

# FRactal Vector Measures in the Case of an Uncountable Iterated Function System

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**ABSTRACT.** In this paper we obtain an extension of the concept of Hutchinson measure (which is the unique fixed point of a contraction on the set of normalized Borel measures on a compact metric space) related to an iterated function system. Our extension means that we consider vector measures (instead of normalized Borel measures) and, also, an uncountable iterated function system instead of a finite one, as in the case of Hutchinson measure.

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## 1. INTRODUCTION

The *Hutchinson measure* (also called *fractal measure*) is related to the attractor of an *iterated function system*. If  $(T, d)$  is a compact metric space, an *iterated function system* (I.F.S.) is a set  $(\omega_i)_{i=1}^n$  of contractions defined and taking values on  $T$ .

If we consider the complete metric space  $(\mathcal{K}(T), \delta)$ , where  $\delta$  is the Hausdorff-Pompeiu metric and  $\mathcal{K}(T)$  is the set of all nonempty compact subsets of  $T$  one can prove that the map  $S : \mathcal{K}(T) \rightarrow \mathcal{K}(T)$ ,  $S(A) = \bigcup_{i=1}^n \omega_i(A)$ , is also a contraction.

The unique fixed point  $F$  of  $S$  is called the *attractor* of the I.F.S. Using the I.F.S. we can consider the so called *Markov operator*, defined and taking values in the set of all normalized Borel measures. This set, with a certain metric, becomes a complete metric space and the Markov operator is a contraction (see [6], pp. 131-133).

The unique fixed point of the Markov operator is called *Hutchinson measure*. This measure has the interesting property that its support is exactly the attractor of the I.F.S.

A first extension of the Hutchinson measure was given in [4] to the case of vector measures. A second extension was given in [5], when we considered a countable iterated function system and vector measures. In this paper we consider again the case of vector measures, but an uncountable iterated function system.

The present paper can be viewed as a continuation of my Ph.D. thesis (unpublished).

## 2. PRELIMINARIES

In this section we will recall some basic facts. For more details, one can consult [1]-[4] and [6].

## 2.1. The sesquilinear vector integral.

Let  $(T, d)$  be a compact metric space and let  $(X, \langle \cdot, \cdot \rangle)$  be a Hilbert space over the field  $K$  (either  $K = \mathbb{R}$  or  $K = \mathbb{C}$ ).

A function  $f : T \rightarrow X$  is called *simple function* if there exists a partition  $(A_i)_{1 \leq i \leq n}$  of  $T$ , with Borel sets, such that  $f = \sum_{i=1}^n \varphi_{A_i} x_i$ , where  $\varphi_{A_i}$  is the characteristic function of the set  $A_i$  and  $x_i \in X$ ,  $i \in \{1, \dots, n\}$ . We denote by  $S(X)$  the set of all simple functions.

A function  $f : T \rightarrow X$  is called *totally measurable function* if there exists a sequence  $(f_n)_n \subset S(X)$  such that  $\lim_{n \rightarrow \infty} f_n = f$ , the convergence being uniform. We denote by  $TM(X)$  the set of all totally measurable functions. The set  $TM(X)$  is the closure of the set  $S(X)$  in the topology given by the norm  $\|f\|_\infty = \max_{t \in T} |f(t)|$ .

We also denote  $C(X) = \{f : T \rightarrow X \mid f \text{ is continuous}\}$ , and obviously  $C(X) \subset TM(X)$ .

Let  $\mathcal{B}$  be the set of Borel subsets of  $T$ . A function  $\mu : \mathcal{B} \rightarrow X$  is called *vector measure* if  $\mu$  is  $\sigma$ -additive, that is,  $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$ , for any sequence  $(A_n)_n \subset \mathcal{B}$ , with  $A_i \cap A_j = \emptyset$ , for  $i \neq j$ .

Let  $\mu : \mathcal{B} \rightarrow X$  be a vector measure and  $A \in \mathcal{B}$ . We call the *variation of  $\mu$  on  $A$*  the number, denoted by  $|\mu|(A)$ , defined as:

$$|\mu|(A) = \sup \left\{ \sum_i \|\mu(A_i)\| \mid (A_i)_i \text{ is a finite partition of } A \text{ with Borel sets} \right\}.$$

One can prove that  $|\mu| : \mathcal{B} \rightarrow \overline{\mathbb{R}}_+$  is a positive measure (see [1], p. 114).

Let  $\mu : \mathcal{B} \rightarrow X$  be a vector measure. If  $|\mu|(T) < \infty$ , we say that  $\mu$  has *bounded variation*. We denote by  $cabv(X)$  the set of vector measures with bounded variation. One can prove the following theorem (see [1], pag. 156).

**Theorem 2.1.** a) The application  $\|\cdot\| : cabv(X) \rightarrow [0, \infty)$ ,  $\|\mu\| \stackrel{\text{def}}{=} |\mu|(T)$  is a norm on  $cabv(X)$ , the called **variational norm**.

b)  $(cabv(X), \|\cdot\|)$  is a Banach space.

Let  $f = \sum_{i=1}^n \varphi_{A_i} x_i$  be a simple function,  $\mu \in cabv(X)$ . We define:

$$\int f d\mu \stackrel{\text{def}}{=} \sum_{i=1}^n \langle \mu(A_i), x_i \rangle.$$

**Remark.** Obviously,  $\left| \int f d\mu \right| \leq \sum_{i=1}^n \|\mu(A_i)\| \|x_i\| \leq \|f\|_\infty \|\mu\|$ .

These inequalities show that the application  $f \mapsto \int f d\mu$  is continuous and has an unique uniform continuous extension to the set  $TM(X)$  (the closure of  $S(X)$  with respect to the topology given by  $\|\cdot\|_\infty$ ).

Let  $f$  be a function such that  $f \in TM(X)$  and  $(f_n)_n \subset S(X)$  such that  $f_n \xrightarrow{u} f$ . We define  $\int f d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu$ , and the limit does not depend on the sequence  $(f_n)_n$  which converges uniformly to  $f$ .

## 2.2. Norms and topologies on measure spaces.

We denote by  $BL(X)$  the set of all Lipschitz functions  $f : T \rightarrow X$ . On this set we introduce the norm:  $\|f\|_{BL} = \|f\|_{\infty} + \|f\|_L$ , where  $\|f\|_L$  is the Lipschitz constant of  $f$ . For the proofs of the results given in this section one can see [3].

We also denote:  $BL_1(X) = \left\{ f \in BL(X) \mid \|f\|_{BL} \leq 1 \right\}$ .

For any  $\mu \in cabv(X)$ , we define:  $\|\mu\|_{MK} = \sup \left\{ \left| \int f d\mu \right| \mid f \in BL_1(X) \right\}$ .

Then the application  $\mu \mapsto \|\mu\|_{MK}$  is a norm on  $cabv(X)$  and  $\|\mu\|_{MK} \leq \|\mu\|$ .

The norm defined above is called *Monge-Kantorovich-type* norm (in the following we will call it, in short, the Monge-Kantorovich norm).

Let  $a > 0$ . We will denote  $B_a(X) = \left\{ \mu \in cabv(X) \mid \|\mu\| \leq a \right\}$ . It can be proved the following result: For any  $a > 0$  and  $n \in \mathbb{N}$ , the set  $B_a(K^n)$  equipped with the metric generated by the Monge-Kantorovich norm is a compact (hence, complete) metric space.

For any  $v \in X$ , we denote  $cabv(X, v) = \left\{ \mu \in cabv(X) \mid \mu(T) = v \right\}$ . Obviously, for any  $A \subset cabv(X, v)$ , we have:

$$A - A \stackrel{\text{def}}{=} \left\{ \mu - \nu \mid \mu, \nu \in A \right\} \subset cabv(X, 0).$$

We define:  $L_1(X) = \left\{ f : T \rightarrow X \mid f \text{ is a Lipschitz function and } \|f\|_L \leq 1 \right\}$ .

For any  $\mu \in cabv(X, 0)$ , we denote  $\|\mu\|_{MK}^* = \sup \left\{ \left| \int f d\mu \right| \mid f \in L_1(X) \right\}$ . Then the application  $\mu \mapsto \|\mu\|_{MK}^*$  is a norm on  $cabv(X, 0)$  and we have:  $\|\mu\|_{MK} \leq \|\mu\|_{MK}^* \leq \|\mu\| \text{diam}(T)$ . This norm is called *the modified Monge-Kantorovich norm*.

For any  $v \in X$  and  $a > 0$  we denote  $B_a(X, v) = \left\{ \mu \in B_a(X) \mid \mu(T) = v \right\}$ .

One can prove the following theorem: For any  $v \in X$ ,  $a > 0$  and  $n \in \mathbb{N}^*$ , the set  $B_a(K^n, v)$  equipped with the metric given by the modified Monge-Kantorovich norm is a compact (hence, complete) metric space.

### 2.3. Fractal vector measures.

Let  $m \in \mathbb{N}^*$ . We consider the linear and continuous operators  $R_i : X \rightarrow X$ ,  $i = 1, 2, \dots, m$ , and the *iterated function system*  $(\omega_i)_{i=1}^n$ , that is,  $\omega_i : T \rightarrow T$  is a contraction of ratio  $r_i < 1$ , for any  $i = 1, 2, \dots, m$ . We define the *Markov-type operator*:

$$H : cabv(X) \rightarrow cabv(X), \quad H(\mu) = \sum_{i=1}^m R_i \circ \mu(\omega_i^{-1}),$$

that means that, for any Borel subset  $A \subset T$ ,  $H(\mu)(A) = \sum_{i=1}^m R_i \left( \mu(\omega_i^{-1}(A)) \right)$ .

It can be proved (see [4]) that:

**a)** The Markov-type operator is correctly defined:  $H(\mu) \in cabv(X)$  for any  $\mu \in cabv(X)$ .

**b)** Let us consider the Banach space  $(cabv(X), \|\cdot\|)$ . Then,  $H$  is linear and continuous and  $\|H\|_0 \leq \sum_{i=1}^m \|R_i\|_0$  (where we denote by  $\|\cdot\|_0$  the operatorial norm).

**Theorem 2.2. (change of variable formula)** For any  $f \in C(X)$ , we have:

$$\int f dH(\mu) = \int g d\mu, \text{ where } g = \sum_{i=1}^m R_i^* \circ f \circ \omega_i,$$

$R_i^*$  being the adjointed operator of  $R_i$ .

*Proof.* Using additivity properties, it will suffice to prove that for any  $R \in \mathcal{L}(X)$ , for any continuous function  $\omega : T \rightarrow T$  and for any  $f \in C(X)$ , we have:

$$(2.1) \quad \int f dH(R)(\mu) = \int g d\mu,$$

where  $H(R) \in cabv(X)$  is given by  $H(R)(\mu) = R \circ \omega(\mu)$  and  $g = R^* \circ f \circ \omega$ .

We construct the canonical sequence for  $f$  (see [2]):

$$f_m = \sum_{i=1}^{K(m)} \varphi_{B_i^m} f(t_i^m), \text{ where } t_i^m \in B_i^m.$$

It is obvious that  $\|f_m\|_\infty \leq \|f\|_\infty$  and that  $(C_i^m)_{1 \leq i \leq K(m)}$  is a partition of  $T$ , where  $C_i^m = \omega^{-1}(B_i^m)$ . We take  $v_i^m \in C_i^m$  such that  $\omega(v_i^m) = t_i^m$ . We have:

$$\begin{aligned} \int f_m dH(R)(\mu) &= \sum_{i=1}^{K(m)} \langle f(t_i^m), H(R)(\mu)(B_i^m) \rangle \\ &= \sum_{i=1}^{K(m)} \langle f(t_i^m), R(\mu(\omega^{-1}(B_i^m))) \rangle \\ &= \sum_{i=1}^{K(m)} \langle f(t_i^m), R(\mu(C_i^m)) \rangle \\ &= \sum_{i=1}^{K(m)} \langle (R^* \circ f)(t_i^m), \mu(C_i^m) \rangle \\ &= \sum_{i=1}^{K(m)} \langle (R^* \circ f)(\omega(v_i^m)), \mu(C_i^m) \rangle \\ (2.2) \quad &= \sum_{i=1}^{K(m)} \langle (R^* \circ f \circ \omega)(v_i^m), \mu(C_i^m) \rangle. \end{aligned}$$

For any  $m \in \mathbb{N}$ , we consider the simple function  $g_m = \sum_{i=1}^{K(m)} \varphi_{C_i^m} (R^* \circ f \circ \omega)(v_i^m)$ .

Then, from (2.2) we obtain:

$$(2.3) \quad \int f_m dH(R)(\mu) = \int g_m d\mu.$$

We also have:  $g_m \xrightarrow[m]{u} g = R^* \circ f \circ \omega$ . Indeed, for any  $t \in T$  one can find an unique  $i \in \{1, 2, \dots, K(m)\}$  such that  $t \in C_i^m$ . Therefore:

$$\|g_m(t) - g(t)\| = \|R^*(f(\omega(v_i^m))) - R^*(f(\omega(t)))\| =$$

$$\begin{aligned}
&= \|R^* \left( f(\omega(v_i^m)) - f(\omega(t)) \right)\| \leq \\
&\leq \|R^*\|_0 \|f(\omega(v_i^m)) - f(\omega(t))\| = \\
&= \|R\|_0 \|f(\omega(v_i^m)) - f(\omega(t))\|.
\end{aligned}$$

However  $\omega(v_i^m) = t_i^m \in B_i^m$ ,  $\omega(t) \in B_i^m$  and  $B_i^m \subset f^{-1}(B(y_i^m, \frac{1}{m}))$  (using the way of construction of the canonical sequence).

Hence  $f(t_i^m) = f(\omega(v_i^m)) \in B(y_i^m, \frac{1}{m})$ ,  $f(\omega(t)) \in B(y_i^m, \frac{1}{m})$ , and this implies that  $\|f(\omega(v_i^m)) - f(\omega(t))\| \leq \frac{2}{m}$ .

Therefore,  $\|g_m(t) - g(t)\| \leq \|R\|_0 \cdot \frac{2}{m}$ , that proves that  $g_m \xrightarrow{u} g$ , and, from that,  $\int g d\mu = \lim_m \int g_m d\mu$ .

Using the equality  $\int f dH(R)(\mu) = \lim_m \int f_m dH(R)(\mu)$ , from (2.3), and taking the limit when  $m \rightarrow \infty$ , we obtain (2.1).  $\square$

The following result can be proved: We consider the space  $(cabv(K^n), \|\cdot\|_{MK})$  and suppose  $\sum_{i=1}^m \|R_i\|_0 (1 + r_i) < 1$ . Let  $a > 0$ ,  $\mu^0 \in cabv(K^n)$  and  $P : cabv(K^n) \rightarrow cabv(K^n)$ , where  $P(\mu) = H(\mu) + \mu^0$ . Let, also,  $\emptyset \neq A \subset B_a(K^n)$ , weak-\* closed, such that  $P(A) \subset A$  and  $\pi : A \rightarrow A$ ,  $\pi(\mu) = P(\mu)$ . Then  $\pi$  is a contraction on  $A$ , with the ratio  $\|\pi\|_L \leq \sum_{i=1}^m \|R_i\|_0 (1 + r_i)$ . Consequently, there exists a unique measure  $\mu^* \in A$  such that  $\pi(\mu^*) = \mu^*$ . This measure  $\mu^*$  is called *invariant vector measure* or *fractal vector measure*.

**Remark. a)** It can be proved (see [3]) that the weak-\* topology on  $B_a(K^n)$  is the same with the topology given by  $\|\cdot\|_{MK}$  on  $B_a(K^n)$ . But  $B_a(K^n)$  is compact in the weak-\* topology, so is compact (hence, complete) in the topology given by  $\|\cdot\|_{MK}$ . If  $A \subset B_a(K^n)$  is weak-\* closed, we conclude that  $A$  is weak-\* compact in this topology, hence is compact (and complete) in the topology given by  $\|\cdot\|_{MK}$ .

**b)** Theorems similar to the previous result can be proved for  $(cabv(K^n, 0), \|\cdot\|_{MK}^*)$  and for  $(cabv(X), \|\cdot\|)$ , when  $X$  is an arbitrary Hilbert space (see [4]).

### 3. RESULTS

#### 3.1. An integral for vector functions with respect to Lebesgue measure.

Let  $a, b \in \mathbb{R}$ ,  $a < b$ ,  $\lambda$  the Lebesgue measure on  $[a, b]$  and  $X$  a Banach space. Notations:

- $S_{[a,b]}(X) = \left\{ f : [a, b] \rightarrow X \mid f \text{ is a simple function} \right\}$
- $TM_{[a,b]}(X) = \left\{ f : [a, b] \rightarrow X \mid (\exists) (f_n)_n \subset S_{[a,b]}(X) \text{ such that } f_n \xrightarrow{u} f \right\}$

**Definition 3.1.** For  $f \in S_{[a,b]}(X)$ ,  $f = \sum_{i=1}^m \varphi_{A_i} x_i$ , where  $(A_i)_{1 \leq i \leq m}$  is a partition with Borel sets of  $[a, b]$  and  $x_i \in X$ , we define  $\int_{[a,b]} f d\lambda \stackrel{\text{def}}{=} \sum_{i=1}^m \lambda(A_i) x_i$ .

$$\text{Obviously, } \left\| \int_{[a,b]} f d\lambda \right\| \leq \sum_{i=1}^m \|x_i\| \lambda(A_i) \leq \|f\|_{\infty} (b-a).$$

From these inequalities we deduce that the application  $f \mapsto \int_{[a,b]} f d\lambda$  is continuous and, consequently, it has an extension to  $\overline{S_{[a,b]}(X)} = TM_{[a,b]}(X)$ . Namely, if  $(f_n)_n \subset S_{[a,b]}(X)$  and  $f_n \xrightarrow{u} f$ , we define  $\int_{[a,b]} f d\lambda = \lim_{n \rightarrow \infty} \int_{[a,b]} f_n d\lambda$ , and the limit is the same for all sequences  $(f_n)_n$  uniformly convergent to  $f$ .

### 3.2. The Markov-type operator.

In this section  $X$  is a Hilbert space and  $(T, d)$  is a compact metric space. For any  $\alpha \in [a, b]$ , we consider:

- $R_{\alpha} : X \rightarrow X$ , linear and continuous;
- $\omega_{\alpha} : X \rightarrow X$ , contractions of ratio  $r_{\alpha} < 1$ .

We suppose that:

- i) The application  $\alpha \mapsto R_{\alpha}$  is continuous on  $[a, b]$ ;
- ii) For any  $\mu \in \text{cabv}(X)$ , the application  $\alpha \mapsto \mu \circ \omega_{\alpha}^{-1}$  is continuous on  $[a, b]$ ;
- iii) For any  $f \in C(X)$ , the application  $\alpha \mapsto f \circ \omega_{\alpha}$  is continuous on  $[a, b]$ ;
- iv) The application  $\alpha \mapsto r_{\alpha}$  is measurable.

**Remark.** For the conditions **i) - iv)**, we consider the topologies:

**a)** on  $[a, b]$ , the trace of the topology given on  $\mathbb{R}$  by the canonical metric:

$$d_{\mathbb{R}}(x, y) = |x - y|;$$

**b)** on  $\mathcal{L}(X)$  (the space of linear and continuous operators on  $X$ ): the topology given by the operatorial norm;

**c)** on  $\text{cabv}(X)$ : the topology given by the variational norm;

**d)** on  $C(X)$ : the topology given by the norm  $\|\cdot\|_{\infty}$ :

$$\|f\|_{\infty} = \sup_{t \in T} \|f(t)\|, \quad (\forall) f \in C(X).$$

**Definition 3.2.** For any  $\mu \in \text{cabv}(X)$ , we define  $H(\mu) : \mathcal{B} \rightarrow X$  via:

$$H(\mu)(A) = \int_{[a,b]} R_{\alpha} \left( \mu \left( \omega_{\alpha}^{-1}(A) \right) \right) d\lambda,$$

for any  $A \in \mathcal{B}$ .

We will use the notation:  $H(\mu) \stackrel{\text{not}}{=} \int_{[a,b]} (R_\alpha \circ \mu \circ \omega_\alpha^{-1}) d\lambda$ .

**Theorem 3.3.** *We consider  $(cabv(X), \|\cdot\|)$ . Then, for any  $\mu \in cabv(X)$ ,  $H(\mu) \in cabv(X)$ , the function  $\mu \rightarrow H(\mu)$  is linear and continuous, and  $\|H\|_0 \leq \int_{[a,b]} \|R_\alpha\|_0 d\lambda$ .*

*Proof.* Let  $(A_j)_{1 \leq j \leq n}$  a partition of  $T$  with Borel sets. We have:

$$\begin{aligned} \sum_{j=1}^n \|H(\mu)(A_j)\| &= \sum_{j=1}^n \left\| \int_{[a,b]} R_\alpha(\mu(\omega_\alpha^{-1}(A_j))) d\lambda \right\| \\ &\leq \sum_{j=1}^n \left( \int_{[a,b]} \|R_\alpha\|_0 \|\mu(\omega_\alpha^{-1}(A_j))\| d\lambda \right) \\ &= \int_{[a,b]} \left( \sum_{j=1}^n \|R_\alpha\|_0 \|\mu(\omega_\alpha^{-1}(A_j))\| \right) d\lambda \\ &= \int_{[a,b]} \left( \|R_\alpha\|_0 \sum_{j=1}^n \|\mu(\omega_\alpha^{-1}(A_j))\| \right) d\lambda \\ &\leq |\mu|(T) \int_{[a,b]} \|R_\alpha\|_0 d\lambda \\ &= \|\mu\| \int_{[a,b]} \|R_\alpha\| d\lambda. \end{aligned}$$

Taking the supremum with respect to all the finite partitions of  $T$  with Borel sets, we obtain

$$\|H(\mu)\|(T) \leq \|\mu\| \left( \int_{[a,b]} \|R_\alpha\|_0 d\lambda \right).$$

Hence  $H(\mu) \in cabv(X)$  and  $\|H(\mu)\| \leq \left( \int_{[a,b]} \|R_\alpha\|_0 d\lambda \right) \|\mu\|$ . We conclude that  $H$  is linear and con-

tinuous and  $\|H\|_0 \leq \int_{[a,b]} \|R_\alpha\| d\lambda$ . □

**Theorem 3.4. (change of variable formula)**

*For any  $f \in C(X)$  and  $\mu \in cabv(X)$  we have  $\int f dH(\mu) = \int g d\mu$ , where  $g : T \rightarrow X$ ,  $g(t) = \int_{[a,b]} R_\alpha^*(f(\omega_\alpha(t))) d\lambda$ .*

*Proof.* For any  $n \in \mathbb{N}^*$  we consider the partition  $(B_i)_{1 \leq i \leq n}$  of the interval  $[a, b]$ , where  $B_i = [x_{i-1}, x_i]$ ,  $x_i = a + i \frac{b-a}{n}$ . Obviously,  $\lambda(B_i) = \frac{b-a}{n}$ . In each set  $B_i$  we choose  $\alpha_i$ , arbitrarily, fixed. According to Theorem 2.2 we have:  $\int f dH_n(\mu) = \int g_n d\mu$ , where  $H_n(\mu) = \frac{b-a}{n} \sum_{i=1}^n R_{\alpha_i} \circ \mu \circ \omega_{\alpha_i}^{-1}$ ,  $g_n = \frac{b-a}{n} \sum_{i=1}^n R_{\alpha_i}^* \circ f \circ \omega_{\alpha_i}$ .

It will be sufficient to prove that:

- A)  $H_n(\mu) \rightarrow H(\mu)$  in  $cabv(X)$ ;
- B)  $g_n \xrightarrow{u} g$  in  $TM(X)$ .

**A)** Let  $\alpha \in [a, b]$ ; there exists  $i \in \{1, \dots, n\}$  such that  $\alpha \in B_i$ . Let also  $(A_j)_{1 \leq j \leq m}$  be a partition of  $T$  with Borel sets. We denote:

$$\mathcal{S} \stackrel{\text{not}}{=} \sum_{j=1}^m \left\| \int_{[a,b]} R_\alpha(\mu(\omega_\alpha^{-1}(A_j))) d\lambda - \frac{b-a}{n} \sum_{i=1}^n R_{\alpha_i}(\mu(\omega_{\alpha_i}^{-1}(A_j))) \right\|.$$

We have:

$$\begin{aligned} \mathcal{S} &= \sum_{j=1}^m \left\| \sum_{i=1}^n \left\{ \int_{B_i} [R_\alpha(\mu(\omega_\alpha^{-1}(A_j))) - R_{\alpha_i}(\mu(\omega_{\alpha_i}^{-1}(A_j)))] d\lambda \right\} \right\| \\ &\leq \sum_{j=1}^m \left\{ \sum_{i=1}^n \left[ \int_{B_i} \left( \|R_\alpha(\mu(\omega_\alpha^{-1}(A_j))) - R_{\alpha_i}(\mu(\omega_{\alpha_i}^{-1}(A_j)))\| \right. \right. \right. \\ &\quad \left. \left. \left. + \|R_\alpha(\mu(\omega_\alpha^{-1}(A_j))) - R_{\alpha_i}(\mu(\omega_{\alpha_i}^{-1}(A_j)))\| \right) d\lambda \right] \right\} \\ &\leq \sum_{j=1}^m \left\{ \sum_{i=1}^n \left[ \int_{B_i} \|R_\alpha\|_0 \|\mu(\omega_\alpha^{-1}(A_j)) - \mu(\omega_{\alpha_i}^{-1}(A_j))\| d\lambda \right. \right. \\ &\quad \left. \left. + \int_{B_i} \|R_\alpha - R_{\alpha_i}\| \|\mu(\omega_\alpha^{-1}(A_j))\| d\lambda \right] \right\}. \end{aligned} \tag{3.1}$$

From **iv)** and **ii)**, we deduce:

**1<sup>0</sup>)**  $(\forall) \varepsilon > 0, (\exists) \delta_\varepsilon > 0$  such that for  $|\alpha - \alpha_0| < \delta_\varepsilon, \|R_\alpha - R_0\| < \varepsilon$ .

**2<sup>0</sup>)**  $(\forall) \varepsilon > 0, (\exists) \eta_\varepsilon > 0$  such that for  $|\alpha - \alpha_0| < \eta_\varepsilon, \|\mu \circ \omega_\alpha^{-1} - \mu \circ \omega_{\alpha_0}^{-1}\| < \varepsilon$ .

Let  $\varepsilon > 0$  and  $n \in \mathbb{N}$  such that  $\frac{b-a}{n} < \min(\delta_\varepsilon, \eta_\varepsilon)$ . Because  $|\alpha - \alpha_i| < \frac{b-a}{n}$ , we have, using (3.1):

$$\begin{aligned} \mathcal{S} &\leq \sum_{i=1}^n \left[ \int_{B_i} \|\mu \circ \omega_\alpha^{-1} - \mu \circ \omega_{\alpha_i}^{-1}\| \|R_\alpha\|_0 d\lambda + \|\mu\| \int_{B_i} \|R_\alpha - R_{\alpha_i}\|_0 d\lambda \right] \\ &\leq \varepsilon \int_{[a,b]} \|R_\alpha\|_0 d\lambda + \varepsilon \|\mu\| (b-a) \stackrel{\text{not}}{=} \varepsilon M \end{aligned}$$



where  $M = \int_{[a,b]} \|R_\alpha\|_0 \, d\lambda + \|\mu\|(b-a)$ .

Taking the supremum for all the finite partitions of  $T$  with Borel sets, we conclude that  $H_n(\mu) \rightarrow H(\mu)$  in  $cabv(X)$ .

**B)** Let  $t \in T$ . We have:

$$\begin{aligned}
\|g_n(t) - g(t)\| &= \left\| \int_{[a,b]} R_\alpha^*(f(\omega_\alpha(t))) \, d\lambda - \frac{b-a}{n} \sum_{i=1}^n R_{\alpha_i}^*(f(\omega_{\alpha_i}(t))) \right\| \\
&= \left\| \sum_{i=1}^n \left\{ \int_{B_i} \left[ R_\alpha^*(f(\omega_\alpha(t))) - R_{\alpha_i}^*(f(\omega_{\alpha_i}(t))) \right] \, d\lambda \right\} \right\| \\
&\leq \sum_{i=1}^n \left[ \int_{B_i} \left\| R_\alpha^*(f(\omega_\alpha(t))) - R_{\alpha_i}^*(f(\omega_{\alpha_i}(t))) \right\| \, d\lambda \right. \\
&\quad \left. + \int_{B_i} \left\| R_\alpha^*(f(\omega_{\alpha_i}(t))) - R_{\alpha_i}^*(f(\omega_{\alpha_i}(t))) \right\| \, d\lambda \right] \\
&\leq \sum_{i=1}^n \left[ \int_{B_i} \|R_\alpha\|_0 \underbrace{\left\| (f \circ \omega_\alpha - f \circ \omega_{\alpha_i})(t) \right\|}_{\leq \|f \circ \omega_\alpha - f \circ \omega_{\alpha_i}\|_\infty} \, d\lambda \right. \\
(3.2) \quad &\quad \left. + \int_{B_i} \|R_\alpha - R_{\alpha_i}\|_0 \underbrace{\|f(\omega_{\alpha_i}(t))\|}_{\leq \|f\|_\infty} \, d\lambda \right].
\end{aligned}$$

Because  $|\alpha - \alpha_i| < \frac{b-a}{n}$ , for  $n \in \mathbb{N}$  big enough, from **iii)** we have  $\|f \circ \omega_\alpha - f \circ \omega_{\alpha_i}\| < \varepsilon$ . Hence, using (3.2), we obtain:

$$\|g_n(t) - g(t)\| \leq \varepsilon \left( \int_{[a,b]} \|R_\alpha\|_0 \, d\lambda + \|f\|_\infty (b-a) \right), \quad (\forall) t \in T,$$

and so  $g_n \xrightarrow{u} g$ . □

**Theorem 3.5.** Let  $f \in L_1(X)$  and  $g$  as in Theorem 3.4 Then  $g$  is a Lipschitz function and  $\|g\|_L \leq \int_{[a,b]} \|R_\alpha\|_0 r_\alpha \, d\lambda$ .

*Proof.* First we remark that the function  $\alpha \mapsto \|R_\alpha\|_0 r_\alpha$  is Lebesgue integrable. This results from the fact that the function is measurable (as a product of measurable functions) and is less than the integrable function  $\alpha \mapsto \|R_\alpha\|_0$ .

Let  $x, y \in T$ . We have:

$$\|g(x) - g(y)\| = \left\| \int_{[a,b]} R_\alpha^* \left[ f(\omega_\alpha(x)) - f(\omega_\alpha(y)) \right] \, d\lambda \right\| \leq$$

$$\begin{aligned}
&\leq \int_{[a,b]} \|R_\alpha^*\|_0 \|f(\omega_\alpha(x)) - f(\omega_\alpha(y))\| \, d\lambda \leq \\
&\leq \int_{[a,b]} \|R_\alpha\|_0 \|\omega_\alpha(x) - \omega_\alpha(y)\| \, d\lambda \leq \\
&\leq \left( \int_{[a,b]} \|R_\alpha\|_0 r_\alpha \, d\lambda \right) d(x, y).
\end{aligned}$$

Then  $g$  is a Lipschitz function and  $\|g_L\| \leq \int_{[a,b]} \|R_\alpha\|_0 r_\alpha \, d\lambda$ . □

**Theorem 3.6.** *We consider the space  $(cabv(X), \|\cdot\|_{MK})$ . Then  $H$  is linear and continuous, and*

$$\|H\|_0 \leq \int_{[a,b]} \|R_\alpha\|_0 (1 + r_\alpha) \, d\lambda.$$

*Proof.* Let  $g$  be as in Theorem 3.4 where  $f \in BL_1(X)$ . For any  $t \in T$ ,

$$\|g(t)\| \leq \int_{[a,b]} \|R_\alpha^*\|_0 \underbrace{\|f(\omega_\alpha(t))\|}_{\leq \|f\|_\infty \leq 1} \, d\lambda \leq \int_{[a,b]} \|R_\alpha\|_0 \, d\lambda.$$

Using Theorem 3.5 we have:

$$\|g\|_{BL} = \|g\|_\infty + \|g\|_L \leq \int_{[a,b]} \|R_\alpha\|_0 (1 + r_\alpha) \, d\lambda.$$

From Theorem 3.4 we know that:

$$\begin{aligned}
\left| \int f \, dH(\mu) \right| &= \left| \int g \, d\mu \right| \underbrace{\leq}_{\text{(see [3])}} \|g\|_{BL} \cdot \|\mu\|_{MK} \\
&\leq \left( \int_{[a,b]} \|R_\alpha\|_0 (1 + r_\alpha) \, d\lambda \right) \|\mu\|_{MK}
\end{aligned}$$

Taking the supremum for all the functions from  $BL_1(X)$  we obtain:

$$\|H(\mu)\|_{MK} \leq \left( \int_{[a,b]} \|R_\alpha\|_0 (1 + r_\alpha) \, d\lambda \right) \|\mu\|_{MK}.$$

$$\text{Hence, } \|H\|_0 \leq \int_{[a,b]} \|R_\alpha\|_0 (1 + r_\alpha) \, d\lambda. \quad \square$$

**Theorem 3.7.** *Let us consider the space  $(cabv(X, 0), \|\cdot\|_{MK}^*)$ . Then:*

**a)** *For any  $\mu \in cabv(X, 0)$ ,  $H(\mu) \in cabv(X, 0)$ ;*

**b)** *We define  $H_1 : cabv(X, 0) \rightarrow cabv(X, 0)$ ,  $H_1(\mu) = H(\mu)$ .*

Then  $H_1$  is linear and continuous and  $\|H_1\|_0 \leq \int_{[a,b]} \|R_\alpha\|_0 r_\alpha d\lambda$ .

*Proof. a)* Let  $\mu \in cabv(X, 0)$ .

$$H(\mu)(T) = \int_{[a,b]} R_\alpha \left( \mu \left( \omega_\alpha^{-1}(T) \right) \right) d\lambda = \int_{[a,b]} R_\alpha \left( \underbrace{\mu(T)}_{=0} \right) d\lambda = 0 \implies H(\mu) \in cabv(X, 0).$$

**b)** Let  $f \in L_1(X)$  and  $g$  be as in Theorems 3.4 and 3.5. Using Theorem 3.5, we have  $\|g\|_L \leq \int_{[a,b]} \|R_\alpha\|_0 r_\alpha d\lambda$ . We can write:

$$\left| \int f dH(\mu) \right| = \left| \int g d\mu \right| \underset{\text{(see [3])}}{\leq} \|g\|_L \cdot \|\mu\|_{MK}^* \leq \left( \int_{[a,b]} \|R_\alpha\|_0 r_\alpha d\lambda \right) \|\mu\|_{MK}^*,$$

and taking the supremum for all  $f \in L_1(X)$ , we obtain that:

$$\|H(\mu)\|_{MK}^* \leq \left( \int_{[a,b]} \|R_\alpha\|_0 r_\alpha d\lambda \right) \|\mu\|_{MK}^*.$$

Hence  $\|H_1\|_0 \leq \int_{[a,b]} \|R_\alpha\|_0 r_\alpha d\lambda$ .  $\square$

### 3.3. Fractal invariant measures.

#### Theorem 3.8. (Fractal Vector Measure for $\|\cdot\|_{MK}$ )

Let us consider the space  $(cabv(K^n), \|\cdot\|_{MK})$ . We suppose that  $\int_{[a,b]} \|R_\alpha\|_0 (1 + r_\alpha) d\lambda < 1$ .

Let  $a > 0$  and  $\mu^0 \in cabv(K^n)$ . We define

$$P : cabv(K^n) \longrightarrow cabv(K^n), \quad P(\mu) = H(\mu) + \mu^0.$$

Let  $\emptyset \neq A \subset B_a(K^n)$ , weak-\* closed, such that  $P(A) \subset A$ . Then the function  $\pi : A \longrightarrow A$ ,  $\pi(\mu) = P(\mu)$ , is a contraction on  $A$  with the ratio  $\|\pi\|_L \leq \int_{[a,b]} \|R_\alpha\|_0 (1 + r_\alpha) d\lambda$ , for the

metric generated by  $\|\cdot\|_{MK}$  on  $A$ . Consequently, there exists a unique measure  $\mu^* \in A$ , such that  $\pi(\mu^*) = \mu^*$ .

*Proof.* The set  $A$ , being weak-\* closed in  $B_a(K^n)$ , is compact, hence complete in this topology. But the weak-\* topology coincides with the one given by  $\|\cdot\|_{MK}$  on  $B_a(K^n)$  (see [3]). So  $A$  is complete in the topology given by  $\|\cdot\|_{MK}$ .

Let  $\mu_1, \mu_2 \in A$ . We have:

$$\|\pi(\mu_1) - \pi(\mu_2)\| = \|H(\mu_1) - H(\mu_2)\| \leq \|H\|_0 \|\mu_1 - \mu_2\|.$$

By using Theorem 3.6 we deduce that

$$\|\pi\|_L \leq \|H\|_0 \leq \int_{[a,b]} \|R_\alpha\|_0 (1 + r_\alpha) d\lambda < 1.$$

Hence  $\pi$  is a contraction on  $A$ . Using the contraction principle, there exists a unique measure  $\mu^* \in A$  with  $\pi(\mu^*) = \mu^*$ .  $\square$

**Theorem 3.9. (Fractal Vector Measure for  $\|\cdot\|_{MK}^*$ )**

Let us consider the space  $(cabv(K^n, 0), \|\cdot\|_{MK}^*)$ . We suppose that:  $\int_{[a,b]} R_\alpha d\lambda = 1_{K^n}$  (that means:  $(\forall) x \in K^n, \int_{[a,b]} R_\alpha(x) d\lambda = x$ ) and  $\int_{[a,b]} \|R_\alpha\|_0 r_\alpha d\lambda < 1$ .

Let  $a > 0, v \in K^n$  such that  $\|v\| \leq a$ . We consider the set  $\emptyset \neq A \subset B_a(K^n, v)$  weak- $*$  closed, such that  $H(A) \subset A$ , and the function  $\pi : A \rightarrow A, \pi(\mu) = H(\mu)$ . Then  $\pi$  is a contraction of ratio  $\|\pi\|_L \leq \int_{[a,b]} \|R_\alpha\|_0 r_\alpha d\lambda$ , for the metric generated by  $\|\cdot\|_{MK}^*$  on  $A$ . Consequently, there exists a unique measure  $\mu^* \in A$  such that  $\pi(\mu^*) = \mu^*$ .

*Proof* Let  $H_1 : cabv(K^n, 0) \rightarrow cabv(K^n, 0), H_1(\mu) = H(\mu)$ . According to Theorem 3.5 we have:  $\|H_1\|_0 \leq \int_{[a,b]} \|R_\alpha\|_0 r_\alpha d\lambda$ . Similar to Theorem 3.8, we deduce:

$$\|\pi\|_L \leq \|H_1\|_0 \leq \int_{[a,b]} \|R_\alpha\|_0 r_\alpha d\lambda < 1,$$

hence  $\pi$  is a contraction for  $\|\mu\|_{MK}^*$ .

But (see [3]) the space  $(B_a(K^n, v), d_{MK^*})$  is compact (where  $d_{MK^*}$  is the metric generated by  $\|\cdot\|_{MK}^*$ ) and  $A$ , being weak- $*$  closed in  $B_a(K^n, v)$  is closed in the topology given by  $d_{MK^*}$ , so is compact in this topology. Hence  $(A, d_{MK^*})$  is a complete metric space. Using the contraction principle, there exists a unique measure  $\mu^* \in A$  such that  $\pi(\mu^*) = \mu^*$ .  $\square$

**Remarks.** a) The condition  $\int_{[a,b]} R_\alpha d\lambda = 1_{K^n}$  ensures that, for any  $\mu \in cabv(K^n, v)$ , we have  $H(\mu) \in cabv(K^n, v)$ . Indeed,

$$H(\mu)(T) = \int_{[a,b]} R_\alpha \left( \mu(\omega_\alpha^{-1}(T)) \right) d\lambda = \int_{[a,b]} R_\alpha (\mu(T)) d\lambda = \int_{[a,b]} R_\alpha(v) d\lambda = v.$$

This means that, for any  $\mu_1, \mu_2 \in cabv(K^n, v), H(\mu_1) - H(\mu_2) \in cabv(K^n, 0)$ , so we can use for  $H(\mu_1) - H(\mu_2)$  the modified Monge-Kantorovich norm.

b) The condition  $\|v\| \leq a$  ensures that  $B_a(K^n, v) \neq \emptyset$ . Let  $t \in T$  and let  $\delta_t$  be the Dirac measure concentrated in  $t$ . We have:  $\delta_t v \in cabv(K^n, v)$  and  $\|\delta_t v\| = \|v\| \leq a$ . Hence  $\delta_t v \in B_a(K^n, v)$ .  $\square$

**Theorem 3.10. (Fractal vector measure for any arbitrary Banach space)**

Let  $X$  a Banach space and consider  $(cabv(X), \|\cdot\|)$ . We suppose:

$$\int_{[a,b]} \|R_\alpha\|_0 \, d\lambda < 1.$$

Let  $\mu^0 \in cabv(X)$  and

$$P : cabv(X) \longrightarrow cabv(X), \quad P(\mu) = H(\mu) + \mu^0.$$

Then there exists an unique measure  $\mu^* \in cabv(X)$  such that  $P(\mu^*) = \mu^*$ .

*Proof.* For any  $\mu_1, \mu_2 \in cabv(X)$ , by using Theorem 3.3, we have:

$$\begin{aligned} \|P(\mu_1) - P(\mu_2)\| &= \|H(\mu_1) - H(\mu_2)\| \leq \|H\|_0 \|\mu_1 - \mu_2\| \\ &\leq \left( \int_{[a,b]} \|R_\alpha\|_0 \, d\lambda \right) \|\mu_1 - \mu_2\|. \end{aligned}$$

So  $P$  is a contraction on the Banach space  $(cabv(X), \|\cdot\|)$ . Using the contraction principle, there exists a unique measure  $\mu^* \in cabv(X)$  such that  $P(\mu^*) = \mu^*$ .  $\square$

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