \rho(F)\}.$$

It is clear that $\alpha < n$. Then, from the definition of the order, for any given ε with $0 < 2\varepsilon < n - \alpha$ and for sufficiently large r, we have

$$|A_j(z)| \leq \exp\left\{r^{\alpha+\varepsilon}\right\},\ j \in I_4, \tag{5.4}$$

$$|F(z)| \leq \exp\left\{r^{\alpha+\varepsilon}\right\}. \tag{5.5}$$

By Lemma 2.3, there exists a set $E \subset [0, 2\pi)$ of linear measure zero, such that whenever $\theta \in [0, 2\pi)\backslash E$, then $\delta(P, \theta) \neq 0$. By Lemma 2.4, there exists a set $E_4 \subset [0, 2\pi)$ which has linear measure zero, such that if $\theta \in [0, 2\pi)\backslash E_4$, then there is a constant $R = R(\theta) > 1$ such that for all z satisfying $\arg z = \theta$ and $|z| \geq R$, we have

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leq |z|^{(j-i)(\rho-1+\varepsilon)} \leq |z|^{k\rho},\ 0 \leq i < j \leq k. \tag{5.6}$$

For any fixed $\theta \in [0, 2\pi) \backslash (E_4 \cup E)$, set

$$\begin{aligned}\delta_s &= \delta((c_s - 1) Q(z), \theta), \ \delta_l = \delta((c_l - 1) Q(z), \theta), \\ \delta_1 &= \max\{\delta((c_j - 1) Q(z), \theta), \ j \in I_1 \backslash \{s\}\}, \\ \delta_3 &= \max\{\delta((c_j - 1) Q(z), \theta), \ j \in I_3 \backslash \{l\}\}.\end{aligned}$$

We can obtain

$$\delta((c_j - 1) Q(z), \theta) = (c_j - 1) \delta(Q(z), \theta) = (1 - c_j) \delta(-Q(z), \theta),$$

then $\delta_s \neq 0$, $\delta_l \neq 0, \delta_1 \neq 0$ and $\delta_3 \neq 0$. We now discuss two cases separately.

Case1. $\delta(-Q(z), \theta) > 0$. We know that

$$\begin{aligned}\text{if } j &\in I_1, \text{ then } \delta((c_j - 1) Q(z), \theta) < 0, \\ \text{if } j &\in I_2, \text{ then } 0 < \delta((c_j - 1) Q(z), \theta) < \delta(-Q(z), \theta), \\ \text{if } j &\in I_3 \backslash \{l\}, \text{ then } 0 < \delta(-Q(z), \theta) < \delta((c_j - 1) Q(z), \theta) \leq \delta_3 < \delta_l.\end{aligned}$$

By Lemma 2.3, for any given ε with $0 < 2\varepsilon < \min\left\{\frac{\delta_l - \delta_3}{\delta_l}, n - \alpha\right\}$, we obtain

$$\left|A_l(z) e^{(c_l - 1)Q(z)}\right| \geq \exp\{(1 - \varepsilon) \delta_l r^n\}, \tag{5.7}$$

$$\left|e^{-Q(z)}\right| \leq \exp\{(1 + \varepsilon) \delta(-Q(z), \theta) r^n\}, \tag{5.8}$$

$$\left|A_j(z) e^{(c_j - 1)Q(z)}\right| \leq \exp\{(1 - \varepsilon) \delta((c_j - 1) Q(z), \theta) r^n\}, \ i \in I_1, \tag{5.9}$$

$$\begin{aligned}\left|A_j(z) e^{(c_j - 1)Q(z)}\right| &\leq \exp\{(1 + \varepsilon) \delta((c_j - 1) Q(z), \theta) r^n\} \\ &\leq \exp\{(1 + \varepsilon) \delta(-Q(z), \theta) r^n\}, \ i \in I_2,\end{aligned} \tag{5.10}$$

$$\begin{aligned}\left|A_j(z) e^{(c_j - 1)Q(z)}\right| &\leq \exp\{(1 + \varepsilon) \delta((c_j - 1) Q(z), \theta) r^n\} \\ &\leq \exp\{(1 + \varepsilon) \delta_3 r^n\}, \ i \in I_3 \backslash \{l\}\end{aligned} \tag{5.11}$$

for sufficiently large r. We now prove that $\frac{\log^+ |f^{(l)}(z)|}{|z|^{\alpha+\varepsilon}}$ is bounded on the ray $\arg z = \theta$. We suppose the contrary. Then by Lemma 2.5, there is a sequence of points $z_m = r_m e^{i\theta}$ $(m = 1, 2, \cdots)$, where $r_m \to +\infty$ such that

$$\frac{\log^+ |f^{(l)}(z_m)|}{|z_m|^{\alpha+\varepsilon}} \to +\infty.$$

Thus, for any sufficiently large number $A > 1$, we have

$$|f^{(l)}(z_m)| > \exp\left\{A|z_m|^{\alpha+\varepsilon}\right\} \tag{5.12}$$

and

$$\left|\frac{f^{(j)}(z_m)}{f^{(l)}(z_m)}\right| \leq \frac{1}{(l - j)!} (1 + o(1)) r_m^{l-j}, \ \ (j = 0, 1, \cdots, l - 1) \tag{5.13}$$

as $m \to +\infty$. From (5.5) and (5.12), we get

$$\left|\frac{F(z_m)}{f^{(l)}(z_m)}\right| \leq \frac{\exp\{|z_m|^{\alpha+\varepsilon}\}}{\exp\{A|z_m|^{\alpha+\varepsilon}\}}.$$

Since $A > 1$, then we have

$$\left|\frac{F(z_m)}{f^{(l)}(z_m)}\right| \to 0 \tag{5.14}$$

for m is large enough. From (5.2), we obtain

$$\left|A_l(z)\,e^{(c_l-1)Q(z_m)}\right| \le \left|e^{-Q(z_m)}\right|\left|\frac{f^{(k)}(z_m)}{f^{(l)}(z_m)}\right| + \sum_{j\in I_1}\left|A_j(z_m)\,e^{(c_j-1)Q(z_m)}\right|\left|\frac{f^{(j)}(z_m)}{f^{(l)}(z_m)}\right|$$

$$+\sum_{j\in I_2}\left|A_j(z_m)\,e^{(c_j-1)Q(z_m)}\right|\left|\frac{f^{(j)}(z_m)}{f^{(l)}(z_m)}\right|$$

$$+\sum_{j\in I_3\setminus\{l\}}\left|A_j(z_m)\,e^{(c_j-1)Q(z_m)}\right|\left|\frac{f^{(j)}(z_m)}{f^{(l)}(z_m)}\right|$$

$$+\sum_{j\in I_4}|A_j(z_m)|\left|\frac{f^{(j)}(z_m)}{f^{(l)}(z_m)}\right| + \left|\frac{F(z_m)}{f^{(l)}(z_m)}\right|. \tag{5.15}$$

Substituting $(5.4)\,,(5.6)\,,(5.7)-(5.11)\,,(5.13)\,,(5.14)$ into (5.15)

$$\exp\{(1-\varepsilon)\,\delta_l r_m^n\} \le \exp\{(1+\varepsilon)\,\delta(-Q(z_m),\theta)\,r_m^n\}\,r_m^{k\rho}$$

$$+\sum_{j\in I_1}\left|\frac{f^{(j)}(z_m)}{f^{(l)}(z_m)}\right|\exp\{(1-\varepsilon)\,\delta((c_j-1)\,Q(z_m),\theta)\,r_m^n\}$$

$$+\sum_{j\in I_2}\left|\frac{f^{(j)}(z_m)}{f^{(l)}(z_m)}\right|\exp\{(1+\varepsilon)\,\delta(-Q(z_m),\theta)\,r_m^n\}$$

$$+\sum_{j\in I_3\setminus\{l\}}\left|\frac{f^{(j)}(z_m)}{f^{(l)}(z_m)}\right|\exp\{(1+\varepsilon)\,\delta_3 r_m^n\}$$

$$+\sum_{j\in I_4}\left|\frac{f^{(j)}(z_m)}{f^{(l)}(z_m)}\right|\exp\left\{r_m^{\alpha+\varepsilon}\right\}+o(1) \le M_0 r_m^{M_1}\exp\left\{r_m^{\alpha+\varepsilon}\right\}\exp\{(1+\varepsilon)\,\delta_3 r_m^n\}, \tag{5.16}$$

where $M_0>0$, $M_1>0$ are some constants. On the other hand, by $0<\varepsilon<\frac{\delta_l-\delta_3}{2\delta_l}$ we have

$$(1-\varepsilon)\,\delta_l-(1+\varepsilon)\,\delta_3=(\delta_l-\delta_3)\left(1-\varepsilon\frac{\delta_l+\delta_3}{\delta_l-\delta_3}\right)$$

$$>(\delta_l-\delta_3)\left(1-\frac{\delta_l-\delta_3}{2\delta_l}\cdot\frac{\delta_l+\delta_3}{\delta_l-\delta_3}\right)=\frac{(\delta_l-\delta_3)^2}{2\delta_l}. \tag{5.17}$$

By (5.16) and (5.17), we can get

$$\exp\left\{\frac{(\delta_l-\delta_3)^2}{2\delta_l}r_m^n\right\}\le M_0 r_m^{M_1}\exp\left\{r_m^{\alpha+\varepsilon}\right\}$$

which is a contradiction because $\alpha+\varepsilon<n$. Therefore, $\frac{\log^+|f^{(l)}(z)|}{|z|^{\alpha+\varepsilon}}$ is bounded on the ray $\arg z=\theta$, hence there is $M>0$, such that

$$|f^{(l)}(z)|\le\exp\left\{M|z|^{\alpha+\varepsilon}\right\}. \tag{5.18}$$

By the same reasoning as in the proof of ([16], Lemma 3.1), we get

$$|f(z)|\le\frac{(1+o(1))}{l!}\left|f^{(l)}(z)\right|r^l. \tag{5.19}$$

By (5.18) and (5.19), we have

$$|f(z)|\le\frac{(1+o(1))}{l!}r^l\exp\left\{M|z|^{\alpha+\varepsilon}\right\}\le C\exp\left\{|z|^{\alpha+2\varepsilon}\right\}$$

with $C>0$, on the ray $\arg z=\theta$.

Case 2. $\delta(-Q(z),\theta)<0$. We know that

$$\delta((c_j-1)Q(z),\theta)=(c_j-1)\delta(Q(z),\theta)=(1-c_j)\delta(-Q(z),\theta).$$

Hence

$$\begin{aligned}\text{if } j &\in I_1\backslash\{s\}, \text{ then } 0<\delta((c_j-1)Q(z),\theta)\\ &\leq \max\{\delta((c_j-1)Q(z),\theta),\ j\in I_1\backslash\{s\}\}=\delta_1<\delta_s,\\ \text{if } j &\in I_2\cup I_3, \text{ then } \delta((c_j-1)Q(z),\theta)<0.\end{aligned}$$

By Lemma 2.3, for any given ε with $0<2\varepsilon<\min\left\{\frac{\delta_s-\delta_1}{\delta_s},n-\alpha\right\}$, we obtain

$$\left|A_s(z)e^{(c_s-1)Q(z)}\right|\geq\exp\{(1-\varepsilon)\delta_s r^n\},\tag{5.20}$$

$$\left|e^{-Q(z)}\right|\leq\exp\{(1-\varepsilon)\delta(-Q(z),\theta)r^n\}<1,\tag{5.21}$$

$$\begin{aligned}\left|A_j(z)e^{(c_j-1)Q(z)}\right| &\leq\exp\{(1+\varepsilon)\delta((c_j-1)Q(z),\theta)r^n\}\\ &\leq\exp\{(1+\varepsilon)\delta_1 r^n\},\ i\in I_1\backslash\{s\},\end{aligned}\tag{5.22}$$

$$\left|A_j(z)e^{(c_j-1)Q(z)}\right|\leq\exp\{(1-\varepsilon)\delta((c_j-1)Q(z),\theta)r^n\}<1,\ i\in I_2\cup I_3\tag{5.23}$$

for sufficiently large r. We now prove that $\frac{\log^+|f^{(s)}(z)|}{|z|^{\alpha+\varepsilon}}$ is bounded on the ray $\arg z=\theta$. We suppose the contrary. Then by Lemma 2.5, there is a sequence of points $z_m=r_me^{i\theta}$ $(m=1,2,\cdots)$, where $r_m\to+\infty$ such that

$$\frac{\log^+|f^{(s)}(z_m)|}{|z_m|^{\alpha+\varepsilon}}\to+\infty.$$

Thus, for any sufficiently large number $A>1$, we have

$$|f^{(s)}(z_m)|>\exp\left\{A|z_m|^{\alpha+\varepsilon}\right\}\tag{5.24}$$

and

$$\left|\frac{f^{(j)}(z_m)}{f^{(s)}(z_m)}\right|\leq\frac{1}{(s-j)!}(1+o(1))r_m^{s-j},\quad(j=0,1,\cdots,s-1)\tag{5.25}$$

as $m\to+\infty$. From (5.5) and (5.24), we get

$$\left|\frac{F(z_m)}{f^{(s)}(z_m)}\right|\leq\frac{\exp\{r_m^{\alpha+\varepsilon}\}}{\exp\left\{Ar_m^{\alpha+\varepsilon}\right\}}.$$

Since $A>1$, then we have

$$\left|\frac{F(z_m)}{f^{(s)}(z_m)}\right|\to0\tag{5.26}$$

for m is large enough. From (5.2), we obtain

$$\begin{aligned}\left|A_s(z)e^{(c_s-1)Q(z_m)}\right| &\leq\left|e^{-Q(z_m)}\right|\left|\frac{f^{(k)}(z_m)}{f^{(s)}(z_m)}\right|\\ &+\sum_{j\in I_1\backslash\{s\}}\left|A_j(z_m)e^{(c_j-1)Q(z_m)}\right|\left|\frac{f^{(j)}(z_m)}{f^{(s)}(z_m)}\right|+\sum_{j\in I_2}\left|A_j(z_m)e^{(c_j-1)Q(z_m)}\right|\left|\frac{f^{(j)}(z_m)}{f^{(s)}(z_m)}\right|\\ &+\sum_{j\in I_3}\left|A_j(z_m)e^{(c_j-1)Q(z_m)}\right|\left|\frac{f^{(j)}(z_m)}{f^{(s)}(z_m)}\right|\end{aligned}$$

$$(5.27) \qquad +\sum_{j\in I_4}|A_j(z_m)|\left|\frac{f^{(j)}(z_m)}{f^{(s)}(z_m)}\right|+\left|\frac{F(z_m)}{f^{(s)}(z_m)}\right|.$$

Substituting $(5.4), (5.6), (5.20)-(5.23), (5.25), (5.26)$ into (5.27)

$$\exp\{(1-\varepsilon)\delta_s r_m^n\} \le \exp\{(1-\varepsilon)\delta(-Q(z_m),\theta)r_m^n\} r_m^{k\rho}$$

$$+\sum_{j\in I_1\backslash\{s\}}\left|\frac{f^{(j)}(z_m)}{f^{(s)}(z_m)}\right|\exp\{(1+\varepsilon)\delta_1 r_m^n\}$$

$$+\sum_{j\in I_2}\left|\frac{f^{(j)}(z_m)}{f^{(s)}(z_m)}\right|\exp\{(1-\varepsilon)\delta((c_j-1)Q(z_m),\theta)r_m^n\}$$

$$+\sum_{j\in I_3}\left|\frac{f^{(j)}(z_m)}{f^{(s)}(z_m)}\right|\exp\{(1-\varepsilon)\delta((c_j-1)Q(z_m),\theta)r_m^n\}$$

$$(5.28) \qquad +\sum_{j\in I_4}\left|\frac{f^{(j)}(z_m)}{f^{(s)}(z_m)}\right|\exp\left\{r_m^{\alpha+\varepsilon}\right\}+o(1)\le M_2 r_m^{M_3}\exp\left\{r_m^{\alpha+\varepsilon}\right\}\exp\{(1+\varepsilon)\delta_1 r_m^n\},$$

where $M_2>0$, $M_3>0$ are some constants. By $0<\varepsilon<\frac{\delta_s-\delta_1}{2\delta_s}$ and (5.28), we can get

$$\exp\left\{\frac{(\delta_s-\delta_1)^2}{2\delta_s}r_m^n\right\}\le M_2 r_m^{M_3}\exp\left\{r_m^{\alpha+\varepsilon}\right\}$$

which is a contradiction because $\alpha+\varepsilon<n$. Therefore, $\frac{\log^+|f^{(s)}(z)|}{|z|^{\alpha+\varepsilon}}$ is bounded on the ray $\arg z=\theta$, hence there is $M>0$, such that

$$|f^{(s)}(z)|\le\exp\left\{M|z|^{\alpha+\varepsilon}\right\}.$$

This implies, as in Case 1, that

$$|f(z)|\le\frac{(1+o(1))}{s!}r^s\exp\left\{M|z|^{\alpha+\varepsilon}\right\}\le C\exp\left\{|z|^{\alpha+2\varepsilon}\right\}$$

with $C>0$, on the ray $\arg z=\theta$, for any given $\theta\in[0,2\pi)\backslash(E_4\cup E)$. Then by Lemma 2.6, we have $\rho(f)\le\alpha+2\varepsilon<n$, which is a contradiction. Hence, every transcendental solution f of (1.6) must be of infinite order. Since

$$\max\left\{\rho\left(A_je^{P_j(z)}\right)\quad(j\in I),\rho\left(e^{Q(z)}F\right)\right\}=n,$$

then by Lemma 2.9 we have $\rho_2(f)\le n$.

6. Proof of Theorem 1.4

Suppose that f is a solution of equation (1.6). Then, by Theorem 1.3 we have $\rho(f)=+\infty$ and $\rho_2(f)\le n$. Set $g(z)=f(z)-\varphi(z)$. Then $g(z)$ is an entire function with $\rho(f)=\rho(g)=+\infty$ and $\rho_2(g)=\rho_2(f)\le n$. Substituting $f=g+\varphi$ into (1.6), we have

$$g^{(k)}+A_{k-1}(z)e^{P_{k-1}(z)}g^{(k-1)}+\cdots+A_1(z)e^{P_1(z)}g'$$

$$(6.1) \qquad +A_0(z)e^{P_0(z)}g=D(z),$$

where

$$D(z)=e^{Q(z)}F(z)-\left[\varphi^{(k)}+\sum_{j=0}^{k-1}A_j(z)e^{P_j(z)}\varphi^{(j)}\right].$$

We prove that $D\not\equiv 0$. In fact, if $D\equiv 0$, then

$$\varphi^{(k)}+A_{k-1}(z)e^{P_{k-1}(z)}\varphi^{(k-1)}+\cdots+A_1(z)e^{P_1(z)}\varphi'+A_0(z)e^{P_0(z)}\varphi=e^{Q(z)}F(z).$$

Hence φ is a solution of equation (1.6), then $\rho(\varphi) = +\infty$, which is a contradiction. Therefore $D \not\equiv 0$. We know that the functions $A_j(z) e^{P_j(z)} (j = 0, 1, \cdots, k-1), D$ are of finite order. By Lemma 2.7 and (6.1), we have

$$\overline{\lambda}(g) = \lambda(g) = \rho(g) = \rho(f) = +\infty,\ \overline{\lambda}_2(g) = \lambda_2(g) = \rho_2(g) = \rho_2(f) \leq n.$$

Then, by f is infinite order solution of equation (1.6) and Lemma 2.7 we obtain

$$\begin{aligned} \overline{\lambda}(f) &= \lambda(f) = \overline{\lambda}(f-\varphi) = \lambda(f-\varphi) = \rho(f) = +\infty, \\ \overline{\lambda}_2(f) &= \lambda_2(f) = \overline{\lambda}_2(f-\varphi) = \lambda_2(f-\varphi) = \rho_2(f) \leq n \end{aligned}$$

which completes the proof.

Acknowledgements. This paper is supported by University of Mostaganem (UMAB) (PRFU Project Code C00L03UN270120180005).

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Department of Mathematics, Laboratory of Pure and Applied Mathematics, University of Mostaganem (UMAB), B. P. 227 Mostaganem-(Algeria)

Email address: saidaniman@yahoo.fr

Department of Mathematics, Laboratory of Pure and Applied Mathematics, University of Mostaganem (UMAB), B. P. 227 Mostaganem-(Algeria)

Email address: benharrat.belaïdi@univ-mosta.dz