ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 1, p.44-49

ON VANISHING OF GENERALIZED LOCAL HOMOLOGY MODULES AND ITS DUALITY

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ABSTRACT. In this paper we study the vanishing and non-vanishing of generalized local cohomology and generalized local homology. In particular for a Noetherian local ring $(R, \mathfrak{m})$ and two non-zero finitely generated R-modules M and N, it is shown that $H_{\mathfrak{m}}^{\dim N}(M, N) \neq 0$.

1. INTRODUCTION

Local cohomology was first defined and studied by Grothendieck [Gro]. Let R be a commutative Noetherian ring with non-zero identity and M be an R-module. For an ideal I of R, the i-th local cohomology modules with support in I is defined as follows:

$$H_I^i(M) = \varinjlim_{t \in \mathbb{N}} \operatorname{Ext}_R^i \left(R/I^t, M\right).$$

On the other hand, a natural generalization of local cohomology modules was introduced by Herzog [Her] as follows: For a pair of R-module (M, N) the i-th generalized local cohomology module of (M, N) with respect to I is the R-module

$$H_I^i(M, N) = \varinjlim_{t \in \mathbb{N}} \operatorname{Ext}_R^i \left(M/I^t M, N\right).$$

Clearly whenever $M = R$, the generalized local cohomology module $H_I^i(R, N)$ is the ordinary local cohomology module $H_I^i(N)$. Moslehi and Bijan-Zadeh introduced a natural generalization of local homology modules [BM]. For $i \in \mathbb{N}_0$, we defined generalized local homology module $U_i^I(M, N)$ of pair (M, N) with respect to I as follows:

$$U_i^I(M, N) = \varprojlim_{t \in \mathbb{N}} \operatorname{Tor}_i^I(M/I^t M, N).$$

Whenever $M = R$, for simplicity of notation we denote $U_i^I(R, N)$ by $U_i^I(N)$.

Two important type of theorems concerning local homology and cohomology are vanishing and non-vanishing results. We collect the known vanishing and non-vanishing results for generalized local homology and cohomology in the following theorems.

Theorem 1.1. *Let M and N be two non-zero finitely generated R-modules such that* $\operatorname{pd} M < \infty$ *(*pd *abbreviates projective dimension).*

(i)*([Yas, 3.7]) Suppose* $\dim N < \infty$*. Then $H_I^i(M, N) = 0$, for all $i > \operatorname{pd} M + \dim (M \otimes_R N)$.*

(ii) *([Bij, 5.5]) Let*

$$t = \operatorname{grade}_N(M/IM) = \inf \left\{i : \operatorname{Ext}_R^i(M/IM, N) \neq 0\right\}.$$

If $t < \infty$, then $H_I^i(M, N) = 0$ for all $i < t$ and $H_I^t(M, N) \neq 0$.

2010 *Mathematics Subject Classification.* 13D45, 14B15, 16E30, 13J99.

Key words and phrases. Generalized local homology, Generalized local cohomology.

(iii) *([Yas, 2.5])*$H_I^i(M,N)=0$, *for all* $i>\operatorname{ara}(I)+\operatorname{pd}M$, *where* $\operatorname{ara}(I)$ *the arithmetic rank of the ideal I is the least number of elements of R required to generate an ideal which has the same radical as I.*

(iv) *([Suz, 2.3]) Let* $(R,\mathfrak{m})$ *be a local ring. Then* $\operatorname{depth}N$ *is the least integer i such that* $H_{\mathfrak{m}}^i(M,N)\neq 0$.

(v) *([Suz, 3.18, 3.21]) Let* $(R,\mathfrak{m})$ *be a Cohen-Macaulay local ring, M and N be two non-zero R-modules such that N is finitely generated and* $\operatorname{Supp}(N)\subseteq\operatorname{Supp}(M)$. *Then* $H_{\mathfrak{m}}^{\dim N}(M,N)\neq 0$.

Definition 1.2. (a) We say that an element $a\in R$ is *M-coregular* if $aM=M$.

(b) The sequence $a_1,a_2,\cdots,a_n$ of R is called an *M-coregular sequence* if

(i) $\operatorname{Ann}_M(a_1,\cdots,a_n)\neq 0$;

(ii) a_i is an $\operatorname{Ann}_M(a_1,\cdots,a_{i-1})$-coregular element, for all $i=1,2,\cdots,n$.

(c) Let M and N be R-modules, where M is finitely generated and N is Artinian. We call the length of any maximal N-coregular sequence contained in $\operatorname{Ann}_R(M)$ the $\operatorname{Cograde}_N(M)$. We note that this is well-defined by [Ooi, 3.10].

Now we recall the concept of *Krull dimension* of an Artinian module, denoted by $\operatorname{Kdim}M$, due to Roberts [Rob]: let M be an Artinian R-module. When $M=0$ we put $\operatorname{Kdim}M=-1$. Then by induction, for any ordinal α, we put $\operatorname{Kdim}M=\alpha$ when (i) $\operatorname{Kdim}M<\alpha$ is false, and (ii) for every ascending chain $M_0\subseteq M_1\subseteq\ldots$ of submodules of M, there exists a positive integer m_0 such that $\operatorname{Kdim}(M_{m+1}/M_m)<\alpha$ for all $m>m_0$. Thus M is non-zero and Noetherian if and only if $\operatorname{Kdim}M=0$.

Theorem 1.3. *Let* $(R,\mathfrak{m})$ *be a local ring, M a finitely generated and N an Artinian R-modules. Then*

(i) *([BM, 4.2])* $\operatorname{Cograde}_N(M/IM)=\inf\{i: U_i^I(M,N)\neq 0\}$.

(ii) *([BM, 4.4]) For all* $i>\operatorname{Kdim}N$, $U_i^{\mathfrak{m}}(M,N)=0$ *and if there exists an element* $x\in I$ *which is N-coregular,* $U_i^I(M,N)=0$.

Theorem 1.4. *([BM, 2.3]) Let* $D(-):=\operatorname{Hom}_R(-,E(R/\mathfrak{m}))$ *be the Matlis dual functor with respect to the injective hull of* $R/\mathfrak{m}$.

(i) $U_i^I(M,D(N))=0$ *if and only if* $H_I^i(M,N)=0$.

(ii) *If N is an Artinian R-module, then* $U_i^I(M,N)=0$ *if and only if* $H_I^i(M,D(N))=0$.

2. Main results

In this section, some results on vanishing and non-vanishing of generalized local homology and cohomology modules are presented. From now on, we assume that R is a Noetherian local ring with a unique maximal ideal $\mathfrak{m}$.

Remark 2.1. Let $\widehat{\ }$ be a $\mathfrak{m}$-adic completion functor. The following items will be required to prove Theorems.

(i) If M is a finitely generated R-module, then $\widehat{R}\otimes_R M\cong\widehat{M}$ by [EJ, 2.5.14].

(ii) If N is an Artinian R-module, then $\widehat{R}\otimes_R N\cong N$ and N is also Artinian as an $\widehat{R}$-module by [Ooi, 3.14].

(iii) If M is a finitely generated and N an Artinian R-modules, then $\operatorname{Hom}_{\widehat{R}}\left(\widehat{M},N\right)\cong\operatorname{Hom}_R(M,N)$ by [EJ, 2.5.15] and [Rot, 11.65].

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(iv) If M is a finitely generated R-module with $\mathrm{pd}_R M = n$, then by [Rot, 11.64],

$$\begin{aligned}\mathrm{Tor}_{n+1}^{\widehat{R}}\left(N, \widehat{M}\right) &\cong \mathrm{Tor}_{n+1}^{\widehat{R}}\left(N, M \otimes_R \widehat{R}\right)\\ &\cong \mathrm{Tor}_{n+1}^{R}(N, M) = 0,\end{aligned}$$

for all $\widehat{R}$-module N. Thus $\mathrm{pd}_{\widehat{R}}\widehat{M} \leqslant n$.

(v) If M is a finitely generated and Cohen-Macaulay R-module, then $\widehat{M}$ is also Cohen-Macaulay as an R-module by [BH, 2.1.8(b)].

(vi) Let M be a finitely generated and N an Artinian R-modules. If $\mathrm{Cosupp}_R(N) \subseteq \mathrm{Supp}_R(M)$, then $\mathrm{Cosupp}_{\widehat{R}}(N) \subseteq \mathrm{Supp}_{\widehat{R}}\left(\widehat{M}\right)$ by [AM, Ch. 3, Exercise 19, viii].

The following Corollary is a consequence of Theorem 1.1.

Corollary 2.2. *Let M be a finitely generated and N an Artinian R-modules such that* $\mathrm{pd}M< \infty$.
(i) *Suppose* $\dim N < \infty$. *Then $U_i^I(M,N) = 0$, for all $i > \mathrm{pd}M + \dim(\mathrm{Hom}_R(M,N))$.*
(ii) *$U_i^I(M,N) = 0$, for all $i > \mathrm{ara}(\widehat{I}) + \mathrm{pd}M$.*
(iii) *Also, let M and N be two non-zero R-module with* $\mathrm{Cosupp}(N) \subseteq \mathrm{Supp}(M)$. *Then if R is Cohen-Macaulay, then $U_{\dim N}^{\mathfrak{m}}(M,N) \neq 0$.*

Proof. (i) By [BM, 2.5], without loss of generality, we may assume that $(R,\mathfrak{m})$ is a complete local ring. Also, $D(N)$ (where $D(-) := \mathrm{Hom}_R(-, E(R/\mathfrak{m}))$ is the Matlis dual functor with respect to the injective hull of $R/\mathfrak{m}$) is a finitely generated R-module and

$$\dim(\mathrm{Hom}_R(M,N)) = \dim(D(\mathrm{Hom}_R(M,N))) = \dim(M \otimes_R D(N))$$

by [Ooi, 1.6(3),(8)]. The assertion is now immediate from Theorem 1.1(i) and 1.4(ii).

(ii) The assertion is now immediate from Theorem 1.1(iii) and 1.4(ii).

(iii) By [Ooi, 2.11], $\mathrm{Cosupp}(N) = \mathrm{Supp}(D(N))$. Therefore $H_{\mathfrak{m}}^{\dim N}(M, D(N)) \neq 0$ by Theorem 1.1(v). The claim now follows from [Ooi, 1.6(2)] and Theorem 1.4(ii). □

Lemma 2.3. *Let M be a finitely generated and N a non-zero, Artinian of dimension d. Then the set*

$$\sum := \{N' : N' \textit{ is a submodule of } N \textit{ and } \dim N/N' < d\}$$

has a minimal element with respect to inclusion. If N_0 is a minimal element of $\sum$, then
(i) $\dim N_0 = d$;
(ii) *N_0 has no non-zero submodule N' such that* $\dim N_0/N' < d$;
(iii) $\mathrm{Att}_R(N_0) = \{\mathfrak{p} \in \mathrm{Att}_R(N) : \dim R/\mathfrak{p} = d\}$; *and*
(iv) $U_d^{\mathfrak{m}}(M,N) \cong U_d^{\mathfrak{m}}(M,N_0)$.

Proof. (i),(ii),(iii) See [Maf, 2.2].

(iv) Since $\dim N/N_0 < d$, it follows from [BM, 4.4(i)] that $U_d^{\mathfrak{m}}(M, N/N_0) = U_{d+1}^{\mathfrak{m}}(M, N/N_0) = 0$. The claim now follows from [BM, 3.2(ii)], by using the exact sequence

$$0 \longrightarrow N_0 \longrightarrow N \longrightarrow N/N_0 \longrightarrow 0.$$

□

Theorem 2.4. *Let M be a non-zero finitely generated and N a non-zero Artinian R-modules with* $\dim N = d$. *Then $U_d^{\mathfrak{m}}(M,N) \neq 0$ and*

$$\mathrm{Ass}_R(U_d^{\mathfrak{m}}(M,N)) = \{\mathfrak{p} \in \mathrm{Att}_R(N) : \dim R/\mathfrak{p} = d\}.$$

Proof. We use induction on d. When $d=0$, the module N has finite length, and so it is annihilated by some power of $\mathfrak{m}$. Hence there exists a positive integer n such that $\mathfrak{m}^n(M\otimes_R N)=0$. Thus

$$U_0^{\mathfrak{m}}(M,N)\cong \widehat{M\otimes_R N}\cong M\otimes_R N,$$

where $\widehat{\ }$ is the completion functor with respect to $\mathfrak{m}$. Hence $U_0^{\mathfrak{m}}(M,N)\neq 0$, by [Ooi, 3.8]. By [Mat, Exercise 6.9],

$$\begin{aligned}\operatorname{Ass}_R\left(U_0^{\mathfrak{m}}(M,N)\right) &= \operatorname{Ass}_R(M\otimes_R N)\\ &= \{\mathfrak{m}\}\\ &= \operatorname{Ass}_R(N)\\ &= \operatorname{Att}_R(N)\\ &= \{\mathfrak{p}\in\operatorname{Att}_R(N):\dim R/\mathfrak{p}=0\}.\end{aligned}$$

Thus the result has been proved in this case. Assume, inductively, that $d>0$ and that the result has been proved for non-zero Artinian R-modules of dimension $d-1$. By Lemma 2.3, we can assume that N has no non-zero homomorphic image of dimension less than d. We shall make this assumption our aim to show that $\operatorname{Ass}_R(U_d^{\mathfrak{m}}(M,N))=\operatorname{Att}_R(N)$ (see Lemma 2.3(iii)). Since $d>0$, we have $\mathfrak{m}\notin\operatorname{Att}_R(N)$, and so there exists N-coregular element x in $\mathfrak{m}$. We suppose that $U_d^{\mathfrak{m}}(M,N)=0$, and look for a contradiction. If $d=1$, we have $1\leq \operatorname{Cograde}_N(M/\mathfrak{m}M)=\operatorname{Width}_{\mathfrak{m}}(N)\leq\dim N=1$ by [Ooi, 3.17]. Thus $\operatorname{Cograde}_N(M/\mathfrak{m}M)=1$ which is impossible by Theorem 1.3. Thus we can assume $d>1$. Now, for each N-coregular element x in $\mathfrak{m}$, the module $(0:_N x)$ (is non-zero and Artinian and) has $\dim(0:_N x)=d-1$, by [Maf, 2.1], and the exact sequence

$$0\longrightarrow(0:_N x)\longrightarrow N\xrightarrow{x}N\longrightarrow 0$$

induces a long exact sequence

$$0\longrightarrow U_{d-1}^{\mathfrak{m}}(M,0:_N x)\longrightarrow U_{d-1}^{\mathfrak{m}}(M,N)\xrightarrow{x}U_{d-1}^{\mathfrak{m}}(M,N).$$

In view of our assumption $U_d^{\mathfrak{m}}(M,N)=0$. Thus, for each N-coregular element x in $\mathfrak{m}$, we have that

$$\left(0:_{U_{d-1}^{\mathfrak{m}}(M,N)}x\right)\cong U_{d-1}^{\mathfrak{m}}(M,0:_N x),$$

which is non-zero, by the inductive hypothesis. Therefore $U_{d-1}^{\mathfrak{m}}(M,N)\neq 0$. Our next step is to prove that $\mathfrak{m}\in\operatorname{Ass}_R\left(U_{d-1}^{\mathfrak{m}}(M,N)\right)$.

We suppose that $\mathfrak{m}\notin\operatorname{Ass}_R\left(U_{d-1}^{\mathfrak{m}}(M,N)\right)$ and look for a contradiction. Then, by the Prime Avoidance Theorem,

$$\mathfrak{m}\nsubseteq\left(\bigcup_{\mathfrak{p}\in\operatorname{Ass}_R\left(U_{d-1}^{\mathfrak{m}}(M,N)\right)}\mathfrak{p}\right)\bigcup\left(\bigcup_{\mathfrak{q}\in\operatorname{Att}_R(N)}\mathfrak{q}\right).$$

Hence there exists an N-coregular element y that belongs to $\mathfrak{m}$ such that $\left(0:_{U_{d-1}^{\mathfrak{m}}(M,N)}y\right)=0$. This is a contradiction (note that, for each N-coregular element y in $\mathfrak{m}$, we have that $\left(0:_{U_{d-1}^{\mathfrak{m}}(M,N)}y\right)\cong U_{d-1}^{\mathfrak{m}}(M,0:_N y)$, which is non-zero, by the inductive hypothesis).

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Thus $\mathfrak{m} \in \operatorname{Ass}_R\left(U^{\mathfrak{m}}_{d-1}(M,N)\right)$. By [BM, 3.1], we can assume that $\mathfrak{p}_1, \ldots \mathfrak{p}_t$ are the remaining members of $\operatorname{Ass}_R\left(U^{\mathfrak{m}}_{d-1}(M,N)\right)$. Again, by the Prime Avoidance Theorem, there exists

$$z \in \mathfrak{m} \setminus \left(\bigcup_{i=1}^{t} \mathfrak{p}_i\right) \bigcup \left(\bigcup_{\mathfrak{q} \in \operatorname{Att}_R(N)} \mathfrak{q}\right).$$

Thus $U^{\mathfrak{m}}_{d-1}(M, 0:_N z) = \left(0:_{U^{\mathfrak{m}}_{d-1}(M,N)} z\right)$, since $z \in \mathfrak{m}$ is an N-coregular element, and by the induction hypothesis, $U^{\mathfrak{m}}_{d-1}(M, 0:_N z) \neq 0$ and

$$\operatorname{Ass}_R\left(U^{\mathfrak{m}}_{d-1}(M, 0:_N z)\right) = \{\mathfrak{p} \in \operatorname{Att}_R(0:_N z) : \dim R/\mathfrak{p} = d-1\}.$$

On the other hand,

$$\operatorname{Ass}_R\left(0:_{U^{\mathfrak{m}}_{d-1}(M,N)} z\right) \subseteq \left\{\mathfrak{p} \in \operatorname{Ass}_R\left(U^{\mathfrak{m}}_{d-1}(M,N)\right) : z \in \mathfrak{p}\right\}$$

and $\mathfrak{m}$ is the only member of this set. Since $d > 1$, we have a contradiction. Thus we have proved that $U^{\mathfrak{m}}_d(M,N) \neq 0$. To complete the inductive step, since N now has no non-zero homomorphic image of dimension less than d, it remains for us to prove that $\operatorname{Ass}_R(U^{\mathfrak{m}}_d(M,N)) = \operatorname{Att}_R(N)$. Since $\operatorname{Cograde}_N(M/\mathfrak{m}M) \geq 1$, there exists a coregular element x in $\mathfrak{m}$ on N. Thus $\dim(0:_N x) = d-1$, which implies that $U^{\mathfrak{m}}_d(M, 0:_N x) = 0$, and we have the long exact sequence induced by the exact sequence

$$0 \longrightarrow (0:_N x) \longrightarrow N \xrightarrow{x} N \longrightarrow 0$$

yields that $\left(0:_{U^{\mathfrak{m}}_d(M,N)} x\right) = 0$. It therefore follows that

$$\mathfrak{m} \setminus \left(\bigcup_{\mathfrak{p} \in \operatorname{Att}_R(N)} \mathfrak{p}\right) \subseteq \mathfrak{m} \setminus \left(\bigcup_{\mathfrak{p} \in \operatorname{Ass}_R\left(U^{\mathfrak{m}}_d(M,N)\right)} \mathfrak{p}\right).$$

Suppose that $\mathfrak{q} \in \operatorname{Ass}_R(U^{\mathfrak{m}}_d(M,N))$. it follows from the above inclusion and by the Prime Avoidance Theorem that $\mathfrak{q} \subseteq \mathfrak{p}$, for some $\mathfrak{p} \in \operatorname{Att}_R(N)$. Since $U^{\mathfrak{m}}_d(M,-)$ is an R-linear functor, it follows that $(0:N) \subseteq (0:U^{\mathfrak{m}}_d(M,N)) \subseteq \mathfrak{q} \subseteq \mathfrak{p}$. As $d = \dim R/\operatorname{Ann}_R(N) = \dim R/\mathfrak{p}$, it follows that $\mathfrak{q} = \mathfrak{p}$. Hence $\operatorname{Ass}_R(U^{\mathfrak{m}}_d(M,N)) \subseteq \operatorname{Att}_R(N)$. To establish the reverse inclusion, let $\mathfrak{p} \in \operatorname{Att}_R(N)$, so that $\dim R/\mathfrak{p} = d$. Thus there exists a $\mathfrak{p}$-secondary submodule Q of N (see [Mac, 5.2]). Note that Q can not have any non-zero homomorphic image of dimension less than d (or else it would have an attached prime other than $\mathfrak{p}$). Now if we use Q rather than N in the above, we have $\operatorname{Ass}_R(U^{\mathfrak{m}}_d(M,Q)) \subseteq \operatorname{Att}_R(Q) = \{\mathfrak{p}\}$ and $U^{\mathfrak{m}}_d(M,Q) \neq 0$. Thus $\operatorname{Ass}_R(U^{\mathfrak{m}}_d(M,Q)) = \{\mathfrak{p}\}$. However, the exact sequence

$$0 \longrightarrow Q \longrightarrow N \longrightarrow N/Q \longrightarrow 0$$

induces a monomorphism $U^{\mathfrak{m}}_d(M,Q) \longrightarrow U^{\mathfrak{m}}_d(M,N)$, since $\dim N/Q \leq d$. It now follows that $\{\mathfrak{p}\} = \operatorname{Ass}_R(U^{\mathfrak{m}}_d(M,Q)) \subseteq \operatorname{Ass}_R(U^{\mathfrak{m}}_d(M,N))$. Hence $\operatorname{Att}_R(N) \subseteq \operatorname{Ass}_R(U^{\mathfrak{m}}_d(M,N))$. This complete the inductive step. □

Corollary 2.5. *Let M and N be two non-zero finitely generated R-modules such that* $\dim N = d$. *Then*

$$H^d_{\mathfrak{m}}(M,N) \neq 0.$$

Proof. The assertion is immediate from [Ooi, 1.6 (2) and (8)] and Theorem 1.4 (i). □

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