BOUNDING THE GRAPHICAL PARAMETERS BY THE INDEPENDENT AND k - DOMINATION NUMBERS

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Abstract. In this note, we present upper bounds for the chromatic number, isoperimetric number, edge connectivity that involve the independent and k - domination numbers of graphs.

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1. Introduction and Main Results

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [1]. Let $G = (V(G), E(G))$ be a graph. We use $n(G)$ and $e(G)$ to denote the number of vertices and edges of the graph G, respectively. For a vertex v_i in a graph G, we use $d_i(G)$ to denote its degree in G. We use $\Delta(G) = d_1(G) \geq d_2(G) \geq \cdots \geq d_n(G) = \delta(G)$ to denote the degree sequence of the graph G. For disjoint subsets S and T of the vertex set $V(G)$ of a graph G, let $E(S,T)$ be the set of the edges in G that join a vertex in S and a vertex in T. We use $G_1 \vee G_2$ to denote the the join of two disjoint graphs G_1 and G_2. The graph consists of p isolated vertices is denoted by E_p. We also use $\chi(G)$, $\kappa'(G)$, $\alpha(G)$ to denote the chromatic number, edge connectivity, and independent number of the graph G, respectively. The eccentricity of a vertex v in a graph G is the greatest distance from v to other vertices. The diameter, denoted $d(G)$, of a graph G is the value of the greatest eccentricity. The radius, denoted $r(G)$, of a graph G is the value of the smallest eccentricity. The girth, denoted $g(G)$, of a graph G is the length of a shortest cycle in the graph G. The definition of the isoperimetric number of a graph can be found in [8] or [9]. The isoperimetric number, denoted $\eta(G)$, of a graph G is defined as $\min\{|E(S, V - S)|/|S|$: where S is a nonempty subset of V with $|S| \leq n/2\}$. The definition of the k - dominating set of a graph can be found in [4] or Page 184 in [5]. Let $k \geq 1$ be an integer, a subset D of the vertex set $V(G)$ of a graph G is called a k - dominating set if $|N(u) \cap D| \geq k$ for each vertex $u \in V(G) - D$. The minimum cardinality of a k - dominating set is called the k - dominating number, denoted $\gamma_k(G)$, of the graph G. The 1 - dominating set of G is also called a dominating set. We also use $\gamma(G)$ to denote $\gamma_1(G)$. A k - dominating set D is minimum if $|D|$ is as small as possible. The eigenvalues $\mu_n(G) \leq \mu_{n-1}(G) \leq ... \leq \mu_1(G)$ of the adjacency matrix $A(G)$ of a graph G are called the eigenvalues of G. The Laplacian of a graph G is defined as $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of the degree sequence of G and $A(G)$ is the adjacency matrix of G. The eigenvalues $0 = \lambda_n(G) \leq \lambda_{n-1}(G) \leq ... \leq \lambda_1(G)$ of $L(G)$ are called the Laplacian eigenvalues of a graph G.

In this note, we present upper bounds for the chromatic number, isoperimetric number, and edge connectivity that involve the independent and k - domination numbers of graphs. The main results are as follows.

Theorem 1. Let G be a graph of order n, size e, and independent number α. Then

$$\chi \leq \frac{(1+\delta)+\sqrt{(1+\delta)^2+4((n-\alpha)(n+\alpha-1)-n\delta)}}{2}$$

with equality if and only if G is K_n or E_n.

Theorem 2. Let G be a graph of order $n \geq 2$, size e, minimum degree $\delta \geq 1$, and dominating number γ. Then

$$\chi \leq \frac{(1+\delta)+\sqrt{(1+\delta)^2+4(n^2-(1+\delta)n-2r^2+2r)}}{2}.$$

Notice that when G is a complete graph the upper bound for χ in Theorem 2 is attainable.

Theorem 3. Let G be a graph of order $n \geq 2$ and D a minimum k - dominating set such that $|E(G[D])|$ is as large as possible. If $\delta(G) \geq 2k-1$ and $|N(u) \cap D| \geq k-1$ for each vertex $u \in D$, then

$$\eta \leq n+1-\left(1+\frac{1}{k}\right)\gamma_k.$$

Theorem 4. Let G be a graph of order $n \geq 2$ and D a minimum k - dominating set such that $|E(G[D])|$ is as large as possible. If $\delta \geq k$ and $|N(u) \cap D| \geq k-1$ for each vertex $u \in D$, then

$$\kappa' \leq \gamma_k\left(n+1-\left(1+\frac{1}{k}\right)\gamma_k\right).$$

2. Lemmas

We need the following results as lemmas to prove the theorems above. The following Lemma 1 is Corollary 8.1.1 on Page 118 in [1].

Lemma 1. If G is a graph with chromatic number χ, then G has at least χ vertices of degree at least $\chi - 1$.

The following Lemma 2 follows from the proof of Theorem 3 on Page 191 in [6].

Lemma 2. Let G be a graph of order $n \geq 2$ and D a minimum k - dominating set such that $|E(G[D])|$ is as large as possible. If $\delta \geq k$ and $|N(u) \cap D| \geq k-1$ for each vertex $u \in D$, then

$$|E(D, V-D)| \leq |D|\left(n-\left(1+\frac{1}{k}\right)|D|+1\right).$$

The following Lemma 3 is Corollary 2 on Page 206 in [2].

Lemma 3. Let G be a graph of order n and k an integer. If $\delta \geq 2k-1$, then $\gamma_k \leq \frac{n}{2}$.

The following Lemma 4 is Corollary 2.16 on Page 52 in [5]. It was proved by Payan in [10].

Lemma 4. If a graph G has no isolated vertices, then

$$\gamma \leq \frac{n+2-\delta}{2}.$$

3. Proofs

Proof of Theorem 1. From Lemma 1, we have that $d_1 \geq d_2 \geq \cdots \geq d_\chi \geq \chi - 1$. Then

$$\chi(\chi-1)+(n-\chi)\delta \leq \sum_{i=1}^{\chi} d_i + \sum_{i=\chi+1}^{n} d_i = \sum_{i=1}^{n} d_i = 2e.$$

Since the independent number of G is α, we have an independent set S in G such that $|S| = \alpha$. Now

$$\begin{aligned} 2e &= 2\left(|E(S, V-S)| + |E(G[V-S])|\right) \\ &\leq 2\left(\alpha(n-\alpha) + \frac{(n-\alpha)(n-\alpha-1)}{2}\right) = (n-\alpha)(n+\alpha-1). \end{aligned}$$

Thus

$$\chi(\chi-1)+(n-\chi)\delta \leq (n-\alpha)(n+\alpha-1).$$

Namely,

$$\chi^2 - (1+\delta)\chi - ((n-\alpha)(n+\alpha-1) - n\delta) \leq 0.$$

Choose a vertex, say x, in S. Then $\delta \leq d(x) \leq n - \alpha$. Therefore

$$n\delta \leq n(n-\alpha) \leq (n-\alpha)(n+\alpha-1).$$

Hence

$$(1+\delta)^2 + 4((n-\alpha)(n+\alpha-1) - n\delta) \geq 0.$$

Solving the above inequality, we have

$$\chi \leq \frac{(1+\delta) + \sqrt{(1+\delta)^2 + 4((n-\alpha)(n+\alpha-1) - n\delta)}}{2}.$$

Suppose that G is K_n. Then we can easily compute that both sides of the inequality in Theorem 1 are equal to n. Suppose that G is E_n. Then we can easily compute that both sides of the inequality in Theorem 1 are equal to 1.

Now suppose that both sides of the inequality in Theorem 1 are equal. From the proofs above, we have that

[*i*] For each vertex $x \in S$ and each vertex $y \in V - S$, $xy \in E$.

[*ii*] $G[V-S]$ is complete.

Thus [*i*] and [*ii*] imply that G is $K_{n-\alpha} \vee E_\alpha$. Thus $\delta = n - \alpha$ and $\chi = n - \alpha + 1$. Therefore $(n-\alpha)n = \chi(\chi-1) + (n-\chi)\delta = 2e = (n-\alpha)(n+\alpha-1)$. Hence $\alpha = 1$ and G is K_n or $n = \alpha$ and G is E_n. □

Proof of Theorem 2. From Lemma 1, we have that $d_1 \geq d_2 \geq \cdots \geq d_\chi \geq \chi - 1$. Then

$$\chi(\chi-1)+(n-\chi)\delta \leq \sum_{i=1}^{\chi} d_i + \sum_{i=\chi+1}^{n} d_i = \sum_{i=1}^{n} d_i = 2e.$$

Let $k = 1$ in Lemma 2. Then we have $|E(D, V-D)| \leq |D|(n - 2|D| + 1)$, where D is the minimum 1 - dominating set (i.e., the minimum dominating set) satisfying the conditions in Lemma 2. Notice that the above inequality was first established by Liu, Lu, and Tian in [7]. It was generalized in [6]. Now

$$\begin{aligned} 2e &= 2\left(|E(G[D])| + |E(D, V-D)| + |E(G[V-D])|\right) \\ &\leq 2\left(\frac{\gamma(\gamma-1)}{2} + \gamma(n-2\gamma+1) + \frac{(n-\gamma)(n-\gamma-1)}{2}\right) \end{aligned}$$

$$= \gamma(\gamma-1) + 2\gamma(n-2\gamma+1) + (n-\gamma)(n-\gamma-1).$$

Thus

$$\chi(\chi-1) + (n-\chi)\delta \leq \gamma(\gamma-1) + 2\gamma(n-2\gamma+1) + (n-\gamma)(n-\gamma-1).$$

Namely,

$$\chi^2 - (1+\delta)\chi - (n^2 - (1+\delta)n - 2r^2 + 2r) \leq 0.$$

Next, we will show that

$$(1+\delta)^2 + 4(n^2 - (1+\delta)n - 2r^2 + 2r) \geq 0$$

so that we can solve the inequality above to obtain the desired upper bound for χ in Theorem 2. It suffices to prove that $(n^2 - (1+\delta)n - 2r^2 + 2r) \geq 0$.

If $\gamma = 1$, then, from $n(n-1) \geq n\delta$, we can derive that $n^2+2 \geq n+2+n\delta$. Namely, $n^2+2\gamma \geq n+2\gamma^2+n\delta$. Thus $(n^2 - (1+\delta)n - 2r^2 + 2r) \geq 0$.

Now we suppose that $\gamma \geq 2$. From Lemma 3, we have that $\gamma \leq \frac{n}{2}$. Thus $\frac{n}{2} \leq (n-\gamma)(\gamma-1)$. This implies that $2\gamma^2 - 2\gamma n + 3n \leq 2\gamma$. From Lemma 4, we have that $\delta \leq n+2-2\gamma$. Therefore

$$n^2 + 2\gamma \geq n + 2\gamma^2 + n^2 - 2\gamma n + 2n$$

$$\geq n + 2\gamma^2 + n(n-2\gamma+2) \geq n + 2\gamma^2 + n\delta.$$

Solving the above inequality, we have that

$$\chi \leq \frac{(1+\delta) + \sqrt{(1+\delta)^2 + 4(n^2 - (1+\delta)n - 2r^2 + 2r)}}{2}.$$

□

Proof of Theorem 3. Notice that $2k-1 \geq k$. Let D be a minimum k - dominating set of G satisfying the conditions in Lemma 2. Then by Lemma 2 we have

$$|E(D, V-D)| \leq |D| \left(n - \left(1 + \frac{1}{k}\right)|D| + 1\right).$$

From Lemma 4, we have that $|D| = \gamma_k \leq \frac{n}{2}$. Thus

$$\eta \leq \frac{|E(D,V-D)|}{|D|} \leq n - \left(1+\frac{1}{k}\right)|D| + 1 = n + 1 - \left(1+\frac{1}{k}\right)\gamma_k.$$

□

Proof of Theorem 4. Let D be a minimum k - dominating set of G satisfying the conditions in Lemma 4. Then $|D| = \gamma_k$. Notice that G is disconnected if we remove all the edges between D and $V-D$. Then by Lemma 2 we have that

$$\kappa' \leq |E(D,V-D)| \leq |D| \left(n - \left(1+\frac{1}{k}\right)|D| + 1\right) = \gamma_k \left(n + 1 - \left(1+\frac{1}{k}\right)\gamma_k\right).$$

□

4. Applications of Some of the Main Results

Applying Theorem 3 with $k = 1$ to a graph of order $n \geq 2$ with $\delta \geq 1$, we have the following corollary.

Corollary 1. Let G be a graph of order $n \geq 2$ with $\delta \geq 1$. Then

$$\eta \leq n + 1 - 2\gamma.$$

From Corollary 1, we can see that each lower bound for γ gives an upper bound for η and each lower bound for η also gives an upper bound for γ. Below are three lower bounds for γ established by DeLaViña, Pepper, and Waller in [3].

Theorem 5. Let G be a connected graph order $n \geq 2$ and diameter d. Then

$$\gamma(G) \geq \frac{d+1}{3}.$$

Theorem 6. Let G be a connected graph of order $n \geq 2$ and radius r. Then

$$\gamma(G) \geq \frac{2}{3}r.$$

Moreover, this bound is sharp.

Theorem 7. Let G be a connected graph of order $n \geq 2$ and girth g. Then

$$\gamma(G) \geq \frac{1}{3}g.$$

Moreover, this bound is sharp.

Below are two lower bounds for η which were obtained by Mohar in [8].

Theorem 8. Let G be a graph of order $n \geq 2$. Then

$$\eta \geq \frac{1}{2}\lambda_{n-1}.$$

Theorem 9. Let G be a graph of order $n \geq 2$. Then

$$\eta \geq \frac{1}{2}(\delta - \mu_2).$$

Then from Corollary 1 and Theorems 5, 6, 7, 8, and 9, we have following corollaries.

Corollary 2. Let G be a connected graph of order $n \geq 2$ and diameter d. Then

$$\eta \leq n + 1 - \frac{2(d+1)}{3}.$$

Corollary 3. Let G be a connected graph of order $n \geq 2$ and radius r. Then

$$\eta \leq n + 1 - \frac{4}{3}r.$$

Corollary 4. Let G be a connected graph of order $n \geq 2$ and girth g. Then

$$\eta \leq n + 1 - \frac{2}{3}g.$$

Corollary 5. Let G be a graph of order $n \geq 2$ with $\delta \geq 1$. Then

$$\gamma \leq \frac{n+1}{2} - \frac{\lambda_{n-1}}{4}.$$

Corollary 6. Let G be a graph of order $n \geq 2$ with $\delta \geq 1$. Then

$$\gamma \leq \frac{n+1}{2} - \frac{\delta(G) - \mu_2}{4}.$$

Finally, Theorem 4 has the following corollary.

Corollary 7. Let G be a graph of order $n \geq 2$ with $\delta \geq 1$. Then

$$\kappa' \leq \gamma(n + 1 - 2\gamma).$$

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