

# ON NEW CESÀRO-ORLICZ DOUBLE DIFFERENCE SEQUENCE SPACE

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ABSTRACT. The aim of this paper is to introduce the Cesàro-Orlicz double difference sequence space  $Ces_M^{(2)}(\Delta, p)$ . We study some topological properties of this space and give some inclusion relations.

## 1. INTRODUCTION

Throughout this work,  $\mathbb{N}$ ,  $\mathbb{R}$ ,  $w$  and  $w^2$  denote the sets of positive integers, real numbers, single real sequences and double real sequences, respectively.

First of all, let us recall preliminary definitions and notations.

A double sequence on a normed linear space  $X$  is a function  $x$  from  $\mathbb{N} \times \mathbb{N}$  into  $X$  and briefly denoted by  $x = (x_{kl})$ . If for every  $\varepsilon > 0$  there exists  $n_\varepsilon \in \mathbb{N}$  such that  $\|x_{kl} - a\|_X < \varepsilon$  whenever  $k, l > n_\varepsilon$  then a double sequence  $(x_{kl})$  is said to be converges (in terms of Pringsheim) to  $a \in X$  [16].

A double series  $\sum_{k,l=1}^{\infty} x_{kl}$  is convergent if and only if its sequence of partial sums  $(s_{nm})$  is convergent (see [2],[3]), where  $s_{nm} = \sum_{k=1}^n \sum_{l=1}^m x_{kl}$  for all  $m, n \in \mathbb{N}$ .

Let  $X$  be a linear space. A function  $p : X \rightarrow \mathbb{R}$  is called *paranorm*, if

i)  $p(0) = 0$ ,

ii)  $p(x) \geq 0$  for all  $x \in X$ ,

iii)  $p(-x) = p(x)$  for all  $x \in X$ ,

iv)  $p(x + y) \leq p(x) + p(y)$  for all  $x, y \in X$ ,

v) if  $(\lambda_n)$  is a sequence of scalars with  $\lambda_n \rightarrow \lambda$  as  $n \rightarrow \infty$  and  $(x_n)$  is a sequence of vectors with  $p(x_n - x) \rightarrow 0$  as  $n \rightarrow \infty$ , then  $p(\lambda_n x_n - \lambda x) \rightarrow 0$  as  $n \rightarrow \infty$  (continuity of scalars multiplication).

A paranorm  $p$  for which  $p(x) = 0$  implies  $x = 0$  is called *total* [12].

An Orlicz function is a function  $M : [0, \infty) \rightarrow [0, \infty)$  which is continuous, non-decreasing and convex with  $M(0) = 0$ ,  $M(x) > 0$  for  $x > 0$  and  $M(x) \rightarrow \infty$  as  $x \rightarrow \infty$ .

An Orlicz function  $M$  can always be represented in the following integral form:  $M(x) = \int_0^x \eta(t) dt$ , where  $\eta$  is known as the kernel of  $M$ , is right differentiable for  $t \geq 0$ ,  $\eta(0) = 0$ ,  $\eta(t) > 0$  for  $t > 0$ ,  $\eta$  is nondecreasing and  $\eta(t) \rightarrow \infty$  as  $t \rightarrow \infty$ .

An Orlicz function  $M$  is said to be satisfied the  $\Delta_2$ -condition if there are  $T > 0$  and  $a > 0$  such that  $M(a) > 0$  and  $M(2u) \leq TM(u)$  for all  $u \in [0, a]$  (see [10]).

For  $1 \leq p < \infty$ , the Cesàro sequence space  $Ces_p$  is defined by

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$$Ces_p = \left\{ x \in w : \sum_{j=1}^{\infty} \left( \frac{1}{j} \sum_{i=1}^j |x_i| \right)^p < \infty \right\},$$

equipped with norm

$$\|x\| = \left( \sum_{j=1}^{\infty} \left( \frac{1}{j} \sum_{i=1}^j |x_i| \right)^p \right)^{\frac{1}{p}}.$$

This space was first introduced by Shiue [18]. It is very useful in the theory of matrix operators and others. Sanhan and Suantai introduced and studied a generalized Cesàro sequence space  $Ces(p)$ , where  $p = (p_j)$  is a bounded sequence of positive real numbers (see [17]). Later, this space was studied by many authors in [4], [7], [9], [11], [14], [15].

The notion of difference sequence space was introduced by Kızmaz in [8] in 1981 as follows:

$$X(\Delta) = \{x = (x_k) \in w : (x_k - x_{k+1}) \in X\}$$

for  $X = \ell_{\infty}, c, c_0$ . Subsequently difference sequence spaces has been discussed in Ahmad and Mursaleen [1], Malkowsky and Parashar [13], Et and Başarır [5], Et and Çolak [6] and others.

In this work, we introduce double sequence spaces  $Ces_M^{(2)}(\Delta, p)$  as follows;

Let  $p = (p_{nm})$  be a bounded double sequence of positive real numbers and  $M$  be an Orlicz function. The space  $Ces_M^{(2)}(\Delta, p)$  is defined by

$$Ces_M^{(2)}(\Delta, p) = \left\{ x \in w^2 : \sum_{n,m=1}^{\infty} \left[ M \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta x_{ij}|}{\rho} \right) \right]^{p_{nm}} < \infty, \exists \rho > 0 \right\},$$

where  $\Delta x_{ij} = x_{i-1,j-1} - x_{i-1,j} - x_{i,j-1} + x_{ij}$  for all  $i, j \in \mathbb{N}$  and the terms with negative subscript are assume zero.

The following inequality will be used throughout this paper. Let  $(p_{nm})$  be a bounded double sequence of strictly positive real numbers and denote  $H = \sup_{n,m} p_{nm}$ . For any complex  $a_{nm}$  and  $b_{nm}$  we have

$$|a_{nm} + b_{nm}|^{p_{nm}} \leq D. (|a_{nm}|^{p_{nm}} + |b_{nm}|^{p_{nm}})$$

where  $D = \max(1, 2^{H-1})$ . Also, for any complex  $\lambda$ ,

$$|\lambda|^{p_{nm}} \leq \max(1, |\lambda|^H).$$

## 2. MAIN RESULTS

**Theorem 1.** *Let  $(p_{nm})$  be bounded. The set  $Ces_M^{(2)}(\Delta, p)$  of double sequences is a linear space over the real field  $\mathbb{R}$ .*

*Proof.* Let  $x, y \in Ces_M^{(2)}(\Delta, p)$  and  $\lambda, \beta \in \mathbb{C}$ . Then there exist  $\rho_1 > 0, \rho_2 > 0$  such that

$$\sum_{n,m=1}^{\infty} \left[ M \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta x_{ij}|}{\rho_1} \right) \right]^{p_{nm}} < \infty$$

and

$$\sum_{n,m=1}^{\infty} \left[ M \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta y_{ij}|}{\rho_2} \right) \right]^{p_{nm}} < \infty.$$

Let  $\alpha, \beta \in \mathbb{R}$  and  $\rho_3 = \max \{2|\alpha|\rho_1, 2|\beta|\rho_2\}$ . Since  $M$  is non-decreasing convex function, we have

$$\begin{aligned} & \sum_{n,m=1}^{\infty} \left[ M \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\alpha\Delta x_{ij} + \beta\Delta y_{ij}|}{\rho_3} \right) \right]^{p_{nm}} \\ & \leq \sum_{n,m=1}^{\infty} \left[ M \left( \frac{|\alpha|}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta x_{ij}|}{\rho_3} + \frac{|\beta|}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta y_{ij}|}{\rho_3} \right) \right]^{p_{nm}} \\ & \leq \sum_{n,m=1}^{\infty} \left[ M \left( \frac{1}{2nm} \sum_{i,j=1}^{n,m} \frac{|\Delta x_{ij}|}{\rho_1} + \frac{1}{2nm} \sum_{i,j=1}^{n,m} \frac{|\Delta y_{ij}|}{\rho_2} \right) \right]^{p_{nm}} \\ & \leq \sum_{n,m=1}^{\infty} \frac{1}{2^{p_{nm}}} \left[ M \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta x_{ij}|}{\rho_1} \right) + M \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta y_{ij}|}{\rho_2} \right) \right]^{p_{nm}} \\ & < \sum_{n,m=1}^{\infty} \left[ M \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta x_{ij}|}{\rho_1} \right) + M \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta y_{ij}|}{\rho_2} \right) \right]^{p_{nm}} \\ & \leq D \cdot \sum_{n,m=1}^{\infty} \left[ M \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta x_{ij}|}{\rho_1} \right) \right]^{p_{nm}} \\ & \quad + D \cdot \sum_{n,m=1}^{\infty} \left[ M \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta y_{ij}|}{\rho_2} \right) \right]^{p_{nm}} \\ & < \infty, \end{aligned}$$

where  $D = \max(1, 2^{H-1})$ . This shows that  $\lambda x + \beta y \in Ces_M^{(2)}(\Delta, p)$  and so  $Ces_M^{(2)}(\Delta, p)$  is a linear space.  $\square$

**Theorem 2.** *The double sequence space  $Ces_M^{(2)}(\Delta, p)$  is a paranormed space with the paranorm*

$$g(x) = \inf \left\{ \rho^{\frac{pqr}{R}} > 0 : \left( \sum_{n,m=1}^{\infty} \left[ M \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta x_{ij}|}{\rho} \right) \right]^{p_{nm}} \right)^{\frac{1}{R}} \leq 1, q, r \in \mathbb{N} \right\}$$

where  $H = \sup_{n,m} p_{nm} < \infty$  and  $R = \max(1, H)$ .

*Proof.* It is clear that  $g(x) = g(-x)$  and  $g(0) = 0$ . For any  $x, y \in Ces_M^{(2)}(\Delta, p)$ , there exist  $\rho_1, \rho_2 > 0$  such that

$$\left( \sum_{n,m=1}^{\infty} \left[ M \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta x_{ij}|}{\rho_1} \right) \right]^{p_{nm}} \right)^{\frac{1}{R}} \leq 1$$

and

$$\left( \sum_{n,m=1}^{\infty} \left[ M \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta y_{ij}|}{\rho_2} \right) \right]^{p_{nm}} \right)^{\frac{1}{R}} \leq 1.$$

Let  $\rho_3 = 2^{\frac{R}{h}}(\rho_1 + \rho_2)$ , where  $h = \inf p_{nm} > 0$ . Since  $M$  is a non-decreasing convex function, we have

$$\begin{aligned} & \left( \sum_{n,m=1}^{\infty} \left[ M \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta x_{ij} + \Delta y_{ij}|}{\rho_3} \right) \right]^{p_{nm}} \right)^{\frac{1}{R}} \\ & \leq \left( \sum_{n,m=1}^{\infty} \left[ M \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta x_{ij}|}{2^{\frac{R}{h}}(\rho_1 + \rho_2)} + \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta y_{ij}|}{2^{\frac{R}{h}}(\rho_1 + \rho_2)} \right) \right]^{p_{nm}} \right)^{\frac{1}{R}} \\ & \leq \left( \sum_{n,m=1}^{\infty} \left[ \frac{\rho_1}{2^{\frac{R}{h}}(\rho_1 + \rho_2)} M \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta x_{ij}|}{\rho_1} \right) \right. \right. \\ & \quad \left. \left. + \frac{\rho_2}{2^{\frac{R}{h}}(\rho_1 + \rho_2)} M \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta y_{ij}|}{\rho_2} \right) \right]^{p_{nm}} \right)^{\frac{1}{R}} \\ & \leq \left( \sum_{n,m=1}^{\infty} \left[ \frac{1}{2^{\frac{R}{h}}} M \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta x_{ij}|}{\rho_1} \right) \right]^{p_{nm}} \right)^{\frac{1}{R}} \\ & \quad + \left( \sum_{n,m=1}^{\infty} \left[ \frac{1}{2^{\frac{R}{h}}} M \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta y_{ij}|}{\rho_2} \right) \right]^{p_{nm}} \right)^{\frac{1}{R}} \\ & = \frac{1}{2} \left( \sum_{n,m=1}^{\infty} \left[ M \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta x_{ij}|}{\rho_1} \right) \right]^{p_{nm}} \right)^{\frac{1}{R}} \\ & \quad + \frac{1}{2} \left( \sum_{n,m=1}^{\infty} \left[ M \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta y_{ij}|}{\rho_2} \right) \right]^{p_{nm}} \right)^{\frac{1}{R}} \\ & \leq 1. \end{aligned}$$

Since  $\rho_1, \rho_2, \rho_3$  are positive real numbers we get

$$g(x+y) = \inf \left\{ \rho_3^{\frac{p_{qr}}{R}} > 0 : \left( \sum_{n,m=1}^{\infty} \left[ M \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta x_{ij} + \Delta y_{ij}|}{\rho_3} \right) \right]^{p_{nm}} \right)^{\frac{1}{R}} \leq 1; q, r \in \mathbb{N} \right\}$$

$$\begin{aligned}
&\leq \inf \left\{ \rho_1^{\frac{pqr}{R}} > 0 : \left( \sum_{n,m=1}^{\infty} \left[ M \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta x_{ij}|}{\rho_1} \right) \right]^{pnm} \right)^{\frac{1}{R}} \leq 1; q, r \in \mathbb{N} \right\} \\
&\quad + \inf \left\{ \rho_2^{\frac{pqr}{R}} > 0 : \left( \sum_{n,m=1}^{\infty} \left[ M \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta y_{ij}|}{\rho_2} \right) \right]^{pnm} \right)^{\frac{1}{R}} \leq 1; q, r \in \mathbb{N} \right\} \\
&= g(x) + g(y).
\end{aligned}$$

Let  $(x^n) = \{x_{ij}^n\}$  be any sequence in the space  $Ces_M^{(2)}(\Delta, p)$  such that  $g(x^n - x) \rightarrow 0$ , as  $n \rightarrow \infty$  and  $(\lambda_n)$  is a sequence of reals with  $\lambda_n \rightarrow \lambda$ , as  $n \rightarrow \infty$ . Then, since the inequality

$$g(x^n) \leq g(x) + g(x^n - x)$$

holds by subadditivity of the function  $g$ ,  $\{g(x^n)\}$  is bounded. Taking into account this fact we therefore derive the inequality

$$g(\lambda_n x^n - \lambda x) \leq |\lambda_n - \lambda| g(x^n) + |\lambda| g(x^n - x)$$

which tends to zero as  $n \rightarrow \infty$ . Hence, the scalar multiplication is continuous.

That is to say that  $g$  is a paranorm on the space  $Ces_M^{(2)}(\Delta, p)$ , as asserted.  $\square$

**Theorem 3.** *The space  $Ces_M^{(2)}(\Delta, p)$  is complete with respect to its paranorm.*

*Proof.* Let  $(x^s) = \{x_{ij}^s\}$  be any Cauchy sequence in the space  $Ces_M^{(2)}(\Delta, p)$ . Since  $(x^s)$  is a Cauchy sequence, we have

$$(1) \quad g(x^s - x^t) \rightarrow 0$$

as  $s, t \rightarrow \infty$ . Hence, we get

$$|\Delta x_{ij}^s - \Delta x_{ij}^t| \rightarrow 0$$

as  $s, t \rightarrow \infty$  for all  $i, j \in \mathbb{N}$ . Then, we have  $\{x_{ij}^s\}$  is a Cauchy sequence in  $\mathbb{R}$  for each fixed  $i, j \in \mathbb{N}$ .

Thus, there exists  $x_{ij} \in \mathbb{R}$  such that  $x_{ij}^s \rightarrow x_{ij}$  as  $s \rightarrow \infty$  and say  $x = x_{ij}$ . Since  $M$  is continuous, by (1) we get

$$g(x^s - x) \rightarrow 0$$

as  $t \rightarrow \infty$ .

Since  $Ces_M^{(2)}(\Delta, p)$  is linear space, we get  $x = \{x_{ij}\} \in Ces_M^{(2)}(\Delta, p)$ . This completes the proof.  $\square$

**Theorem 4.** *Let  $0 < p_{nm} \leq q_{nm} < \infty$ . Then  $Ces_M^{(2)}(\Delta, p) \subset Ces_M^{(2)}(\Delta, q)$ .*

*Proof.* Let  $x \in Ces_M^{(2)}(\Delta, p)$ , then there exists  $\rho > 0$  such that

$$\sum_{n,m=1}^{\infty} \left[ M \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta x_{ij}|}{\rho} \right) \right]^{pnm} < \infty.$$

Hence we have  $M \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta x_{ij}|}{\rho} \right) < 1$  for large values of  $n, m$ . Then, we get

$$\sum_{n,m=1}^{\infty} \left[ M \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta x_{ij}|}{\rho} \right) \right]^{qnm} \leq \sum_{n,m=1}^{\infty} \left[ M \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta x_{ij}|}{\rho} \right) \right]^{pnm} < \infty$$

and so  $x \in Ces_M^{(2)}(\Delta, q)$ .  $\square$

**Theorem 5.** Let  $M_1$  and  $M_2$  be Orlicz functions satisfying  $\Delta_2$ -condition. Then

- (a)  $Ces_{M_1}^{(2)}(\Delta, p) \subset Ces_{M_2 \circ M_1}^{(2)}(\Delta, p)$ ,
- (b)  $Ces_{M_1}^{(2)}(\Delta, p) \cap Ces_{M_2}^{(2)}(\Delta, p) \subset Ces_{M_1+M_2}^{(2)}(\Delta, p)$ .

*Proof.* (a) Let  $x \in Ces_{M_1}^{(2)}(\Delta, p)$ . Then there exists  $\rho > 0$  such that

$$\sum_{n,m=1}^{\infty} \left[ M_1 \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta x_{ij}|}{\rho} \right) \right]^{p_{nm}} < \infty.$$

Since  $M_1$  is a continuous function, we can find a real number  $\delta$  with  $0 < \delta < 1$  such that  $M_1(t) < \varepsilon$ . Let  $y_{nm} = M_1 \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta x_{ij}|}{\rho} \right)$ . Hence we write

$$\sum_{n,m=1}^{\infty} [M_2(y_{nm})]^{p_{nm}} = \sum_{y_{nm} \leq \delta} [M_2(y_{nm})]^{p_{nm}} + \sum_{y_{nm} > \delta} [M_2(y_{nm})]^{p_{nm}}.$$

By the properties of  $M_2$ , we have

$$(2) \quad \sum_{y_{nm} \leq \delta} [M_2(y_{nm})]^{p_{nm}} \leq \max \{1, M_2(1)^H\} \sum_{y_{nm} \leq \delta} [y_{nm}]^{p_{nm}}.$$

Also,

$$M_2(y_{nm}) < M_2 \left( 1 + \frac{y_{nm}}{\delta} \right) < \frac{1}{2} M_2(2) + \frac{1}{2} \left( \frac{2y_{nm}}{\delta} \right)$$

for  $y_{nm} > \delta$ . Since  $M_2$  satisfying  $\Delta_2$ -condition and  $\frac{y_{nm}}{\delta} > 1$ , there exists  $T > 0$  such that

$$M_2(y_{nm}) < \frac{1}{2} T \frac{y_{nm}}{\delta} M_2(2) + \frac{1}{2} T \frac{y_{nm}}{\delta} M_2(2) = T \frac{y_{nm}}{\delta} M_2(2).$$

Therefore we have

$$(3) \quad \sum_{y_{nm} > \delta} [M_2(y_{nm})]^{p_{nm}} \leq \max \left\{ 1, \left( T \frac{M_2(2)}{\delta} \right)^H \right\} \sum_{y_{nm} > \delta} [y_{nm}]^{p_{nm}}.$$

Hence by the (2), (3), we get

$$\begin{aligned} \sum_{n,m=1}^{\infty} \left[ (M_2 \circ M_1) \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta x_{ij}|}{\rho} \right) \right]^{p_{nm}} &= \sum_{n,m=1}^{\infty} [M_2(y_{nm})]^{p_{nm}} \\ &\leq B \cdot \sum_{y_{nm} \leq \delta} [y_{nm}]^{p_{nm}} \\ &\quad + F \cdot \sum_{y_{nm} > \delta} [y_{nm}]^{p_{nm}} \\ &< \infty, \end{aligned}$$

where  $B = \max \{1, M_2(1)^H\}$  and  $F = \max \left\{ 1, \left( T \frac{M_2(2)}{\delta} \right)^H \right\}$ . Hence  $Ces_{M_1}^{(2)}(\Delta, p) \subset Ces_{M_2 \circ M_1}^{(2)}(\Delta, p)$ .

(b) Let  $x \in Ces_{M_1}^{(2)}(\Delta, p) \cap Ces_{M_2}^{(2)}(\Delta, p)$ . Then there exists  $\rho > 0$  such that

$$\sum_{n,m=1}^{\infty} \left[ M_1 \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta x_{ij}|}{\rho} \right) \right]^{p_{nm}} < \infty$$

and

$$\sum_{n,m=1}^{\infty} \left[ M_2 \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta x_{ij}|}{\rho} \right) \right]^{p_{nm}} < \infty.$$

Hence we get

$$\begin{aligned} & \sum_{n,m=1}^{\infty} \left( (M_1 + M_2) \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta x_{ij}|}{\rho} \right) \right)^{p_{nm}} \\ & \leq A \cdot \sum_{n,m=1}^{\infty} \left( M_1 \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta x_{ij}|}{\rho} \right) \right)^{p_{nm}} \\ & \quad + A \cdot \sum_{n,m=1}^{\infty} \left( M_2 \left( \frac{1}{nm} \sum_{i,j=1}^{n,m} \frac{|\Delta x_{ij}|}{\rho} \right) \right)^{p_{nm}} \\ & < \infty, \end{aligned}$$

where  $A = \max \{1, 2^{H-1}\}$ . This completes the proof.  $\square$

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