

# ON THE UPPER BOUND OF THE ENERGY OF A CONNECTED GRAPH

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**ABSTRACT.** New upper bounds for the energy of a connected graph are presented in this note. The upper bounds involve the independence number of the graph.

**Mathematics Subject Classification (2010):** 05C50

**Keywords:** Upper bound, Energy, Eigenvalue.

*Article history:*

Received 23 November 2015

Received in revised form 23 December 2015

Accepted 28 December 2015

## 1. INTRODUCTION

All the graphs considered in this note are undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [1]. Let  $G$  be a graph of order  $n$  with  $e$  edges. We use  $\delta = \delta(G)$  and  $\Delta = \Delta(G)$  to denote the minimum and maximum degrees of  $G$ , respectively. The independence number, denoted  $\alpha = \alpha(G)$ , is defined as the size of the largest independent set in  $G$ . The 2 - degree, denoted  $t(v)$ , of a vertex  $v$  in  $G$  is defined as the sum of degrees of vertices adjacent to  $v$ . We use  $T = T(G)$  to denote the maximum 2 - degree of  $G$ . Obviously,  $T(G) \leq (\Delta(G))^2$ . A bipartite graph  $G$  is called semiregular if all the vertices in the same vertex part of a bipartition of the vertex set of  $G$  have the same degree. The eigenvalues  $\mu_1(G) \geq \mu_2(G) \geq \dots \geq \mu_n(G)$  of the adjacency matrix  $A(G)$  of  $G$  are called the eigenvalues of  $G$ . The spread, denoted  $Spr(G)$ , of  $G$  is defined as  $\mu_1(G) - \mu_n(G)$ . The energy, denoted  $Eng(G)$ , of  $G$  is defined as  $\sum_{i=1}^n |\mu_i(G)|$  (see [7]).

Several authors have obtained the upper bounds for the energy of a graph (see [5], [8], [9], [12], [13]). In this note, we will present new upper bounds for the energy of a connected graph. The results are as follows.

**Theorem 1.** *Let  $G$  be a connected graph with  $n \geq 2$  vertices and  $e$  edges. Then*

$$Eng(G) \leq 2\sqrt{e} + 2\sqrt{(n - \alpha - 1) \left( e + \sqrt{T \lfloor \frac{n}{2} \rfloor \lfloor \frac{n}{2} \rfloor} - \frac{2\delta^2\alpha}{n - \alpha} \right)}$$

*with equality if and only if  $G$  is  $K_{1,1}$  or  $K_{1,2}$ .*

Obviously, Theorem 1 has the following corollary.

**Corollary 1.** *Let  $G$  be a connected graph with  $n \geq 2$  vertices and  $e$  edges. Then*

$$Eng(G) \leq 2\sqrt{e} + 2\sqrt{(n - \alpha - 1) \left( e + \Delta \sqrt{\lfloor \frac{n}{2} \rfloor \lfloor \frac{n}{2} \rfloor} - \frac{2\delta^2\alpha}{n - \alpha} \right)}$$

*with equality if and only if  $G$  is  $K_{1,1}$ .*

## 2. LEMMAS

In order to prove Theorem 1, we need the following lemmas. Lemma 1 below is Theorem 3.14 on Pages 88 and 89 in [4].

**Lemma 1.** *Let  $G$  be a graph. If the number of eigenvalues of  $G$  which are greater than, less than, and equal to zero are  $p$ ,  $q$ , and  $r$ , respectively, then*

$$\alpha \leq r + \min\{p, q\},$$

where  $\alpha$  is the independence number of  $G$ .

Lemma 2 below is Theorem 1.5 on Page 26 in [6].

**Lemma 2.** *For a graph  $G$  with  $n$  vertices and  $e$  edges,*

$$Spr(G) \leq \mu_1 + \sqrt{2e - \mu_1^2} \leq 2\sqrt{e}.$$

*Equality holds throughout if and only if equality holds in the first inequality; equivalently, if and only if  $e = 0$  or  $G$  is  $K_{a,b}$  for some  $a, b$  with  $e = ab$  and  $a + b \leq n$ .*

Lemma 3 below is obvious.

**Lemma 3.** *If  $x \geq 0$  and  $y \geq 0$ , then  $\sqrt{x} + \sqrt{y} \leq \sqrt{2(x+y)}$  with equality if and only if  $x = y$ .*

Lemma 4 below is Corollary 3.4 on Page 2731 in [10].

**Lemma 4.** *Let  $G$  be a graph. Then  $Spr(G) \geq 2\delta\sqrt{\frac{\alpha(G)}{n-\alpha(G)}}$ . If equality holds, then  $G$  is a semiregular bipartite graph.*

Lemma 5 is Theorem 1 on Page 5 in [2].

**Lemma 5.** *Let  $G$  be a connected graph. Then  $\mu_1 \leq \sqrt{T(G)}$  with equality if and only if  $G$  is either a regular graph or a semiregular bipartite graph.*

Lemma 6 follows from Proposition 2 on Page 174 in [3].

**Lemma 6.** *Let  $G$  be a graph. Then  $\mu_n \geq -\sqrt{\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor}$  with equality if and only if  $G$  is  $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ .*

## 3. PROOFS

Next, we will present proofs for Theorem 1.

**Proof of Theorem 1.** Let  $\mu_1 \geq \mu_2 \geq \dots \geq \mu_p$  be the  $p$  positive eigenvalues of  $G$  and let  $\rho_q \geq \rho_{q-1} \geq \dots \geq \rho_1$  be the  $q$  negative eigenvalues of  $G$ . Then  $G$  has  $n - p - q$  eigenvalues which are equal to zero. From Lemma 1, we have

$$\alpha \leq (n - p - q) + \min\{p, q\}.$$

Thus  $\alpha \leq (n - p - q) + q$  and  $\alpha \leq (n - p - q) + p$ . Namely,  $p \leq n - \alpha$  and  $q \leq n - \alpha$ . Since  $\sum_{i=1}^p \mu_i + \sum_{i=1}^q \rho_i = 0$ , we have that

$$Eng(G) = 2 \sum_{i=1}^p \mu_i = 2 \sum_{i=1}^q |\rho_i|.$$

From Cauchy - Schwarz inequality, we have that

$$\frac{Eng(G)}{2} = \sum_{i=1}^p \mu_i \leq \mu_1 + \sqrt{(p-1) \sum_{i=2}^p \mu_i^2} = \mu_1 + \sqrt{(p-1) \left( \sum_{i=1}^p \mu_i^2 - \mu_1^2 \right)}.$$

Similarly, we have that

$$\frac{Eng(G)}{2} = \sum_{i=1}^q |\rho_i| \leq |\rho_1| + \sqrt{(q-1) \sum_{i=2}^q \rho_i^2} = |\rho_1| + \sqrt{(q-1) \left( \sum_{i=1}^q \rho_i^2 - \rho_1^2 \right)}.$$

Hence we get that

$$\begin{aligned} Eng(G) &= \frac{Eng(G)}{2} + \frac{Eng(G)}{2} \\ &\leq \mu_1 + \sqrt{(p-1) \left( \sum_{i=1}^p \mu_i^2 - \mu_1^2 \right)} + |\rho_1| + \sqrt{(q-1) \left( \sum_{i=1}^q \rho_i^2 - \rho_1^2 \right)}. \end{aligned}$$

Then by Lemmas 2 and 3 it follows that

$$\begin{aligned} Eng(G) &\leq 2\sqrt{e} + \sqrt{n-\alpha-1} \left( \sqrt{\left( \sum_{i=1}^p \mu_i^2 - \mu_1^2 \right)} + \sqrt{\left( \sum_{i=1}^q \rho_i^2 - \rho_1^2 \right)} \right) \\ &\leq 2\sqrt{e} + \sqrt{n-\alpha-1} \sqrt{2 \left( \sum_{i=1}^p \mu_i^2 - \mu_1^2 + \sum_{i=1}^q \rho_i^2 - \rho_1^2 \right)}. \end{aligned}$$

Since  $\sum_{i=1}^p \mu_i^2 + \sum_{i=1}^q \rho_i^2 = \text{the trace of } A^2 = \text{the sum of diagonal entries of } A^2 = \text{the sum of degrees of vertices in } G = 2e$ , we get that

$$\begin{aligned} Eng(G) &\leq 2\sqrt{e} + \sqrt{2(n-\alpha-1)(2e - \mu_1^2 - \rho_1^2)} \\ &= 2\sqrt{e} + \sqrt{2(n-\alpha-1)(2e - (\mu_1 - \rho_1)^2 - 2\mu_1\rho_1)}. \end{aligned}$$

Then by Lemmas 4, 5, and 6 we get that

$$\begin{aligned} Eng(G) &\leq 2\sqrt{e} + \sqrt{2(n-\alpha-1) \left( 2e + 2\sqrt{T\left[\frac{n}{2}\right] \lfloor \frac{n}{2} \rfloor} - \frac{4\delta^2\alpha}{n-\alpha} \right)} \\ &= 2\sqrt{e} + 2\sqrt{(n-\alpha-1) \left( e + \sqrt{T\left[\frac{n}{2}\right] \lfloor \frac{n}{2} \rfloor} - \frac{2\delta^2\alpha}{n-\alpha} \right)}. \end{aligned}$$

If  $G$  is  $K_{1,1}$  or  $K_{1,2}$ , it is trivial to verify that

$$Eng(G) = 2\sqrt{e} + 2\sqrt{(n-\alpha-1) \left( e + \sqrt{T\left[\frac{n}{2}\right] \lfloor \frac{n}{2} \rfloor} - \frac{2\delta^2\alpha}{n-\alpha} \right)}.$$

If

$$Eng(G) = 2\sqrt{e} + 2\sqrt{(n-\alpha-1) \left( e + \sqrt{T\left[\frac{n}{2}\right] \lfloor \frac{n}{2} \rfloor} - \frac{2\delta^2\alpha}{n-\alpha} \right)},$$

then, from the proofs above, we have that  $p = q = n - \alpha$  and  $G = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ . Since  $G$  is connected, its adjacency matrix is irreducible. From Perron - Frobenius theorem, we have that  $p = 1$  (see [11]). Thus  $\alpha = n - 1$ . Hence  $G$  must be  $K_{1,1}$  or  $K_{1,2}$ .  $\square$

#### 4. ACKNOWLEDGMENTS

The author would like to thank the referee for his or her suggestions which improve the original manuscript.

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