

# DISTANCE ROMAN DOMINATION IN RANDOM GRAPHS

ELAHE SHARIFI, NADER JAFARI RAD

**ABSTRACT.** For a positive integer  $k$ , a subset  $D \subseteq V(G)$  is called a *distance- $k$  dominating set* of  $G$  if every vertex in  $V(G) - D$  is within distance  $k$  from some vertex of  $D$ . The minimum cardinality among all distance- $k$  dominating sets of  $G$  is called the *distance- $k$  domination number* of  $G$ . For any positive integer  $r$ , a function  $f : V(G) \rightarrow \{0, 1, 2\}$  is a *Roman  $r$ -dominating function* if every vertex  $u$  for which  $f(u) = 0$  is adjacent to at least  $r$  vertices  $v$  for which  $f(v) = 2$ . The weight of a Roman  $r$ -dominating function is the value  $f(V(G)) = \sum_{u \in V(G)} f(u)$ . The *Roman  $r$ -domination number* of a graph  $G$  is the minimum weight of a Roman  $r$ -dominating function on  $G$ . We study distance- $k$  domination number and Roman  $r$ -domination number in Random graphs by considering a combined variant namely distance- $k$  Roman  $r$ -domination number.

**Mathematics Subject Classification (2010):** 05C69

**Keywords:** Domination; Roman domination; Distance domination; Random graph.

*Article history:*

Received 9 June 2016

Received in revised form 9 August 2016

Accepted 10 August 2016

## 1. INTRODUCTION

Let  $G = (V, E)$  be a finite, undirected and simple graph with vertex set  $V = V(G)$  and edge set  $E = E(G)$ . The number of vertices  $|V|$  is called the order of  $G$  and is denoted by  $n = n(G)$ . We denote the *open neighborhood* of a vertex  $v$  of  $G$  by  $N_G(v)$ , or just  $N(v)$ , and its *closed neighborhood* by  $N_G[v]$  or  $N[v]$ . For a vertex set  $S \subseteq V(G)$ , we denote  $N(S) = \cup_{v \in S} N(v)$  and  $N[S] = \cup_{v \in S} N[v]$ . The *degree* of a vertex  $x$ ,  $\deg(x)$  (or  $\deg_G(x)$  to refer  $G$ ) in a graph  $G$  denotes the number of neighbors of  $x$  in  $G$ . We refer  $\delta(G)$  as the *minimum degree* of the vertices of  $G$ . A set of vertices  $S$  in  $G$  is a *dominating set*, if  $N[S] = V(G)$ . The *domination number*,  $\gamma(G)$ , of  $G$  is the minimum cardinality of a dominating set of  $G$ . For references and also terminology on domination in graphs see for example [10, 12].

For a graph  $G$ , let  $f : V(G) \rightarrow \{0, 1, 2\}$  be a function, and let  $(V_0, V_1, V_2)$  be the ordered partition of  $V(G)$  induced by  $f$ , where  $V_i = \{v \in V(G) : f(v) = i\}$  and for  $i = 0, 1, 2$ . There is a 1-1 correspondence between the functions  $f : V(G) \rightarrow \{0, 1, 2\}$  and the ordered partition  $(V_0, V_1, V_2)$  of  $V(G)$ . So we will write  $f = (V_0, V_1, V_2)$ . A function  $f : V(G) \rightarrow \{0, 1, 2\}$  is a *Roman dominating function* (RDF) if every vertex  $u$  for which  $f(u) = 0$  is adjacent to at least one vertex  $v$  for which  $f(v) = 2$ . The weight of an RDF  $f$  is the value  $f(V(G)) = \sum_{u \in V(G)} f(u)$ . The *Roman domination number* of a graph  $G$ , denoted by  $\gamma_R(G)$ , is the minimum weight of an RDF on  $G$ . Roman domination numbers have been studied, for example, in [4, 17, 18].

For a positive integer  $k$ , a subset  $D \subseteq V(G)$  is called a *distance- $k$  dominating set* of  $G$  if every vertex in  $V(G) - D$  is within distance  $k$  from some vertex of  $D$ . The minimum cardinality among all distance- $k$  dominating sets of  $G$  is called the *distance- $k$  domination number* of  $G$ . In this paper we denote the distance- $k$  domination number of  $G$  by  $\gamma^k(G)$ . The concept of distance- $k$  domination in graphs was introduced by Henning et al. [11] and further studied for example in [8, 15, 16, 19, 20]. Fink and Jacobson

[6, 7] introduced the concept of  $r$ -domination for a positive integer  $r$ . A subset  $D \subseteq V(G)$  is called an  $r$ -dominating set of  $G$  if every vertex in  $V(G) - D$  is adjacent to at least  $r$  vertices of  $D$ . The minimum cardinality among all  $r$ -dominating set of  $G$  is called the  $r$ -domination number of  $G$  and is denoted by  $\gamma_r(G)$ . This concept was further studied, for example in [3, 5, 21, 22].

Kammerling and Volkmann [14] extended the concept of Roman domination to *Roman  $r$ -domination*, for any positive integer  $r$ . A function  $f : V(G) \rightarrow \{0, 1, 2\}$  is a *Roman  $r$ -dominating function* if every vertex  $u$  for which  $f(u) = 0$  is adjacent to at least  $r$  vertex  $v$  for which  $f(v) = 2$ . The weight of a Roman  $r$ -dominating function is the value  $f(V(G)) = \sum_{u \in V(G)} f(u)$ . The *Roman  $r$ -domination number* of a graph  $G$ , denoted by  $\gamma_{rR}(G)$ , is the minimum weight of a Roman  $r$ -dominating function on  $G$ .

Several authors studied domination parameters in Random graphs, see for example [1, 2, 13, 23]. Our aim in this paper is to study the concepts of Roman  $r$ -domination and distance- $k$  domination in Random graphs. For this purpose we define a new invariant namely distance- $k$  Roman  $r$ -domination which is a generalization of Roman  $r$ -domination and distance- $k$  domination. A function  $f : V(G) \rightarrow \{0, 1, 2\}$  is a *distance- $k$  Roman  $r$ -dominating function* if every vertex  $u$  for which  $f(u) = 0$  is within distance  $k$  of at least  $r$  vertex  $v$  for which  $f(v) = 2$ . The weight of a distance- $k$  Roman  $r$ -dominating function is the value  $f(V(G)) = \sum_{u \in V(G)} f(u)$ . The *distance- $k$  Roman  $r$ -domination number* of a graph  $G$ , denoted by  $\gamma_R^{(k,r)}(G)$ , is the minimum weight of a distance- $k$  Roman  $r$ -dominating function on  $G$ . It is obvious that  $\gamma_R^{(1,r)}(G) = \gamma_{rR}(G)$ . Also if a graph  $G$  has a distance- $k$  Roman 1-dominating function  $f = (V_0, V_1, V_2)$  with  $V_1 = \emptyset$ , then  $\gamma_R^{(k,1)}(G) \geq 2\gamma^k(G)$ , and thus  $\gamma^k(G) = \frac{1}{2}\gamma_R^{(k,1)}(G)$ , since clearly  $\gamma_R^{(k,1)}(G) \leq 2\gamma^k(G)$ . Throughout this paper we assume that  $r < \frac{n}{2}$ .

## 2. MAIN RESULTS

Let  $n$  be a positive integer and  $0 < p < 1$ . The *random graph*  $G(n, p)$  is a probability space over the set of graphs on the vertex set  $[n] = \{1, \dots, n\}$  determined by  $Pr[\{i, j\} \in E(G)] = p$  with these events mutually independent. We say that an event holds *asymptotically almost surely* (a.a.s.) if the probability that it holds tends to 1 as  $n$  tends to infinity. Note that by definition the weigh of any distance- $k$  Roman  $r$ -dominating set must be at least  $2r$ . It is well known that for constant  $p < 1$ , the diameter of  $G(n, p)$  is two a.a.s. Thus if  $p$  is constant and  $k \geq 2$  then a.a.s.  $\gamma_R^{(2,r)}(G(n, p)) = 2r$ . The case  $p$  constant and  $k = 1$  will be addressed as an open problem. We next assume that  $p$  is not constant.

**Theorem 2.1** (Bollobas, [2]). *Let  $c$  be a positive constant,  $d = d(n) \geq 2$  a natural number, and define  $p = p(n, c, d)$ ,  $0 < p < 1$ , by  $p^d n^{d-1} = \log(n^2/c)$ . Suppose that  $pn/(\log n)^3 \rightarrow \infty$ . Then in  $G(n, p)$ , we have*

$$(1) \quad \lim_{n \rightarrow \infty} Pr(\text{diam } G = d) = e^{-c/2},$$

$$(2) \quad \lim_{n \rightarrow \infty} Pr(\text{diam } G = d + 1) = 1 - e^{-c/2}.$$

From Theorem 2.1, the following can be obtained readily.

**Theorem 2.2.** *For any positive integers  $k \geq 3$  and  $r$ , in a random graph  $G(n, p)$  with  $p = \sqrt[k]{\frac{\log(n^2/c)}{n^{k-1}}}$ , a.a.s  $\gamma_R^{(k,r)}(G(n, p)) = 2r$ .*

Next we consider the case  $k = 2$ .

**Theorem 2.3** (Hopcraft and Kannan, [13]). *Let  $p = c\sqrt{\frac{\ln n}{n}}$ . For  $c > \sqrt{2}$ ,  $G(n, p)$  almost surely has diameter less than or equal to two.*

From Theorem 2.3 for  $p \geq \sqrt{2}\sqrt{\frac{\ln n}{n}}$  we obtain that  $\gamma_R^{(2,r)}(G(n, p)) = 2r$  a.a.s. We will weaken the minimum value of  $p$  from  $\sqrt{2}\sqrt{\frac{\ln n}{n}}$  to  $p \geq c\sqrt{\frac{\ln n}{n}}$ , for a fixed constant  $c > 1$ .

**Theorem 2.4.** *Let  $c > 1$  be a fixed constant. For any positive integer  $r$ , in a random graph  $G(n, p)$  with  $p \geq c\sqrt{\frac{\ln n}{n}}$ , a.a.s.  $\gamma_R^{(2,r)}(G(n, p)) = 2r$ .*

*Proof.* Let  $D \subseteq V(G(n, p))$  be a subset with  $|D| = r$ . Let the vertices in  $D$  be labeled as  $v_1, v_2, \dots, v_r$ . The probability that a vertex  $u \in V(G(n, p)) \setminus D$  is not within distance-2 from a vertex  $v_i \in D$  is given by  $Pr(u \notin N_2(v_i)) \leq (1 - p^2)^{n-2}$ . Let  $X$  be a random variable that denotes the number of vertices  $u \in V(G(n, p)) \setminus D$ , where the number vertices of  $D$  within distance 2 from  $u$  is less than  $r$ . We show that  $Pr(X > 0) \rightarrow 0$  as  $n \rightarrow \infty$ .

A fixed vertex  $u$  is defined *bad*, if there is less than  $r$  vertices in  $D$  within distance two from  $u$ . By the linearity property of the expectation we have

$$(2.1) \quad E(X) = (n - r)Pr(\text{fixed } u \text{ is bad}).$$

Let  $X_u$  be a random variable that denotes the number of vertices in  $D$  that are not within distance two from  $u$ . Then  $E(X_u) \leq r(1 - p^2)^{n-2} \leq re^{-p^2(n-2)}$ . By the Markov's inequality we have  $Pr(X_u > 0) \leq E(X_u) \leq re^{-p^2(n-2)}$ . Thus,

$$(2.2) \quad Pr(\text{fixed } u \text{ is bad}) = Pr(X_u > 0) \leq re^{-p^2(n-2)}.$$

By (2.1) and (2.2), we have  $E(X) \leq (n - r)re^{-p^2(n-2)}$ . By the Markov's inequality we obtain,

$$(2.3) \quad Pr(X > 0) \leq E(X) \leq (n - r)re^{-p^2(n-2)} < nre^{-p^2(n-2)}.$$

Since  $n \rightarrow \infty$ , in (2.3), we have  $e^{p^2(n-2)} > rn$  for sufficiently large  $n$ . This implies that  $p^2(n-2) > \ln rn$  and so  $p > \sqrt{\frac{\ln rn}{n-2}}$ . We conclude that  $p > \sqrt{\frac{\ln n}{n}}$ . Let  $p > c\sqrt{\frac{\ln n}{n}}$ , where  $c > 1$  is a constant. We determine the value of  $e^{p^2(n-2)}$ .

$$(2.4) \quad e^{p^2(n-2)} \geq (e^{\ln n})^{c^2(\frac{n-2}{n})} \geq n^{c^2(1-\frac{2}{n})}.$$

From (2.3) and (2.4) we have

$$(2.5) \quad nre^{-p^2(n-2)} \leq \frac{nr}{n^{c^2(1-\frac{2}{n})}} = \frac{r}{n^{c^2(1-\frac{2}{n})-1}}.$$

Since  $c^2 > 1$  as  $n \rightarrow \infty$ ,  $c^2(1 - \frac{2}{n}) > 1$ , and hence,  $c^2(1 - \frac{2}{n}) - 1 > 0$ . Thus, as  $n \rightarrow \infty$ ,

$$\frac{r}{n^{c^2(1-\frac{2}{n})-1}} \rightarrow 0.$$

Therefore, from (2.3) and (2.5) we have  $Pr(X > 0) \rightarrow 0$  as  $n \rightarrow \infty$ . □

Thus the remaining case is  $k = 1$ . We propose the following problem.

**Problem 2.5.** *For  $k = 1$  and  $p \in (0, 1)$  ( $p$  is not necessarily constant) determine  $\gamma_R^{(1,r)}(G(n, p))$  a.a.e.*

### 3. CONCLUDING REMARKS

We end the paper with stating some probabilistic bounds for the distance- $k$  Roman  $r$ -domination number in graphs using similar results on Roman domination and  $r$ -domination numbers. It is obvious that  $\gamma_R^{(k,r)}(G) = 2r$  if  $\text{diam}(G) \leq k$ . Thus we assume that  $\text{diam}(G) > k$ . For a vertex  $v$  let  $N_k(v)$  be the set of all vertices  $u$  such that  $u \neq v$  and is within distance- $k$  from  $v$ , and let  $\delta_k = \delta_k(G) = \min\{|N_k(v)| : v \in V(G)\}$ . We also define the  $k$ -graph  $G^k$  as the graph with vertex set  $V(G^k) = V(G)$ , and  $E(G^k) = \{xy : d_G(x, y) \leq k\}$ . Note that  $G^1 = G$ . Hansberg and Volkmann, [9] proved that if  $G$  is a graph on  $n$  vertices with  $\delta(G) \geq r$ , where  $r$  is a positive integer, and  $\frac{\delta(G)+1+2\ln 2}{\ln(\delta(G)+1)} \geq 2r$ , then  $\gamma_{rR}(G) \leq \left(\frac{2r \ln(\delta(G)+1) - \ln 4 + 2}{\delta(G)+1}\right)n$ . It is obvious that  $\gamma_R^{(k,r)}(G) = \gamma_{rR}(G^k)$ . Thus from the above upper bound and with an identical proof as the proof of Theorem 11 of [23], we obtain the following.

**Theorem 3.1.** If  $\frac{\delta_k+1+2\ln 2}{\ln(\delta_k+1)} \geq 2r$  and  $\delta_k \geq r$ , then

$$\gamma_R^{(k,r)}(G) \leq \left( \frac{2r \ln(\delta_k + 1) - \ln 4 + 2}{\delta_k + 1} \right) n.$$

This bound is asymptotically best possible.

#### REFERENCES

- [1] N. Alon and J. Spencer, *The Probabilistic Method*, John Wiley, New York, (1992).
- [2] B. Bollobas, *Random Graphs*, Cambridge University Press (2001).
- [3] B. Chen and S. Zhou, Upper bounds for  $f$ -domination number of graphs, *Discrete Math.* 185 (1998), 239-243.
- [4] E. J. Cockayne, P. M. Dreyer Jr., S. M. Hedetniemi, and S. T Hedetniemi, Roman domination in graphs, *Discrete Math.* 278 (2004), 11-22.
- [5] O. Favaron, A. Hansberg and L. Volkmann, On  $k$ -domination and minimum degree in graphs, *J. Graph Theory* 57 (2008), 33-40.
- [6] J. F. Fink and M. S. Jacobson, *n*-domination in graphs *Graph Theory with Applications to Algorithms and Computer Science*, John Wiley and Sons, New York, 1985, pp. 283-300.
- [7] J. F. Fink and M. S. Jacobson, *On n*-domination, *n*-dependence and forbidden subgraphs, *Graph Theory with Applications to Algorithms and Computer Science*, John Wiley and Sons, New York, 1985, pp. 301-311.
- [8] A. Hansberg, D. Meierling and L. Volkmann, Distance Domination and Distance Irredundance in Graphs, *Elec. J. Combin.* (2007), R35.
- [9] A. Hansberg and L. Volkmann, Upper bounds on the  $k$ -domination number and the  $k$ -Roman domination number, *Discrete Appl. Math.* 157 (2009), 1634–1639.
- [10] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, (1998).
- [11] M. A. Henning, O. R. Oellermann, and H. C. Swart, Bounds on distance domination parameters, *J. Combin. Inform. System Sci.* 16 (1991), 11–18.
- [12] M.A. Henning, and A. Yeo, *Total domination in graphs*, Springer Monographs in Mathematics, Springer, New York, (2013).
- [13] J. Hopcroft, and R. Kannan, Mathematics for modern computing, Preprint (2013).
- [14] K. Kammerling and L. Volkmann, Roman  $k$ -domination in graphs, *J. Korean Math. Soc.* 46 (2009), No. 6, pp. 1309-1318.
- [15] J. Raczek, M. Lemanska, and J. Cyman, Lower bound on the distance  $k$ -domination number of a tree, *Math. Slovaca* 56 (2006), 235–243.
- [16] J. Raczek, Distance paired domination numbers of graphs, *Discrete Math.* 308 (2008), 2473–2483.
- [17] C. S. ReVelle, and K. E. Rosing, Defendens imperium romanum: a classical problem in military strategy, *Amer. Math. Monthly* 107 (2000), 585-594.
- [18] I. Stewart, Defend the Roman Empire!, *Sci. Amer.* 281 (1999), 136-139.
- [19] N. Sridharan, V. S.A. Subramanian and M.D. Elias, Bounds on the distance two-domination number of a graph, *Graphs and Combin.* 18 (2002), 667-675.
- [20] F. Tian and J. M. Xu, A note on distance domination number of graphs, *Australas. J. Combin.* 43 (2009), 181-190.
- [21] L. Volkmann, A bound on the  $k$ -domination number of a graph, *Czech. Math. J.* 60 (2010), 77–83.
- [22] S. Zhou, Inequalities involving independence domination,  $f$ -domination, connected and total  $f$ -domination numbers, *Czech. Math. J.* 50 (2000), 321-330.
- [23] V. Zverovich, A. Poghosyan, On Roman, Global and Restrained Domination in Graphs, *Graphs and Combin.* 27 (2011), 755–768.

DEPARTMENT OF MATHEMATICS, SHAHROOD UNIVERSITY OF TECHNOLOGY, SHAHROOD, IRAN  
*E-mail address:* [n.jafarirad@gmail.com](mailto:n.jafarirad@gmail.com)