# DERIVATIONS ON PSEUDO-Q-ALGEBRAS

ŞULE AYAR ÖZBAL, ALEV FIRAT

Abstract. In this paper, we study derivations on pseudo-$Q$-algebras and investigate some of their properties.

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## 1. Introduction

BCK-algebras and BCI-algebras were introduced by Imai and Iseki as two classes of abstract algebras in 1966 [6], [7]. It is known that the class of BCK-algebras is a proper subclass of BCI-algebras. In 1983, BCH-algebras as a wide class of abstract algebras were introduced by Hu and Li [11], [16]. In their study, it is given that the class of BCI-algebras are proper subclasses of BCH-algebras. In 1999, the notion of d-algebras that is another useful generalization of BCK-algebras was introduced by Neggers and Kim [17]. In 2001, a new notion called a Q-algebras was introduced by J. Neggers, S. S. Ahn and H. S. Kim [10]. At the same time pseudo-BCK-algebras as an extension of BCK-algebras was introduced by G. Geordscu, and A. Iorgulescu [3]. In 2008, pseudo-BCK-algebras as a natural generalization of BCI-algebras and pseudo-BCK-algebras were introduced by W. A. Dudek and Y. B. Jun [12]. These algebras have also connections with other algebras of logics such as pseudo-MV-algebras and pseudo-BL-algebras defined by G. Georgesuc and A. Iorgulescu [5] and [4], respectively. As a generalization of many algebras, these pseudo algebras has been studied by many researchers [1], [2], [14] and [8], [13]. In the last decade, many papers about derivations on different algebras are written by many researchers. Derivations is a very interesting and important area of researchers in the theory of algebraic structures in mathematics. Indeed the concept has been inspired by derivations on rings. In this paper we apply the notion of derivation in ring and near-ring theory to pseudo-Q algebras, and using the notion of derivation we investigate some of its properties.

## 2. Pseudo-Q-algebras

**Definition 2.1.** [15] *A pseudo-Q-algebra is a non-empty set $X$ with a constant $0$ and two binary operations $*$ and $\circ$ satisfying the following axioms:*

*(PQ1)* $x * x = x \circ x = 0$;

*(PQ2)* $x * 0 = x \circ 0 = x$;

*(PQ3)* $(x * y) \circ z = (x \circ z) * y$, *for all* $x, y, z \in X$.

**Definition 2.2.** [15] *Let* $(X;*,\circ,0)$ *be a pseudo-Q-algebras and let* $\emptyset \neq I \subset X$*.* $I$ *is called a pseudo subalgebra of* $X$ *if* $x*y, x\circ y \in I$ *whenever* $x, y \in I$*.* $I$ *is called ideal of* $X$ *if it satisfies:*

*(I1)* $0 \in I$*;*

*(I2)* $x*y$ *or* $x\circ y \in I$ *and* $y \in I$ *imply* $x \in I$ *for all* $x, y \in X$*.*

**Proposition 2.3.** [15] *If* $(X;*,\circ,0)$ *is a pseudo-Q-algebra, then (PQ4)* $(x*(x\circ y))\circ y = (x\circ(x*y))*y = 0$*, for any* $x, y \in X$*.*

**Definition 2.4.** [15] *Let* $(X;*,\circ,0)$ *be a pseudo-Q-algebras. Define the relation "* $\leq$ *" on* $X$ *by*

$x \leq y$ *if and only if* $x*y = 0$ *(or equivalently,* $x\circ y = 0$*)*

*for all* $x, y \in X$*.*

**Proposition 2.5.** [15] *In a pseudo-Q-algebra* $(X;*,\circ,0)$ *the following properties hold for all* $x, y \in X$*:*

*(1)* $x \leq 0 \Rightarrow x = 0$*;*

*(2)* $x*(x\circ y) \leq y, x\circ(x*y) \leq y$*;*

*(3)* $0*x = 0\circ x$*;*

*(4)* $x \leq y \Rightarrow 0*x = 0\circ y$*;*

*(5)* $0\circ(0*(0\circ x)) = 0\circ x$*,* $0*(0\circ(0*x)) = 0*x$*;*

*(6)* $0*(x*y) = (0\circ x)\circ(0*y)$*;*

*(7)* $0\circ(x\circ y) = (0*x)*(0\circ y)$*.*

**Theorem 2.6.** [18] *Let* $(X;*,\circ,0)$ *be a pseudo-Q-algebra. The following statements are equivalent:*

*(i)* $x*(y*z) = (x*y)*z$*, for all* $x, y, z \in X$*;*

*(ii)* $0*x = x = 0\circ x$*, for every* $x \in X$*;*

*(iii)* $x*y = x\circ y = y*x$*, for all* $x, y \in X$*;*

*(iv)* $x\circ(y\circ z) = (x\circ y)\circ z$*, for all* $x, y, z \in X$*.*

**Theorem 2.7.** [15] *For any pseudo Q-algebra* $X$*, the set*

$$K(X) = \{x \in X | 0 \preceq x\}$$

*is a pseudo subalgebra of* $X$*.*

**Definition 2.8.** [15] *A pseudo Q-algebra* $X$ *is said to be* $\circ$*-medial if it satisfies the following identity*

$$(x*y)\circ(z*u) = (x*z)\circ(y*u)$$

*for all* $x, y, z, u$ *in* $X$*.*

**Proposition 2.9.** [15] *Every $\circ$-medial pseudo $Q$-algebra $X$ satisfies the following identities: For any $x, y \in X$*

**(i):** $x * y = 0 \circ (y * x)$
**(ii):** $0 \circ (0 * x) = x$
**(iii):** $x \circ (x * y) = y$

[15] Let $X$ be a pseudo $Q$-algebra. For any non-empty subset $S$ of $X$, we define

$$G(S) := \{x \in S | 0 * x = x = 0 \circ x\}$$

.

In particular, if $S = X$ then we say $G(X)$ is the $G$-part of $X$.

**Definition 2.10.** [15] *An element $a$ of a pseudo $Q$-algebra $X$ is called a pseudo atom of $X$ if for every $x \in X$, $x \preceq a$ implies $x = a$.*

**Theorem 2.11.** [15] *Let $X$ be a pseudo $Q$-algebra. Then the following are equivalent for all $x, y, z, w, u \in X$*

**(i):** *$w$ is a pseudo atom of $X$.*
**(ii):** $w = x \circ (x * w)$ *and* $w = x * (x \circ w)$
**(iii):** $(x * y) \circ (x * w) = w * y$ *and* $(x \circ y) * (x \circ w) = w \circ y$
**(iv):** $w * (x \circ y) = y \circ (x * w)$ *and* $w \circ (x * y) = y * (x \circ w)$
**(v):** $0 \circ (y * w) = w * y$ *and* $0 * (y \circ w) = w \circ y$
**(vi):** $0 \circ (0 * w) = w$ *and* $0 * (0 \circ w) = w$
**(vii):** $0 \circ (0 * (w \circ z)) = w \circ z$ *and* $0 * (0 \circ (w * z)) = w * z$
**(viii):** $z \circ (z * (w \circ u)) = w \circ u$ *and* $z * (z \circ (w * u)) = w * u$

## 3. Derivations on Pseudo Q-Algebras

In this section we introduce the notion of a derivation for a pseudo Q-algebra $X$. For simplicity we define $x \wedge y = y * (y \circ x)$ for all $x, y \in X$.

**Definition 3.1.** *Let $X$ be a pseudo $Q$-algebra. A map $d$: $X \to X$ is said to be a left-right derivation (briefly, a $(l, r)$-derivation) of $X$, if it satisfies the identity $d(x*y) = (d(x)*y) \wedge (x*d(y))$ for all $x, y \in X$.*

*If $d$ satisfies the identity $d(x * y) = (x * d(y)) \wedge (d(x) * y)$ for all $x, y \in X$ then $d$ is said to be a right-left derivation (briefly, $(r, l)$-derivation) of $X$. Moreover, if $d$ is both $(l, r)$ and $(r, l)$-derivation, then it is said that $d$ is an derivation.*

**Example 3.2.** *Let $X = \{0, a, b, c\}$ be the pseudo $Q-algebra$ with Cayley tables given as*

| $*$ | 0 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| $a$ | $a$ | 0 | 0 | 0 |
| $b$ | $b$ | $b$ | 0 | $a$ |
| $c$ | $c$ | $c$ | 0 | 0 |

| $\circ$ | 0 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| $a$ | $a$ | 0 | 0 | 0 |
| $b$ | $b$ | $c$ | 0 | $c$ |
| $c$ | $c$ | $c$ | 0 | 0 |

*Define a map $d : X \to X$ by*

$$d(x) = \begin{cases} 0, & x{=}0 \\ a, & x{=}a \\ 0, & x{=}b \\ 0, & x{=}c \end{cases}$$

*Then it is easily checked that $d$ is a derivation of $X$.*

**Definition 3.3.** *A derivation $d$ of a pseudo $Q$-algebra $X$ is said to be regular if $d(0) = 0$.*

**Proposition 3.4.** *Let $d$ be a $(l, r)$-derivation of a pseudo $Q$-algebra $X$. Then the followings hold:*

*(i) $d(x) = d(x) \wedge (x * d(0))$ for all $x \in X$.*

*(ii) $d(0) = d(0) \wedge (0 * d(0))$.*

*(iii) If $d$ is regular then $d(x) = d(x) \wedge x$ for all $x \in X$.*

*Proof.* i) Let $d$ be a $(l, r)$-derivation of a pseudo $Q$-algebra $X$. Then

$$\begin{aligned} d(x) &= d(x * 0) \quad (by\ \ (PQ2)) \\ &= (d(x) * 0) \wedge (x * d(0)) \quad (by\ \ Definition\, 3.1) \\ &= d(x) \wedge (x * d(0)) \quad (by\ \ (PQ2)) \end{aligned}$$

Therefore, $d(x) = d(x) \wedge (x * d(0))$.

ii) Let $d$ be a $(l, r)$-derivation of a pseudo $Q$-algebra $X$. Then

$$\begin{aligned} d(0) &= d(0 * 0) \quad (by\ \ (PQ1)) \\ &= (d(0) * 0) \wedge (0 * d(0)) \quad (by\ \ Definition\, 3.1) \\ &= d(0) \wedge (0 * d(0)) \quad (by\ \ (PQ2)) \end{aligned}$$

Hence, $d(0) = d(0) \wedge (0 * d(0))$.

iii) Let $d$ be a regular $(l, r)$-derivation of a pseudo $Q$-algebra $X$. Then

$$\begin{aligned} d(x) &= d(x * 0) \quad (by\ \ (PQ2)) \\ &= (d(x) * 0) \wedge (x * d(0)) \quad (by\ \ Definition\, 3.1) \\ &= d(x) \wedge (x * d(0)) \quad (by\ \ (PQ2)) \\ &= d(x) \wedge (x * 0) \quad (by\ \ d\ \ is\ \ regular) \\ &= d(x) \wedge x \quad (by\ \ (PQ2)) \end{aligned}$$

Therefore, $d(x) = d(x) \wedge x$ .

□

**Proposition 3.5.** *Let $d$ be a $(r, l)$-derivation of a pseudo $Q$-algebra $X$. Then the followings hold:*

*(i) $d(x) = (x * d(0)) \wedge d(x)$ for all $x \in X$.*

*(ii) $d(0) = (0 * d(0)) \wedge d(0)$.*

*(iii) If $d$ is regular then $d(x) = x \wedge d(x)$ for all $x \in X$.*

*Proof.* i) Let $d$ be a $(r, l)$-derivation of a pseudo $Q$-algebra $X$. Then

$$\begin{aligned} d(x) &= d(x * 0) \quad (by\ \ (PQ2)) \\ &= (x * d(0)) \wedge (d(x) * 0) \quad (by\ \ (Definition\, 3.1)) \\ &= (x * d(0)) \wedge d(x) \quad (by\ \ (PQ2)) \end{aligned}$$

Therefore, $d(x) = (x * d(0)) \wedge d(x)$.

ii) Let $d$ be a $(r, l)$-derivation of a pseudo $Q$-algebra $X$. Then

$$\begin{aligned} d(0) = d(0 * 0) \quad & (by \ \ (PQ1)) \\ &= (0 * d(0)) \wedge (d(0) * 0) \quad (by \ \ Definition\, 3.1) \\ &= (0 * d(0)) \wedge d(0) \quad (by \ \ (PQ2)) \end{aligned}$$

Hence, $d(0) = (0 * d(0)) \wedge d(0)$.

iii) Let $d$ be a regular $(r, l)$-derivation of a pseudo $Q$-algebra $X$. Then

$$\begin{aligned} d(x) = d(x * 0) \quad & (by \ \ (PQ2)) \\ &= (x * d(0)) \wedge (d(x) * 0) \quad (by \ \ Definition\, 3.1) \\ &= (x * 0) \wedge d(x) \quad (by \ \ d \ \ is \ \ regular \ \ and \ \ by \ \ (PQ2)) \\ &= x \wedge d(x) \quad (by \ \ (PQ2)) \end{aligned}$$

Therefore, $d(x) = x \wedge d(x)$.

□

**Proposition 3.6.** *Let $X$ be a pseudo $Q$-algebra and $d$ be a derivation of $X$. If $d(0) \in K(X)$ then $d(0) = 0$.*

*Proof.* Let $d(0) \in K(X)$. Then $0 \leq d(0)$. This is to mean that $0 * d(0) = 0$.
By using the definition of $(r, l)$-derivation of a pseudo $Q$-algebra $X$ we get

$$d(0) = d(0) * (0 * d(0)) = d(0) \wedge 0 = 0 * (0 \circ d(0)) = 0 * 0 = 0$$

Therefore we get $d(0) = 0$.
Similarly, we can get $d(0) = 0$ by using the definition of $(l, r)$-derivation of a pseudo $Q$-algebra $X$. □

**Proposition 3.7.** *Let $X$ be a pseudo $Q$-algebra and $d$ be a (r, l)-derivation of $X$. If $x \leq d(0)$ for all $x \in X$ then $d = 0$.*

*Proof.* Let $x \leq d(0)$ for any element $x$ of a pseudo $Q$-algebra $X$ where $d$ is a (r, l)-derivation of $X$. Then we have $x * d(0) = 0$. By using the definition of a (r, l)-derivation of $X$ and by (PQ2), (PQ1) we have

$$d(x) = d(x * 0) = (x * d(0)) \wedge d(x) = 0 \wedge d(x) = d(x) * (d(x) \circ 0) = d(x) * d(x) = 0$$

Hence we get $d(x) = 0$.

Additionally, if $d$ is a regular derivation that is $x * d(0) = x * 0 = 0$ we have $x = 0$ i.e. $X = \{0\}$.

□

**Proposition 3.8.** *Let $d$ be a (l,r)-derivation of a pseudo $Q$-algebra $X$. If $d(x) \leq 0$ for all $x \in X$ then $d = 0$.*

*Proof.* Let $d(x) \leq 0$ for all $x$ in a pseudo $Q$-algebra $X$. By (PQ2) we have
$d(x) * 0 = d(x) = 0$ for all $x \in X$.

□

**Proposition 3.9.** *Let $d$ be a (r,l)-derivation of a pseudo $Q$-algebra $X$. If $d(0) = 0$ then $d(x) \leq x$ for all $x \in X$.*

*Proof.* Let $d$ be a regular (r,l)-derivation of a pseudo $Q$-algebra $X$. Then

$$\begin{aligned} d(x) \circ x = (x \wedge d(x)) \circ x \quad & (by\ \ Proposition\ \ 3.5\ \ (iii)) \\ &= (d(x) * (d(x) \circ x)) \circ x \quad (by\ \ Definition\ \ of\ \ \wedge) \\ &= (d(x) \circ x) * (d(x) \circ x) \\ &= 0 \end{aligned}$$

Hence $d(x) \circ x = 0$ i.e. $d(x) \leq x$.

□

**Definition 3.10.** *Let $d$ be a derivation of a pseudo $Q$-algebra $X$. A pseudo ideal $A$ of $X$ is said to be $d$-invariant if $d(A) \subseteq A$.*

**Proposition 3.11.** *Let $d$ be a regular (r,l)-derivation of a pseudo $Q$-algebra $X$ then every pseudo ideal $A$ of $X$ is $d$-invariant; that is $d(A) \subset A$.*

*Proof.* By Proposition 3.9 we have $d(x) \leq x$ for all $x \in X$. Let $y \in d(A)$ then $d(x) = y$ for some $x \in A$. Then we get

$$y * x = d(x) * x = 0.$$

This is to say that $y \leq x$. $x \in A$ and $A$ is a pseudo ideal of $X$. Then $y \in A$. So $d(A) \subset A$.

□

**Proposition 3.12.** *Let $d$ be a (r,l)-derivation of a $\circ$-medial pseudo $Q$-algebra $X$ then $d(a) \in G(X)$ for all $a \in G(X)$.*

*Proof.* For any $a \in G(X)$ we have

$$d(a) = d(0 * a) = (0 * d(a)) \wedge (d(0) * a) = (d(0) * a) * ((d(0) * a) \circ (0 * d(a))) = 0 * d(a)$$

Hence we get $d(a) \in G(X)$.

□

**Theorem 3.13.** *Let $d$ be a derivation of a pseudo $Q$-algebra $X$. Then $d$ is regular if and only if every ideal of $X$ is $d$-invariant.*

*Proof.* Let $d$ be a regular $(r, l)$-derivation of a pseudo $Q$-algebra $X$. Then every pseudo ideal $A$ is $d$-invariant.

Conversely, let $X$ be a pseudo $Q$-algebra that is every ideal of $X$ is $d$-invariant. By our assumption 0 is an ideal of $X$ that is $d$-invariant. So we have $d(\{0\}) \subseteq \{0\}$. That implies $d(0) = 0$.

□

**Proposition 3.14.** *Let $d$ be a regular $(l, r)$- derivation of a pseudo $Q$-algebra $X$ and $w$ be a pseudo atom in $X$. Then $d(w)$ is a pseudo atom of $X$.*

*Proof.* Let $d$ be a regular $(l, r)$- derivation of a pseudo $Q$-algebra $X$ and $w$ be a pseudo atom in $X$. Then

$$d(w) = d(0 * (0 \circ w)) = (d(0) * (0 \circ w)) \wedge (0 * d(0 \circ w))$$

$$= (0 * (0 \circ w)) \wedge (0 * d(0 \circ w)) = w \wedge (0 * d(0 \circ w))$$

$$= (0 * d(0 \circ w)) * ((0 * d(0 \circ w) \circ w) = w.$$

Therefore, $d(w) = w$. So, $0 * (0 \circ d(w)) = 0 * (0 \circ w) = w = d(w)$. Hence, $d(w)$ is an atom. □

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Faculty of Science and Letter, Department of Mathematics, Yaşar Üníversítesí, ízmír, Türkíye

*E-mail address*: sule.ayar@yasar.edu.tr

Faculty of Science, Department of Mathematics, EgeÜníversítesí, ízmír, Türkíye

*E-mail address*: alev.firat@ege.edu.tr