

BI-CONJUGATIVE RELATIONS

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ABSTRACT. In this paper the concept of bi-conjugative relations on sets is introduced. Characterizations of this relations are obtained. In addition, particularly we show that the anti-order relation $\not\leq$ in poset (L, \leq) is not a bi-conjugative relation.

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1. INTRODUCTION AND PRELIMINARIES

The regularity of binary relations was first characterized by Zareckiĭ ([11]). Further criteria for regularity were given by Markowsky ([8]), Schein ([10]) and Xu Xiao-quan and Liu Yingming ([12]) (see also [1] and [2]). The concepts of conjugative relations, dually conjugative relations and dually normal relations were introduced by Guanghao Jiang and Luoshan Xu ([3], [4]), and a characterization of normal relations was introduced and analyzed by Jiang Guanghao, Xu Luoshan, Cai Jin and Han Guiwen in [5].

In this paper, we introduce and analyze bi-conjugative relations on sets.

The following are some basic concepts needed in the sequel, for other nonexplicitly stated elementary notions please refer to papers [1] – [6] and [11], and to book [7].

For a set X , we call ρ a binary relation on X , if $\rho \subseteq X \times X$. Let $\mathcal{B}(X)$ denote the set of all binary relations on X . For $\alpha, \beta \in \mathcal{B}(X)$, define

$$\beta \circ \alpha = \{(x, z) \in X \times X : (\exists y \in X)((x, y) \in \alpha \wedge (y, z) \in \beta)\}.$$

The relation $\beta \circ \alpha$ is called the composition of α and β . It is well known that $(\mathcal{B}(X), \circ)$ a semigroup. The relation $\Delta_X = \{(x, x) : x \in X\}$ is the identity. For a binary relation α on a set X , define $\alpha^{-1} = \{(x, y) \in X \times X : (y, x) \in \alpha\}$ and $\alpha^C = X \times X \setminus \alpha$.

The following classes of elements in the semigroup $\mathcal{B}(X)$, given in the following definition, have been investigated:

Definition 1.1. For relation $\alpha \in \mathcal{B}(X)$ we say that it is:

– *regular* if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha \circ \beta \circ \alpha.$$

– *normal* ([5]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha \circ \beta \circ (\alpha^C)^{-1}.$$

– *dually normal* ([4]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = (\alpha^C)^{-1} \circ \beta \circ \alpha.$$

– *conjugative* ([3]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha^{-1} \circ \beta \circ \alpha.$$

– *dually conjugative* ([3]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha \circ \beta \circ \alpha^{-1}.$$

– *quasi-regular* ([9]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha^C \circ \beta \circ \alpha.$$

– *dually quasi-regular* ([9]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha \circ \beta \circ \alpha^C.$$

Besides that, for $\alpha, \beta \in \mathcal{B}(X)$ and $x, y \in X$, we define the *box product* of relation α and relation β by

$$\begin{aligned} (\alpha \square \beta)(x, y) &= \alpha x \times \beta y \\ &= \{(u, v) \in X \times X : u \in \alpha x \wedge v \in \beta y\}. \end{aligned}$$

Let $\alpha, \beta, \gamma \in \mathcal{B}(X)$ be arbitrary relations, then

$$(1.1) \quad \gamma \circ \beta \circ \alpha = (\alpha \square \gamma^{-1})(\beta)$$

holds. Indeed, we have

$$\begin{aligned} (u, v) \in \gamma \circ \beta \circ \alpha &\iff (\exists a, b \in X)((u, a) \in \alpha \wedge (a, b) \in \beta \wedge (b, v) \in \gamma) \\ &\iff (\exists(a, b) \in \beta)(u \in \alpha a \wedge v \in \gamma^{-1}b) \\ &\iff (\exists(a, b) \in \beta)((u, v) \in \alpha a \times \gamma^{-1}b) \\ &\iff (\exists(a, b) \in \beta)((u, v) \in (\alpha \square \gamma^{-1})(a, b)) \\ &\iff (u, v) \in (\alpha \square \gamma^{-1})(\beta). \end{aligned}$$

Now, we can equations, introduced in Definition 1.1, represent in the new way. For example, a conjugative relation α satisfies the following equation $\alpha = (\alpha \square \alpha)(\beta)$, and if α is a dually conjugative relation, then the following equation $\alpha = (\alpha^{-1} \square \alpha^{-1})(\beta)$ holds. Analogously, a normal relation α is described by $\alpha = ((\alpha^C)^{-1} \square \alpha^{-1})(\beta)$, and a dually normal relation α satisfies the following equation $\alpha = (\alpha \square \alpha^C)(\beta)$. Descriptions of quasi-regular relations and dually quasi-regular relations now appear in the following way: $\alpha = (\alpha \square (\alpha^C)^{-1})(\beta)$ and $\alpha = (\alpha^C \square \alpha^{-1})(\beta)$.

2. BI-CONJUGATIVE RELATIONS

Put $\alpha^1 = \alpha$. It is easy to see that $(\alpha^{-1})^C = (\alpha^C)^{-1}$ holds. Definition 1.1 describes equalities

$$\alpha = (\alpha^a)^i \circ \beta \circ (\alpha^b)^j$$

for some $\beta \in \mathcal{B}(X)$ where $i, j \in \{-1, 1\}$ and $a, b \in \{1, C\}$. We should investigate all other possibilities since some of possibilities given in the previous equation have been investigated. According to this attitude, in the following definition we introduce a new class of elements in $\mathcal{B}(X)$.

Definition 2.1. For relation $\alpha \in \mathcal{B}(X)$ we say that it is a *bi-conjugative* relation on X if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$(2.1) \quad \alpha = \alpha^{-1} \circ \beta \circ \alpha^{-1}.$$

It is easy to see that the dual of a bi-conjugative relation α is again a bi-conjugative relation. Besides, for bi-conjugative relation α on a set X the following $Dom(\alpha) = R(\alpha)$ holds.

The family $\mathcal{BC}(X)$ of all bi-conjugative relations on set X is not empty. For example, $\Delta_X \in \mathcal{BC}(X)$ and $\nabla_X = \Delta_X^C \in \mathcal{BC}(X)$. Besides, since for any bijective relation ψ on X

$$\psi = \Delta_X \circ \psi \circ \Delta_X = (\psi^{-1} \circ \psi) \circ \psi \circ (\psi \circ \psi^{-1}) = \psi^{-1} \circ (\psi \circ \psi \circ \psi) \circ \psi^{-1}$$

holds, we have $\psi \in \mathcal{BC}(X)$. For symmetric and idempotent relation α on set X we have

$$\alpha = \alpha^2 = \alpha \circ \Delta_X \circ \alpha = \alpha^{-1} \circ \Delta_X \circ \alpha^{-1}.$$

Therefore, this relation is a bi-conjugative relation on X . Further on, the following implication $\alpha \in \mathcal{BC}(X) \implies \alpha^{-1} \in \mathcal{BC}(X)$ holds also.

According to the equation (1.1), the condition (2.1) is equivalent to the following condition

$$(2.2) \quad \alpha = (\alpha^{-1} \square \alpha)(\beta).$$

Our first proposition is an adaptation of Schein's result exposed in [10], Theorem 1. (See, also, [2], Lemma 1.)

Theorem 2.2. *For a binary relation $\alpha \in \mathcal{B}(X)$, relation*

$$\alpha^* = (\alpha \circ \alpha^C \circ \alpha)^C$$

is the maximal element in the family of all relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha^{-1} \circ \beta \circ \alpha^{-1} \subseteq \alpha.$$

Proof. First, remember ourself that

$$\max\{\beta \in \mathcal{B}(X) : \alpha^{-1} \circ \beta \circ \alpha^{-1} \subseteq \alpha\} = \cup\{\beta \in \mathcal{B}(X) : \alpha^{-1} \circ \beta \circ \alpha^{-1} \subseteq \alpha\}.$$

Let $\beta \in \mathcal{B}(X)$ be an arbitrary relation such that $\alpha^{-1} \circ \beta \circ \alpha^{-1} \subseteq \alpha$. We will prove that $\beta \subseteq \alpha^*$. If not, there is $(x, y) \in \beta$ such that $\neg((x, y) \in \alpha^*)$. The last gives:

$$(x, y) \in \alpha \circ \alpha^C \circ \alpha \iff$$

$$(\exists u, v \in X)((x, u) \in \alpha \wedge (u, v) \in \alpha^C \wedge (v, y) \in \alpha) \iff$$

$$(\exists u, v \in X)((u, x) \in \alpha^{-1} \wedge (u, v) \in \alpha^C \wedge (y, v) \in \alpha^{-1}) \implies$$

$$(\exists u, v \in X)((u, x) \in \alpha^{-1} \wedge (x, y) \in \beta \wedge (y, v) \in \alpha^{-1} \wedge (u, v) \in \alpha^C) \implies$$

$$(\exists u, v \in X)((u, v) \in \alpha^{-1} \circ \beta \circ \alpha^{-1} \subseteq \alpha \wedge (u, v) \in \alpha^C)$$

We got a contradiction. So, there must be $\beta \subseteq \alpha^*$.

On the other hand, we should prove that

$$\alpha^{-1} \circ \alpha^* \circ \alpha^{-1} \subseteq \alpha.$$

Let $(x, y) \in \alpha^{-1} \circ \alpha^* \circ \alpha^{-1}$ be an arbitrary element. Then, there are elements $u, v \in X$ such that $(x, u) \in \alpha^{-1}$, $(u, v) \in \alpha^*$ and $(v, y) \in \alpha^{-1}$. So, from

$$(u, x) \in \alpha, \neg((u, v) \in \alpha \circ \alpha^C \circ \alpha), (y, v) \in \alpha,$$

we have $\neg((x, y) \in \alpha^C)$. Suppose that $(x, y) \in \alpha^C$. Then, we have $(u, v) \in \alpha \circ \alpha^C \circ \alpha$, which is impossible. Hence, we have to $(x, y) \in \alpha$ and therefore, $\alpha^{-1} \circ \alpha^* \circ \alpha^{-1} \subseteq \alpha$.

Finally, we conclude that α^* is the maximal element of the family of all relations $\beta \in \mathcal{B}(X)$ such that $\alpha^{-1} \circ \beta \circ \alpha^{-1} \subseteq \alpha$. \square

It is easy to see that holds

$$\begin{aligned}\alpha^* &= \{(x, y) \in X \times X : \alpha^{-1} \circ \{(x, y)\} \circ \alpha^{-1} \subseteq \alpha\} \\ &= \{(x, y) \in X \times X : \alpha^{-1}x \times \alpha^{-1}y \subseteq \alpha\}.\end{aligned}$$

Also, we have $\alpha^* = ((\alpha \square \alpha^{-1})(\alpha^C))^C$ by the concept exposed in the equation (1.1).

In the following proposition we give a characterization of bi-conjugative relations. It is our adaptation of concept exposed in [6], Theorem 7.2.

Theorem 2.3. *For a binary relation α on a set X , the following conditions are equivalent:*

- (1) α is a bi-conjugative relation.
- (2) For all $x, y \in X$, if $(x, y) \in \alpha$, there exist $u, v \in X$ such that:
 - (a) $(u, x) \in \alpha \wedge (y, v) \in \alpha$,
 - (b) $(\forall s, t \in X)((u, s) \in \alpha \wedge (t, v) \in \alpha \implies (s, t) \in \alpha)$.
- (3) $\alpha \subseteq \alpha^{-1} \circ \alpha^* \circ \alpha^{-1}$.

Proof. (1) \implies (2). Let α be a bi-conjugative relation, i.e. let there exists a relation β such that $\alpha = \alpha^{-1} \circ \beta \circ \alpha^{-1}$. Let $(x, y) \in \alpha$. Then there exist elements $u, v \in X$ such that

$$(x, u) \in \alpha^{-1}, (u, v) \in \beta, (v, y) \in \alpha^{-1}.$$

From this follows that there exist elements $u, v \in X$ such that

$$(u, x) \in \alpha \wedge (y, v) \in \alpha.$$

This proves condition (a).

Now, we check the condition (b). Let $s, t \in X$ be arbitrary elements such that $(u, s) \in \alpha$ and $(t, v) \in \alpha$. Now, from $(s, u) \in \alpha^{-1}$, $(u, v) \in \beta$ and $(v, t) \in \alpha^{-1}$ follows $(s, t) \in \alpha^{-1} \circ \beta \circ \alpha^{-1} = \alpha$.

(2) \implies (1). Define a binary relation

$$\alpha' = \{(u, v) \in X \times X : (\forall s, t \in X)((u, s) \in \alpha \wedge (t, v) \in \alpha \implies (s, t) \in \alpha)\}$$

and show that $\alpha^{-1} \circ \alpha' \circ \alpha^{-1} = \alpha$ is valid. Let $(x, y) \in \alpha$. Then there exist elements $u, v \in X$ such that the conditions (a) and (b) are hold. We have $(u, v) \in \alpha'$ by definition of relation α' .

Further, from $(x, u) \in \alpha^{-1}$, $(u, v) \in \alpha'$ and $(v, y) \in \alpha^{-1}$ follows $(x, y) \in \alpha^{-1} \circ \alpha' \circ \alpha^{-1}$. Hence, we have $\alpha \subseteq \alpha^{-1} \circ \alpha' \circ \alpha^{-1}$. Contrary, let $(x, y) \in \alpha^{-1} \circ \alpha' \circ \alpha^{-1}$ be an arbitrary pair. There exist elements $u, v \in X$ such that $(x, u) \in \alpha^{-1}$, $(u, v) \in \alpha'$ and $(v, y) \in \alpha^{-1}$, i.e. such that $(u, x) \in \alpha$ and $(y, v) \in \alpha$. Hence, by definition of relation α' , follows $(x, y) \in \alpha$ since $(u, v) \in \alpha'$. Therefore, $\alpha^{-1} \circ \alpha' \circ \alpha^{-1} \subseteq \alpha$. So, the relation α is a bi-conjugative relation on X since there exists a relation α' such that $\alpha^{-1} \circ \alpha' \circ \alpha^{-1} = \alpha$.

(1) \iff (3). Let α be a bi-conjugative relation. Then there a relation β such that $\alpha = \alpha^{-1} \circ \beta \circ \alpha^{-1}$. Since $\alpha^* = \max\{\beta \in \mathcal{B}(X) : \alpha^{-1} \circ \beta \circ \alpha^{-1} \subseteq \alpha\}$, we have $\beta \subseteq \alpha^*$ and

$\alpha = \alpha^{-1} \circ \beta \circ \alpha^{-1} \subseteq \alpha^{-1} \circ \alpha^* \circ \alpha^{-1}$. Contrary, let holds $\alpha \subseteq \alpha^{-1} \circ \alpha^* \circ \alpha^{-1}$, for a relation α . Then, we have $\alpha \subseteq \alpha^{-1} \circ \alpha^* \circ \alpha^{-1} \subseteq \alpha$. So, the relation α is bi-conjugative relation on set X . \square

Corollary 2.4. *Let (L, \leq) be a poset. Relation $\not\leq$ is not a bi-conjugative relation on L .*

Proof. Let $\not\leq$ be a bi-conjugative relation on set X , and let $x, y \in X$ be elements such that $x \not\leq y$. Then, by previous theorem, there exist elements $u, v \in X$ such that:

- (a) $u \not\leq x \wedge y \not\leq v$;
- (b) $(\forall s, t \in L)((u \not\leq s \wedge t \not\leq v) \implies s \not\leq t)$.

Let z be an arbitrary element and if we put $z = s = t$ in formula (b), we have

$$(u \not\leq z \wedge z \not\leq v) \implies z \not\leq z.$$

It is a contradiction. Hence, $\neg(u \not\leq z \wedge z \not\leq v)$. Follows $u \leq z \vee z \leq v$. Further on, let $s, t \in L$ be arbitrary elements such that $u \not\leq s$ and $t \not\leq v$. For $z = s$, from the last disjunction we have $u \leq s \vee s \leq v$ and also for $z = t$ we have $u \leq t \vee t \leq v$. So, there are fourth possibilities:

- (1) $u \leq s \wedge u \leq t \wedge, u \not\leq s \wedge t \not\leq v$.
- (2) $u \leq s \wedge t \leq v \wedge u \not\leq s \wedge t \not\leq v$.
- (3) $s \leq v \wedge t \leq v \wedge u \not\leq s \wedge t \not\leq v$.
- (4) $s \leq v \wedge u \leq t \wedge u \not\leq s \wedge t \not\leq v$.

Since, options (1), (2) and (3) are contradictions, it left the possibility (4). In this case, since $u \not\leq s \implies (u \not\leq t \vee t \not\leq s)$ holds as a contraposition of the transitivity $(u \leq t \wedge t \leq s) \implies u \leq s$, we have $s \leq v \wedge u \leq t \wedge (u \not\leq t \vee t \not\leq s) \wedge t \not\leq v$. Finally, since the option $u \not\leq t$ is in contradiction with $u \leq t$, we have to $t \not\leq s$ which is in contradiction with the consequence $s \not\leq t$ of implication (b). Therefore, the relation $\not\leq$ cannot satisfies the condition (b) of Theorem 2.3. \square

Example 2.5. Let α be a bi-conjugative relation on set X . Then there exists a relation β on X such that $\alpha = \alpha^{-1} \circ \beta \circ \alpha^{-1}$. If θ is an equivalence relation on X and $\gamma \in \mathcal{B}(X)$, we define relation

$$\gamma/\theta = \{(a\theta, b\theta) \in X/\theta \times X/\theta : (\exists a' \in X)(\exists b' \in X)((a, a') \in \theta \wedge (a', b') \in \gamma \wedge (b, b') \in \theta)\}.$$

It is easy to that

$$\alpha/\theta = (\alpha/\theta)^{-1} \circ \beta/\theta \circ (\alpha/\theta)^{-1}$$

holds. So, the relation α/θ is a bi-conjugative relation on X/θ . Therefore, for any equivalence relation θ on X there is a correspondence $\Phi_\theta : \mathcal{BC}(X) \longrightarrow \mathcal{BC}(X/\theta)$.

Example 2.6. Let α' be a bi-conjugative element in $\mathcal{B}(X')$. Then there exists a relation $\beta' \in \mathcal{B}(X')$ such that $\alpha' = (\alpha')^{-1} \circ \beta' \circ (\alpha')^{-1}$. For a mapping $f : X \longrightarrow X'$ and a relation $\gamma' \in \mathcal{B}(X')$ we define $f^{-1}(\gamma')$ by

$$(x, y) \in f^{-1}(\gamma') \iff (f(x), f(y)) \in \gamma'.$$

If f is a surjective mapping, we have:

$$(x, y) \in f^{-1}(\alpha') \iff (x, y) \in (f^{-1}(\alpha'))^{-1} \circ f^{-1}(\beta') \circ (f^{-1}(\alpha'))^{-1}.$$

So, the relation $f^{-1}(\alpha')$ is a bi-conjugative relation in $\mathcal{B}(X)$. Since for any equivalence relation θ on X , the mapping $\pi : X \rightarrow X/\theta$ is a surjective, there is a correspondence $\Psi_\theta : \mathcal{BC}(X/\theta) \rightarrow \mathcal{BC}(X)$ also.

Further on, if $\mathcal{E}(X)$ is the family of all equivalence relations on set X , then for any bi-conjugative relation α in X there is the family $\mathcal{BC}(\alpha) = \{\pi^{-1}(\alpha/\theta) : \theta \in \mathcal{E}(X)\}$ of bi-conjugative relations on X . Such that subfamily is this one $\mathcal{BC}(\nabla_{X/\theta}) = \{\pi^{-1}(\nabla_{X/\theta}) : \theta \in \mathcal{E}(X)\}$.

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