

# FINITELY BI-CONJUGATIVE RELATIONS

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ABSTRACT. In this paper, the concept of finitely bi-conjugative relations is introduced. A characterization of this relations is obtained. Particular we show when the anti-order relation  $\not\leq$  is a finitely bi-conjugative relation.

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## 1. INTRODUCTION

The concept of finitely conjugative relations was introduced by Guanghai Jiang and Luoshan Xu in [1], the concept of finitely dual normal relations was introduced and analyzed by Jiang Guanghai and Xu Luoshan in [2] and the concept of finitely quasi-conjugative relations was introduced and analyzed by these authors in [3]. In this article, we introduce and analyze the notion of finitely bi-conjugative relations as a continuation of our article [4].

For a set  $X$ , we call  $\rho$  a relation on  $X$ , if  $\rho \subseteq X \times X$ . Let  $\mathcal{B}(X)$  be denote the set of all binary relations on  $X$ . For  $\alpha, \beta \in \mathcal{B}(X)$ , define

$$\beta \circ \alpha = \{(x, z) \in X \times X : (\exists y \in X)((x, y) \in \alpha \wedge (y, z) \in \beta)\}.$$

The relation  $\beta \circ \alpha$  is called the composition of  $\alpha$  and  $\beta$ . It is well known that  $(\mathcal{B}(X), \circ)$  is a semigroup. The latter family, with the composition, is not only a semigroup, but also a monoid. Namely,  $\Delta_X = \{(x, x) : x \in X\}$  is its identity element. For a relation  $\alpha$  on a set  $X$ , define  $\alpha^{-1} = \{(x, y) \in X \times X : (y, x) \in \alpha\}$  and  $\alpha^C = (X \times X) \setminus \alpha$ .

Let  $A$  and  $B$  be subsets of  $X$ . For  $\alpha \in \mathcal{B}(X)$ , set

$$A\alpha = \{y \in X : (\exists a \in A)((a, y) \in \alpha)\}, \quad \alpha B = \{x \in X : (\exists b \in B)((x, b) \in \alpha)\}.$$

It is easy to see that  $A\alpha = \alpha^{-1}A$  holds. Specially, we put  $a\alpha$  if  $\{a\}\alpha$  and  $\alpha b$  if  $\alpha\{b\}$ .

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## 2. BI-CONJUGATIVE RELATIONS

The following classes of elements in the semigroup  $\mathcal{B}(X)$  have been investigated:

- *regular* if there exists a relation  $\beta \in \mathcal{B}(X)$  such that

$$\alpha = \alpha \circ \beta \circ \alpha.$$

- *dually normal* ([2]) if there exists a relation  $\beta \in \mathcal{B}(X)$  such that

$$\alpha = (\alpha^C)^{-1} \circ \beta \circ \alpha.$$

- *conjugative* ([1]) if there exists a relation  $\beta \in \mathcal{B}(X)$  such that

$$\alpha = \alpha^{-1} \circ \beta \circ \alpha.$$

- *dually conjugative* ([1]) if there exists a relation  $\beta \in \mathcal{B}(X)$  such that

$$\alpha = \alpha \circ \beta \circ \alpha^{-1}.$$

The notion of *bi-conjugative relation* was introduced in the paper [4] by the following way:

**Definition 2.1.** For a relation  $\alpha \in \mathcal{B}(X)$  we say that it is a *bi-conjugative* relation on  $X$  if there exists a relation  $\beta \in \mathcal{B}(X)$  such that

$$\alpha = \alpha^{-1} \circ \beta \circ \alpha^{-1}.$$

The family  $\mathcal{BC}(X)$  of all bi-conjugative relations on set  $X$  is not empty. For example,  $\Delta_X \in \mathcal{BC}(X)$  and  $\nabla_X = \Delta_X^C \in \mathcal{BC}(X)$ .

## 3. FINITELY BI-CONJUGATIVE RELATIONS

In this section we introduce the concept of finitely bi-conjugative relations and give a characterization of this relations. For that we need the concept of *finite extension* of a relation. That notion and belonging notation we borrow from articles [1] and [2]. For any set  $X$ , let  $X^{(<\omega)} = \{F \subseteq X : F \text{ is finite and nonempty}\}$ .

**Definition 3.1.** ([1], Definition 3.3; [2], Definition 3.4) Let  $\alpha$  be a binary relation on a set  $X$ . Define a binary relation  $\alpha^{(<\omega)}$  on  $X^{(<\omega)}$ , called the *finite extension* of  $\alpha$ , such that

$$(\forall F, G \in X^{(<\omega)})((F, G) \in \alpha^{(<\omega)} \iff G \subseteq F\alpha).$$

From Definition 3.1, we immediately obtain that

$$(\forall F, G \in X^{(<\omega)})((F, G) \in (\alpha^{-1})^{(<\omega)} \iff G \subseteq F\alpha^{-1} = \alpha F).$$

Now, we can introduce concept of *finitely bi-conjugative relation*.

**Definition 3.2.** A relation  $\alpha$  on a set  $X$  is called *finitely bi-conjugative* if there exists a relation  $\beta^{(<\omega)}$  on  $X^{(<\omega)}$  such that

$$\alpha^{(<\omega)} = (\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^{-1})^{(<\omega)}.$$

Although it seems, in accordance with Definition 2.1, it would be better to call a relation  $\alpha$  on  $X$  to be finitely bi-conjugative if its finite extension to  $X^{(<\omega)}$  is a bi-conjugative relation, we will not use that option. That concept is different from our concept given by Definition 3.2.

Now we give an essential characterization of finitely bi-conjugative relations.

**Theorem 3.1.** *A relation  $\alpha$  on a set  $X$  is a finitely bi-conjugative relation if and only if for all  $F, G \in X^{(<\omega)}$ , if  $G \subseteq F\alpha$ , then there are  $U, V \in X^{(<\omega)}$ , such that*

- (i)  $U \subseteq \alpha F$ ,  $G \subseteq \alpha V$ , and
- (ii) for all  $S, T \in X^{(<\omega)}$ , if  $U \subseteq \alpha S$  and  $T \subseteq \alpha V$  then  $T \subseteq S\alpha$ .

*Proof.* (Necessity) Let  $\alpha$  be a finitely bi-conjugative relation on set  $X$ . Then there is a relation  $\beta^{(<\omega)} \subseteq (X^{(<\omega)})^2$  such that  $(\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^{-1})^{(<\omega)} = \alpha^{(<\omega)}$ . For all  $(F, G) \in (X^{(<\omega)})^2$ , if  $G \subseteq F\alpha$ , i.e.,  $(F, G) \in \alpha^{(<\omega)}$ , thus  $(F, G) \in (\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^{-1})^{(<\omega)}$ . Therefore, there are  $U$  and  $V$  in  $(X^{(<\omega)})$  such that  $(F, U) \in (\alpha^{-1})^{(<\omega)}$ ,  $(U, V) \in \beta^{(<\omega)}$  and  $(V, G) \in (\alpha^{-1})^{(<\omega)}$ , i.e.,  $U \subseteq F\alpha^{-1} = \alpha F$ ,  $G \subseteq V\alpha^{-1} = \alpha V$ . Hence we have got the condition (i).

Now we check the condition (ii). For all  $(S, T) \in (X^{(<\omega)})^2$ , if  $U \subseteq \alpha S$  and  $T \subseteq \alpha V$ , i.e.,  $(S, U) \in (\alpha^{-1})^{(<\omega)}$  and  $(V, T) \in (\alpha^{-1})^{(<\omega)}$ , then by  $(U, V) \in \beta^{(<\omega)}$ , we have  $(S, T) \in (\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^{-1})^{(<\omega)}$ , i.e.,  $(S, T) \in \alpha^{(<\omega)}$ . Hence  $T \subseteq S\alpha$ .

(Sufficiency) Let  $\alpha$  be a relation on a set  $X$  such that for  $F, G \in X^{(<\omega)}$  with  $G \subseteq F\alpha$  there are  $U, V \in X^{(<\omega)}$  such that conditions (i) and (ii) hold. Define a binary relation  $\beta^{(<\omega)} \subseteq X^{(<\omega)} \times X^{(<\omega)}$  by

$$(F, G) \in \beta \iff (\forall S, T \in X^{(<\omega)})(F \subseteq \alpha S \wedge T \cap \alpha G \neq \emptyset \implies T \cap S\alpha \neq \emptyset).$$

First, check that

$$(3.1) \quad (\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^{-1})^{(<\omega)} \subseteq \alpha^{(<\omega)}$$

holds. For all  $H, W \in X^{(<\omega)}$ , if  $(H, W) \in (\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^{-1})^{(<\omega)}$ , then there are  $F, G \in X^{(<\omega)}$  with  $(H, F) \in (\alpha^{-1})^{(<\omega)}$ ,  $(F, G) \in \beta^{(<\omega)}$  and  $(G, W) \in (\alpha^{-1})^{(<\omega)}$ . Then  $F \subseteq H\alpha^{-1} = \alpha H$  and  $W \subseteq G\alpha^{-1} = \alpha G$ . For all  $w \in W$ , let  $S = H$ ,  $T = \{w\}$ . Then  $F \subseteq \alpha S$  and  $\alpha G \cap T \neq \emptyset$  because  $w \in T$  and  $w \in \alpha G$ . Since  $(F, G) \in \beta^{(<\omega)}$ , we have that  $F \subseteq \alpha S \wedge \alpha G \cap T \neq \emptyset$  implies  $T \cap S\alpha \neq \emptyset$ . Hence,  $w \in S\alpha$ , i.e.  $W \subseteq S\alpha$ . So, we have  $(H, W) = (S, W) \in \alpha^{(<\omega)}$ . Therefore, we have  $(\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^{-1})^{(<\omega)} \subseteq \alpha^{(<\omega)}$ .

The second, check that

$$(3.2) \quad \alpha^{(<\omega)} \subseteq (\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^{-1})^{(<\omega)}$$

holds. For all  $H, W \in X^{(<\omega)}$ , if  $(H, W) \in \alpha^{(<\omega)}$  (i.e.,  $W \subseteq H\alpha$ ), there are  $A, B \in X^{(<\omega)}$  such that:

- (i')  $A \subseteq \alpha H$ ,  $W \subseteq \alpha B$ , and
- (ii') for all  $S, T \in X^{(<\omega)}$ , if  $A \subseteq \alpha S$  and  $T \subseteq \alpha B$ , then  $T \subseteq S\alpha$ .

Now, we have to show that  $(A, B) \in \beta^{(<\omega)}$ . Let be for all  $(C, D) \in (X^{(<\omega)})^2$  the following  $A \subseteq \alpha D$  and  $D \cap \alpha B \neq \emptyset$  hold. From  $D \cap \alpha B \neq \emptyset$  follows that there exists an element  $d \in D \cap \alpha B (\neq \emptyset)$ . So,  $d \in D$  and  $d \in \alpha B$ . Put  $S = C$  and  $T = \{d\}$ . Then, by (ii'), we have

$$(A \subseteq \alpha S \wedge T = \{d\} \subseteq \alpha B) \implies \{d\} = T \subseteq S\alpha,$$

i.e.  $\emptyset \neq D \cap S\alpha = T \cap S\alpha$ . Therefore,  $(A, B) \in \beta^{(<\omega)}$  by definition of  $\beta^{(<\omega)}$ . Finally, for  $(H, A) \in (\alpha^{-1})^{(<\omega)}$ ,  $(A, B) \in \beta^{(<\omega)}$  and  $(B, W) \in (\alpha^{-1})^{(<\omega)}$  follows that  $(H, W) \in (\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^{-1})^{(<\omega)}$ .

By assertion (3.1) and (3.2), we have  $\alpha^{(<\omega)} = (\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^{-1})^{(<\omega)}$ . So,  $\alpha$  is a finitely bi-conjugative relation on  $X$ .  $\square$

Particulary, if we put  $F = \{x\}$  and  $G = \{y\}$  in the Theorem 3.1, we give the following consequent.

**Corollary 3.1.** *Let  $\alpha$  be a relation on a set  $X$ . Then  $\alpha$  is a finitely bi-conjugative relation on  $X$  if and only if for all elements  $x, y \in X$  such that  $(x, y) \in \alpha$  there are finite subsets  $U = \{u_1, u_2, \dots, u_m\}$  and  $V = \{v_1, v_2, \dots, v_n\}$  in  $X^{(<\omega)}$  such that*

(1<sup>0</sup>)  $(\forall i \in \{1, 2, \dots, m\})((u_i, x) \in \alpha) \wedge (\exists j \in \{1, 2, \dots, n\})((y, v_j) \in \alpha)$ , and

(2<sup>0</sup>) for all  $S = \{s_1, s_2, \dots, s_p\}$  in  $X^{(<\omega)}$  and  $t \in X$  holds

$$(U \subseteq \alpha S \wedge (\exists v_j \in V)((t, v_j) \in \alpha)) \implies (\exists s_k \in S)((s_k, t) \in \alpha) .$$

*Proof.* Let  $\alpha$  be a finitely bi-conjugative relation on  $X$  and let  $x, y$  be elements of  $X$  such that  $(x, y) \in \alpha$ . If we put  $F = \{x\}$  and  $G = \{y\}$  in Theorem 3.1, then there exist finite subsets  $U = \{u_1, u_2, \dots, u_m\}$  and  $V = \{v_1, v_2, \dots, v_n\}$  in  $X^{(<\omega)}$  such that conditions (1<sup>0</sup>) and (2<sup>0</sup>) hold.

Opposite, let for all elements  $x, y$  in  $X$  such that  $(x, y) \in \alpha$  there are  $U = \{u_1, u_2, \dots, u_m\}$  and  $V = \{v_1, v_2, \dots, v_n\}$  in  $X^{(<\omega)}$  such that conditions (1<sup>0</sup>) and (2<sup>0</sup>) hold. Define binary relation  $\beta^{(<\omega)} \subseteq X^{<\omega} \times X^{<\omega}$  by

$$(A, B) \in \beta^{(<\omega)} \iff (\forall S \in X^{<\omega})(\forall t \in X)((A \subseteq \alpha S \wedge t \in \alpha B) \implies t \in S\alpha).$$

The proof that the equality  $(\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^{-1})^{(<\omega)} = \alpha^{(<\omega)}$  holds is some as in the Theorem 3.1. So,  $\alpha$  is a finitely bi-conjugative relation.  $\square$

At end of this note, we show when the anti-order relation  $\not\leq$  on poset  $(L, \leq)$  is a finitely bi-conjugative idempotent relation.

**Theorem 3.2.** *Let  $(L, \leq)$  be a poset. Then the relation  $\not\leq$  on  $L$  is a finitely be-conjugative relation if and only if for all  $x, y \in L$  such that  $x \not\leq y$ , there exist finite subsets  $U = \{u_1, u_2, \dots, u_m\}$  and  $V = \{v_1, v_2, \dots, v_n\}$  of  $L$  such that*

(a)  $(\forall i \in \{1, 2, \dots, m\})(u_i \not\leq x)$  and  $(\exists j \in \{1, 2, \dots, n\})(y \not\leq v_j)$  and

(b)  $(\forall z \in L)((\exists i \in \{1, 2, \dots, m\})(u_i \leq z) \vee (\forall j \in \{1, 2, \dots, n\})(z \leq v_j))$ .

*Proof.* Let  $x, y \in L$  such that  $x \not\leq y$ . Then, since  $\not\leq$  is a finitely bi-conjugative relation on  $L$ , there exist finite subsets  $U = \{u_1, u_2, \dots, u_m\}$  and  $V = \{v_1, v_2, \dots, v_n\}$  of  $L$  such that

(1)  $(\forall i \in \{1, 2, \dots, m\})(u_i \not\leq x) \wedge (\exists j \in \{1, 2, \dots, n\})(y \not\leq v_j)$ , and

(2) for all  $S = \{s_1, s_2, \dots, s_p\}$  in  $L^{(<\omega)}$  and  $t \in L$  the following holds

$$((\forall u_i \in U)(\exists s_k \in S)(u_i \not\leq s_k) \wedge (\exists v_j \in V)(t \not\leq v_j)) \implies (\exists s_{k'} \in S)(s_{k'} \not\leq t).$$

For  $z \in L$ , let  $S = \{z\} = \{t\}$ . Then by (2), from

$$(\forall u_i \in U)(u_i \not\leq z) \wedge (\exists v_j \in V)(z \not\leq v_j)$$

implies  $z \not\leq z$ . It is a contradiction. Hence, we have

$$\neg((\forall u_i \in U)(u_i \not\leq z) \wedge (\exists v_j \in V)(z \not\leq v_j)).$$

So, finally, we have

$$(\exists i \in \{1, 2, \dots, m\})(u_i \leq z) \vee (\forall j \in \{1, 2, \dots, n\})(z \leq v_j).$$

Let for  $(x, y) \in L^2$  be  $x \not\leq y$  holds and let there exist finite subsets  $U = \{u_1, u_2, \dots, u_m\}$  and  $V = \{v_1, v_2, \dots, v_n\}$  of  $L$  satisfying conditions (a) and (b). So, the condition (a) is the condition (1<sup>0</sup>) in Corollary 3.1.

Let  $S \in L^{(<\omega)}$  and  $t \in L$  with

$$(\forall u_i \in U)(\exists s_k \in S)(u_i \not\leq s_k) \text{ and } (\exists v_j \in V)(t \not\leq v_j)$$

holds. Suppose that  $(\forall s_k \in S)(s_k \leq t)$  holds. Then, by (b), for  $S = \{s\}$  and  $z = s$ , we have

$$(\exists u_i \in U)(u_i \leq s) \vee (\forall v_j \in V)(s \leq v_j).$$

The first option is impossible because  $(\forall u_i)(u_i \not\leq s)$ . Let the option  $(\forall v_j \in V)(s \leq v_j)$  be valid. Then from  $(\exists v_j \in V)(t \not\leq v_j)$  and  $s \leq t$  follows  $(\exists v_j \in V)(s \not\leq v_j)$ . It is in contradiction with  $(\forall v_j \in V)(s \leq v_j)$ . So, must to be  $\neg(\forall s_k \in S)(s_k \leq t)$ . Thus  $(\exists s_k \in S)(s_k \not\leq t)$ . Hence  $\not\leq$  satisfies also condition (2<sup>0</sup>) in Corollary 3.1. Finally, the relation  $\not\leq$  is a finitely bi-conjugative relation on  $L$ .  $\square$

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