

# BI-CONJUGATIVE RELATIONS

DANIEL ABRAHAM ROMANO

ABSTRACT. In this paper the concept of bi-conjugative relations on sets is introduced. Characterizations of this relations are obtained. In addition, particularly we show that the anti-order relation  $\not\leq$  in poset  $(L, \leq)$  is not a bi-conjugative relation.

**Mathematics Subject Classification (2010):** Primary 03E02, 06A11; Secondary 20M20

**Key words:** relations, bi-conjugative relations, semigroup of all binary relations

## 1. INTRODUCTION AND PRELIMINARIES

The regularity of binary relations was first characterized by Zareckiĭ ([11]). Further criteria for regularity were given by Markowsky ([8]), Schein ([10]) and Xu Xiao-quan and Liu Yingming ([12]) (see also [1] and [2]). The concepts of conjugative relations, dually conjugative relations and dually normal relations were introduced by Guanghao Jiang and Luoshan Xu ([3], [4]), and a characterization of normal relations was introduced and analyzed by Jiang Guanghao, Xu Luoshan, Cai Jin and Han Guiwen in [5].

In this paper, we introduce and analyze bi-conjugative relations on sets.

The following are some basic concepts needed in the sequel, for other nonexplicitly stated elementary notions please refer to papers [1] – [6] and [11], and to book [7].

For a set  $X$ , we call  $\rho$  a binary relation on  $X$ , if  $\rho \subseteq X \times X$ . Let  $\mathcal{B}(X)$  denote the set of all binary relations on  $X$ . For  $\alpha, \beta \in \mathcal{B}(X)$ , define

$$\beta \circ \alpha = \{(x, z) \in X \times X : (\exists y \in X)((x, y) \in \alpha \wedge (y, z) \in \beta)\}.$$

The relation  $\beta \circ \alpha$  is called the composition of  $\alpha$  and  $\beta$ . It is well known that  $(\mathcal{B}(X), \circ)$  a semigroup. The relation  $\Delta_X = \{(x, x) : x \in X\}$  is the identity. For a binary relation  $\alpha$  on a set  $X$ , define  $\alpha^{-1} = \{(x, y) \in X \times X : (y, x) \in \alpha\}$  and  $\alpha^C = X \times X \setminus \alpha$ .

The following classes of elements in the semigroup  $\mathcal{B}(X)$ , given in the following definition, have been investigated:

**Definition 1.1.** For relation  $\alpha \in \mathcal{B}(X)$  we say that it is:

– *regular* if there exists a relation  $\beta \in \mathcal{B}(X)$  such that

$$\alpha = \alpha \circ \beta \circ \alpha.$$

– *normal* ([5]) if there exists a relation  $\beta \in \mathcal{B}(X)$  such that

$$\alpha = \alpha \circ \beta \circ (\alpha^C)^{-1}.$$

– *dually normal* ([4]) if there exists a relation  $\beta \in \mathcal{B}(X)$  such that

$$\alpha = (\alpha^C)^{-1} \circ \beta \circ \alpha.$$

– *conjugative* ([3]) if there exists a relation  $\beta \in \mathcal{B}(X)$  such that

$$\alpha = \alpha^{-1} \circ \beta \circ \alpha.$$

– *dually conjugative* ([3]) if there exists a relation  $\beta \in \mathcal{B}(X)$  such that

$$\alpha = \alpha \circ \beta \circ \alpha^{-1}.$$

– *quasi-regular* ([9]) if there exists a relation  $\beta \in \mathcal{B}(X)$  such that

$$\alpha = \alpha^C \circ \beta \circ \alpha.$$

– *dually quasi-regular* ([9]) if there exists a relation  $\beta \in \mathcal{B}(X)$  such that

$$\alpha = \alpha \circ \beta \circ \alpha^C.$$

Besides that, for  $\alpha, \beta \in \mathcal{B}(X)$  and  $x, y \in X$ , we define the *box product* of relation  $\alpha$  and relation  $\beta$  by

$$\begin{aligned} (\alpha \square \beta)(x, y) &= \alpha x \times \beta y \\ &= \{(u, v) \in X \times X : u \in \alpha x \wedge v \in \beta y\}. \end{aligned}$$

Let  $\alpha, \beta, \gamma \in \mathcal{B}(X)$  be arbitrary relations, then

$$(1.1) \quad \gamma \circ \beta \circ \alpha = (\alpha \square \gamma^{-1})(\beta)$$

holds. Indeed, we have

$$\begin{aligned} (u, v) \in \gamma \circ \beta \circ \alpha &\iff (\exists a, b \in X)((u, a) \in \alpha \wedge (a, b) \in \beta \wedge (b, v) \in \gamma) \\ &\iff (\exists (a, b) \in \beta)(u \in \alpha a \wedge v \in \gamma^{-1} b) \\ &\iff (\exists (a, b) \in \beta)((u, v) \in \alpha a \times \gamma^{-1} b) \\ &\iff (\exists (a, b) \in \beta)((u, v) \in (\alpha \square \gamma^{-1})(a, b)) \\ &\iff (u, v) \in (\alpha \square \gamma^{-1})(\beta). \end{aligned}$$

Now, we can equations, introduced in Definition 1.1, represent in the new way. For example, a conjugative relation  $\alpha$  satisfies the following equation  $\alpha = (\alpha \square \alpha)(\beta)$ , and if  $\alpha$  is a dually conjugative relation, then the following equation  $\alpha = (\alpha^{-1} \square \alpha^{-1})(\beta)$  holds. Analogously, a normal relation  $\alpha$  is described by  $\alpha = ((\alpha^C)^{-1} \square \alpha^{-1})(\beta)$ , and a dually normal relation  $\alpha$  satisfies the following equation  $\alpha = (\alpha \square \alpha^C)(\beta)$ . Descriptions of quasi-regular relations and dually quasi-regular relations now appear in the following way:  $\alpha = (\alpha \square (\alpha^C)^{-1})(\beta)$  and  $\alpha = (\alpha^C \square \alpha^{-1})(\beta)$ .

## 2. BI-CONJUGATIVE RELATIONS

Put  $\alpha^1 = \alpha$ . It is easy to see that  $(\alpha^{-1})^C = (\alpha^C)^{-1}$  holds. Definition 1.1 describes equalities

$$\alpha = (\alpha^a)^i \circ \beta \circ (\alpha^b)^j$$

for some  $\beta \in \mathcal{B}(X)$  where  $i, j \in \{-1, 1\}$  and  $a, b \in \{1, C\}$ . We should investigate all other possibilities since some of possibilities given in the previous equation have been investigated. According to this attitude, in the following definition we introduce a new class of elements in  $\mathcal{B}(X)$ .

**Definition 2.1.** For relation  $\alpha \in \mathcal{B}(X)$  we say that it is a *bi-conjugative* relation on  $X$  if there exists a relation  $\beta \in \mathcal{B}(X)$  such that

$$(2.1) \quad \alpha = \alpha^{-1} \circ \beta \circ \alpha^{-1}.$$

It is easy to see that the dual of a bi-conjugative relation  $\alpha$  is again a bi-conjugative relation. Besides, for bi-conjugative relation  $\alpha$  on a set  $X$  the following  $Dom(\alpha) = R(\alpha)$  holds.

The family  $\mathcal{BC}(X)$  of all bi-conjugative relations on set  $X$  is not empty. For example,  $\Delta_X \in \mathcal{BC}(X)$  and  $\nabla_X = \Delta_X^C \in \mathcal{BC}(X)$ . Besides, since for any bijective relation  $\psi$  on  $X$

$$\psi = \Delta_X \circ \psi \circ \Delta_X = (\psi^{-1} \circ \psi) \circ \psi \circ (\psi \circ \psi^{-1}) = \psi^{-1} \circ (\psi \circ \psi \circ \psi) \circ \psi^{-1}$$

holds, we have  $\psi \in \mathcal{BC}(X)$ . For symmetric and idempotent relation  $\alpha$  on set  $X$  we have

$$\alpha = \alpha^2 = \alpha \circ \Delta_X \circ \alpha = \alpha^{-1} \circ \Delta_X \circ \alpha^{-1}.$$

Therefore, this relation is a bi-conjugative relation on  $X$ . Further on, the following implication  $\alpha \in \mathcal{BC}(X) \implies \alpha^{-1} \in \mathcal{BC}(X)$  holds also.

According to the equation (1.1), the condition (2.1) is equivalent to the following condition

$$(2.2) \quad \alpha = (\alpha^{-1} \square \alpha)(\beta).$$

Our first proposition is an adaptation of Schein's result exposed in [10], Theorem 1. (See, also, [2], Lemma 1.)

**Theorem 2.2.** *For a binary relation  $\alpha \in \mathcal{B}(X)$ , relation*

$$\alpha^* = (\alpha \circ \alpha^C \circ \alpha)^C$$

*is the maximal element in the family of all relation  $\beta \in \mathcal{B}(X)$  such that*

$$\alpha^{-1} \circ \beta \circ \alpha^{-1} \subseteq \alpha.$$

*Proof.* First, remember ourself that

$$\max\{\beta \in \mathcal{B}(X) : \alpha^{-1} \circ \beta \circ \alpha^{-1} \subseteq \alpha\} = \cup\{\beta \in \mathcal{B}(X) : \alpha^{-1} \circ \beta \circ \alpha^{-1} \subseteq \alpha\}.$$

Let  $\beta \in \mathcal{B}(X)$  be an arbitrary relation such that  $\alpha^{-1} \circ \beta \circ \alpha^{-1} \subseteq \alpha$ . We will prove that  $\beta \subseteq \alpha^*$ . If not, there is  $(x, y) \in \beta$  such that  $\neg((x, y) \in \alpha^*)$ . The last gives:

$$(x, y) \in \alpha \circ \alpha^C \circ \alpha \iff$$

$$(\exists u, v \in X)((x, u) \in \alpha \wedge (u, v) \in \alpha^C \wedge (v, y) \in \alpha) \iff$$

$$(\exists u, v \in X)((u, x) \in \alpha^{-1} \wedge (u, v) \in \alpha^C \wedge (y, v) \in \alpha^{-1}) \implies$$

$$(\exists u, v \in X)((u, x) \in \alpha^{-1} \wedge (x, y) \in \beta \wedge (y, v) \in \alpha^{-1} \wedge (u, v) \in \alpha^C) \implies$$

$$(\exists u, v \in X)((u, v) \in \alpha^{-1} \circ \beta \circ \alpha^{-1} \subseteq \alpha \wedge (u, v) \in \alpha^C)$$

We got a contradiction. So, there must be  $\beta \subseteq \alpha^*$ .

On the other hand, we should prove that

$$\alpha^{-1} \circ \alpha^* \circ \alpha^{-1} \subseteq \alpha.$$

Let  $(x, y) \in \alpha^{-1} \circ \alpha^* \circ \alpha^{-1}$  be an arbitrary element. Then, there are elements  $u, v \in X$  such that  $(x, u) \in \alpha^{-1}$ ,  $(u, v) \in \alpha^*$  and  $(v, y) \in \alpha^{-1}$ . So, from

$$(u, x) \in \alpha, \neg((u, v) \in \alpha \circ \alpha^C \circ \alpha), (y, v) \in \alpha,$$

we have  $\neg((x, y) \in \alpha^C)$ . Suppose that  $(x, y) \in \alpha^C$ . Then, we have  $(u, v) \in \alpha \circ \alpha^C \circ \alpha$ , which is impossible. Hence, we have to  $(x, y) \in \alpha$  and therefore,  $\alpha^{-1} \circ \alpha^* \circ \alpha^{-1} \subseteq \alpha$ .

Finally, we conclude that  $\alpha^*$  is the maximal element of the family of all relations  $\beta \in \mathcal{B}(X)$  such that  $\alpha^{-1} \circ \beta \circ \alpha^{-1} \subseteq \alpha$ .  $\square$

It is easy to see that holds

$$\begin{aligned}\alpha^* &= \{(x, y) \in X \times X : \alpha^{-1} \circ \{(x, y)\} \circ \alpha^{-1} \subseteq \alpha\} \\ &= \{(x, y) \in X \times X : \alpha^{-1}x \times \alpha^{-1}y \subseteq \alpha\}.\end{aligned}$$

Also, we have  $\alpha^* = ((\alpha \square \alpha^{-1})(\alpha^C))^C$  by the concept exposed in the equation (1.1).

In the following proposition we give a characterization of bi-conjugative relations. It is our adaptation of concept exposed in [6], Theorem 7.2.

**Theorem 2.3.** *For a binary relation  $\alpha$  on a set  $X$ , the following conditions are equivalent:*

- (1)  $\alpha$  is a bi-conjugative relation.
- (2) For all  $x, y \in X$ , if  $(x, y) \in \alpha$ , there exist  $u, v \in X$  such that:
  - (a)  $(u, x) \in \alpha \wedge (y, v) \in \alpha$ ,
  - (b)  $(\forall s, t \in X)((u, s) \in \alpha \wedge (t, v) \in \alpha \implies (s, t) \in \alpha)$ .
- (3)  $\alpha \subseteq \alpha^{-1} \circ \alpha^* \circ \alpha^{-1}$ .

*Proof.* (1)  $\implies$  (2). Let  $\alpha$  be a bi-conjugative relation, i.e. let there exists a relation  $\beta$  such that  $\alpha = \alpha^{-1} \circ \beta \circ \alpha^{-1}$ . Let  $(x, y) \in \alpha$ . Then there exist elements  $u, v \in X$  such that

$$(x, u) \in \alpha^{-1}, (u, v) \in \beta, (v, y) \in \alpha^{-1}.$$

From this follows that there exist elements  $u, v \in X$  such that

$$(u, x) \in \alpha \wedge (y, v) \in \alpha.$$

This proves condition (a).

Now, we check the condition (b). Let  $s, t \in X$  be arbitrary elements such that  $(u, s) \in \alpha$  and  $(t, v) \in \alpha$ . Now, from  $(s, u) \in \alpha^{-1}$ ,  $(u, v) \in \beta$  and  $(v, t) \in \alpha^{-1}$  follows  $(s, t) \in \alpha^{-1} \circ \beta \circ \alpha^{-1} = \alpha$ .

(2)  $\implies$  (1). Define a binary relation

$$\alpha' = \{(u, v) \in X \times X : (\forall s, t \in X)((u, s) \in \alpha \wedge (t, v) \in \alpha \implies (s, t) \in \alpha)\}$$

and show that  $\alpha^{-1} \circ \alpha' \circ \alpha^{-1} = \alpha$  is valid. Let  $(x, y) \in \alpha$ . Then there exist elements  $u, v \in X$  such that the conditions (a) and (b) are hold. We have  $(u, v) \in \alpha'$  by definition of relation  $\alpha'$ .

Further, from  $(x, u) \in \alpha^{-1}$ ,  $(u, v) \in \alpha'$  and  $(v, y) \in \alpha^{-1}$  follows  $(x, y) \in \alpha^{-1} \circ \alpha' \circ \alpha^{-1}$ . Hence, we have  $\alpha \subseteq \alpha^{-1} \circ \alpha' \circ \alpha^{-1}$ . Contrary, let  $(x, y) \in \alpha^{-1} \circ \alpha' \circ \alpha^{-1}$  be an arbitrary pair. There exist elements  $u, v \in X$  such that  $(x, u) \in \alpha^{-1}$ ,  $(u, v) \in \alpha'$  and  $(v, y) \in \alpha^{-1}$ , i.e. such that  $(u, x) \in \alpha$  and  $(y, v) \in \alpha$ . Hence, by definition of relation  $\alpha'$ , follows  $(x, y) \in \alpha$  since  $(u, v) \in \alpha'$ . Therefore,  $\alpha^{-1} \circ \alpha' \circ \alpha^{-1} \subseteq \alpha$ . So, the relation  $\alpha$  is a bi-conjugative relation on  $X$  since there exists a relation  $\alpha'$  such that  $\alpha^{-1} \circ \alpha' \circ \alpha^{-1} = \alpha$ .

(1)  $\iff$  (3). Let  $\alpha$  be a bi-conjugative relation. Then there a relation  $\beta$  such that  $\alpha = \alpha^{-1} \circ \beta \circ \alpha^{-1}$ . Since  $\alpha^* = \max\{\beta \in \mathcal{B}(X) : \alpha^{-1} \circ \beta \circ \alpha^{-1} \subseteq \alpha\}$ , we have  $\beta \subseteq \alpha^*$  and

$\alpha = \alpha^{-1} \circ \beta \circ \alpha^{-1} \subseteq \alpha^{-1} \circ \alpha^* \circ \alpha^{-1}$ . Contrary, let holds  $\alpha \subseteq \alpha^{-1} \circ \alpha^* \circ \alpha^{-1}$ , for a relation  $\alpha$ . Then, we have  $\alpha \subseteq \alpha^{-1} \circ \alpha^* \circ \alpha^{-1} \subseteq \alpha$ . So, the relation  $\alpha$  is bi-conjugative relation on set  $X$ .  $\square$

**Corollary 2.4.** *Let  $(L, \leq)$  be a poset. Relation  $\not\leq$  is not a bi-conjugative relation on  $L$ .*

*Proof.* Let  $\not\leq$  be a bi-conjugative relation on set  $X$ , and let  $x, y \in X$  be elements such that  $x \not\leq y$ . Then, by previous theorem, there exist elements  $u, v \in X$  such that:

- (a)  $u \not\leq x \wedge y \not\leq v$ ;
- (b)  $(\forall s, t \in L)((u \not\leq s \wedge t \not\leq v) \implies s \not\leq t)$ .

Let  $z$  be an arbitrary element and if we put  $z = s = t$  in formula (b), we have

$$(u \not\leq z \wedge z \not\leq v) \implies z \not\leq z.$$

It is a contradiction. Hence,  $\neg(u \not\leq z \wedge z \not\leq v)$ . Follows  $u \leq z \vee z \leq v$ . Further on, let  $s, t \in L$  be arbitrary elements such that  $u \not\leq s$  and  $t \not\leq v$ . For  $z = s$ , from the last disjunction we have  $u \leq s \vee s \leq v$  and also for  $z = t$  we have  $u \leq t \vee t \leq v$ . So, there are fourth possibilities:

- (1)  $u \leq s \wedge u \leq t \wedge, u \not\leq s \wedge t \not\leq v$ .
- (2)  $u \leq s \wedge t \leq v \wedge u \not\leq s \wedge t \not\leq v$ .
- (3)  $s \leq v \wedge t \leq v \wedge u \not\leq s \wedge t \not\leq v$ .
- (4)  $s \leq v \wedge u \leq t \wedge u \not\leq s \wedge t \not\leq v$ .

Since, options (1), (2) and (3) are contradictions, it left the possibility (4). In this case, since  $u \not\leq s \implies (u \not\leq t \vee t \not\leq s)$  holds as a contraposition of the transitivity  $(u \leq t \wedge t \leq s) \implies u \leq s$ , we have  $s \leq v \wedge u \leq t \wedge (u \not\leq t \vee t \not\leq s) \wedge t \not\leq v$ . Finally, since the option  $u \not\leq t$  is in contradiction with  $u \leq t$ , we have to  $t \not\leq s$  which is in contradiction with the consequence  $s \not\leq t$  of implication (b). Therefore, the relation  $\not\leq$  cannot satisfies the condition (b) of Theorem 2.3.  $\square$

**Example 2.5.** Let  $\alpha$  be a bi-conjugative relation on set  $X$ . Then there exists a relation  $\beta$  on  $X$  such that  $\alpha = \alpha^{-1} \circ \beta \circ \alpha^{-1}$ . If  $\theta$  is an equivalence relation on  $X$  and  $\gamma \in \mathcal{B}(X)$ , we define relation

$$\gamma/\theta = \{(a\theta, b\theta) \in X/\theta \times X/\theta : (\exists a' \in X)(\exists b' \in X)((a, a') \in \theta \wedge (a', b') \in \gamma \wedge (b, b') \in \theta)\}.$$

It is easy to that

$$\alpha/\theta = (\alpha/\theta)^{-1} \circ \beta/\theta \circ (\alpha/\theta)^{-1}$$

holds. So, the relation  $\alpha/\theta$  is a bi-conjugative relation on  $X/\theta$ . Therefore, for any equivalence relation  $\theta$  on  $X$  there is a correspondence  $\Phi_\theta : \mathcal{BC}(X) \longrightarrow \mathcal{BC}(X/\theta)$ .

**Example 2.6.** Let  $\alpha'$  be a bi-conjugative element in  $\mathcal{B}(X')$ . Then there exists a relation  $\beta' \in \mathcal{B}(X')$  such that  $\alpha' = (\alpha')^{-1} \circ \beta' \circ (\alpha')^{-1}$ . For a mapping  $f : X \longrightarrow X'$  and a relation  $\gamma' \in \mathcal{B}(X')$  we define  $f^{-1}(\gamma')$  by

$$(x, y) \in f^{-1}(\gamma') \iff (f(x), f(y)) \in \gamma'.$$

If  $f$  is a surjective mapping, we have:

$$(x, y) \in f^{-1}(\alpha') \iff (x, y) \in (f^{-1}(\alpha'))^{-1} \circ f^{-1}(\beta') \circ (f^{-1}(\alpha'))^{-1}.$$

So, the relation  $f^{-1}(\alpha')$  is a bi-conjugative relation in  $\mathcal{B}(X)$ . Since for any equivalence relation  $\theta$  on  $X$ , the mapping  $\pi : X \rightarrow X/\theta$  is a surjective, there is a correspondence  $\Psi_\theta : \mathcal{BC}(X/\theta) \rightarrow \mathcal{BC}(X)$  also.

Further on, if  $\mathcal{E}(X)$  is the family of all equivalence relations on set  $X$ , then for any bi-conjugative relation  $\alpha$  in  $X$  there is the family  $\mathcal{BC}(\alpha) = \{\pi^{-1}(\alpha/\theta) : \theta \in \mathcal{E}(X)\}$  of bi-conjugative relations on  $X$ . Such that subfamily is this one  $\mathcal{BC}(\nabla_{X/\theta}) = \{\pi^{-1}(\nabla_{X/\theta}) : \theta \in \mathcal{E}(X)\}$ .

**Acknowledgement:** The author is grateful to an anonymous referee for helpful comments and suggestions which improved the paper.

#### REFERENCES

- [1] H.J.Bandelt: "Regularity and complete distributivity." Semigroup Forum 19: 123-126, 1980
- [2] H.J.Bandelt: "On regularity classes of binary relations". In: Universal Algebra and Applications. Banach Center Publications, vol. 9: pp. 329-333, 1982
- [3] Jiang Guanghao and Xu Luoshan: "Conjugative Relations and Applications". Semigroup Forum, 80(1): 85-91, 2010
- [4] Jiang Guanghao and Xu Luoshan: "Dually normal relations on sets"; Semigroup Forum, 85(1): 75-80, 2012
- [5] Jiang Guanghao, Xu Luoshan, Cai Jin and Han Guiwen: "Normal Relations on Sets and Applications"; Int. J. Contemp. Math. Sciences, 6(15): 721 - 726, 2011
- [6] D.Hardy and M.Petrich: "Binary relations as lattice isomorphisms"; Ann. Mat. Pura Appl, 177(1): 195-224, 1999
- [7] J.M.Howie: "An introduction to semigroup theory"; Academic press, 1976.
- [8] G.Markowsky: "Idempotents and product representations with applications to the semigroup of binary relations". Semigroup Forum, 5: 95-119, 1972
- [9] D.A.Romano: "Quasi-regular relation on sets - a new class of relations on sets", Publications de l'Institut Mathematique, 93(107): 127-132, 2013
- [10] B.M.Schein: "Regular elements of the semigroup of all binary relations". Semigroup Forum 13: 95-102,1976
- [11] A. Zareckiĭ: "The semigroup of binary relations". Mat. Sb. 61(3): 291-305, 1963 (In Russian)
- [12] Xu Xiao-quan and Liu Yingming. "Relational representations of hypercontinuous lattices", in: *Domain Theory, Logic, and Computation*, Kluwer Academic Publisher, pp. 65-74, 2003

FACULTY OF EDUCATION, EAST SARAJEVO UNIVERSITY, B.B, SEMBERSKI RATARI STREET, 76300 BIJELJINA, BOSNIA AND HERZEGOVINA

*E-mail address:* bato49@hotmail.com