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NEMYTSKIJ OPERATORS IN LEBESGUE SPACES WITH A VARIABLE EXPONENT

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Abstract: In this paper we prove a result concerning sufficient conditions for the continuity of the general nonlinear superposition operator (generalized Nemytskij operator) acting in Lebesgue spaces with a variable exponent. We also provide an application to the study of the Fréchet-differentiability of the gradient norm on a Sobolev space with a variable exponent.
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1. Introduction

Suppose that $\Omega \subset \mathbf{R}^N$ is a bounded domain. Let $f:\Omega\times\mathbf{R}^M \to \mathbf{R}$ be a function satisfying the *Carathéodory conditions*:

(i) for each $s\in\mathbf{R}^M$, the function $x\to f(x,s)$ is Lebesgue measurable in Ω;

(ii) for almost all $x\in\Omega$, the function $s\to f(x,s)$ is continuous in $\mathbf{R}^M$.

To such a function we associate the *Nemytskij operator*

$$(N_f u)(x) := f(x,u(x)) \text{ for each } x\in\Omega,$$

defined on classes of vector functions $u:\Omega\to\mathbf{R}^M$, $u(x)=(u_1(x),u_2(x),\dots,u_M(x))$.

Let us make the following convention for the Carathéodory function, the assertion ”$x\in\Omega$” is to be understood in the sense ”almost all $x\in\Omega$”.

It is well known that, for any measurable function $u:\Omega\to\mathbf{R}^M$, the function $\Omega\ni x\mapsto f(x,u(x))\in\mathbf{R}$ is also measurable.

We now review some definitions and properties related to Lebesgue spaces with variable exponents needed throughout the paper. For proofs and references see [3].

Given a function $p(\cdot)\in L^\infty(\Omega)$ that satisfies

$$1\le p^- := \operatorname{ess}\inf_{x\in\Omega} p(x) \le \operatorname{ess}\sup_{x\in\Omega} p(x) =: p^+ < \infty,$$

the *Lebesgue space* $L^{p(\cdot)}(\Omega)$ *with variable exponent* $p(\cdot)$ is defined as

$$L^{p(\cdot)}(\Omega) := \left\{ v:\Omega\to\mathbf{R};\ v \textit{ is measurable and } \rho_{p(\cdot)}(v) := \int_\Omega |v(x)|^{p(x)}\,dx < \infty \right\}.$$

Equipped with the norm

$$u \in L^{p(\cdot)}(\Omega) \to \|u\|_{p(\cdot)} := \inf\left\{\lambda > 0; \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \le 1\right\},$$

the space $L^{p(\cdot)}(\Omega)$ is a separable Banach space.

Given $p(\cdot) \in L^\infty(\Omega)$ such that $p^- > 1$, let $p'(\cdot) \in L^\infty(\Omega)$ be defined by

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1 \text{ for almost all } x \in \Omega.$$

For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, the following inequality holds:

(1) $$\int_\Omega |u(x)v(x)|\,\mathrm{d}x \le \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}.$$

If $v, w \in L^{p(\cdot)}(\Omega)$, then:

(2) $$\rho_{p(\cdot)}(v+w) \le 2^{p^+}\left(\rho_{p(\cdot)}(v) + \rho_{p(\cdot)}(w)\right).$$

The following theorem summarizes the relations between the norm $\|\cdot\|_{0,p(\cdot)}$ and the convex modular $\rho_{p(\cdot)}$.

Theorem 1. *Let $p(\cdot) \in L^\infty(\Omega)$ be such that $p^- \ge 1$ and let $u \in L^{p(\cdot)}(\Omega)$. Then:*

(a) *If $u \ne 0$, then $\|u\|_{p(\cdot)} = a$ if and only if $\rho_{p(\cdot)}(a^{-1}u) = 1$.*

(b) $\|u\|_{p(\cdot)} < 1$ *(resp. $=1$ or >1) if and only if $\rho_{p(\cdot)}(u) < 1$ (resp. $=1$ or >1).*

(c) $\|u\|_{p(\cdot)} > 1$ *implies* $\|u\|_{p(\cdot)}^{p^-} \le \rho_{p(\cdot)}(u) \le \|u\|_{p(\cdot)}^{p^+}$.

(d) $\|u\|_{p(\cdot)} < 1$ *implies* $\|u\|_{p(\cdot)}^{p^+} \le \rho_{p(\cdot)}(u) \le \|u\|_{p(\cdot)}^{p^-}$.

(e) *Let $u \in L^{p(\cdot)}(\Omega)$ and $u_n \in L^{p(\cdot)}(\Omega)$, $n = 1, 2, \ldots$. The following statements are equivalent:*

(i) $\|u - u_n\|_{p(\cdot)} \to 0$ *as* $n \to \infty$.

(ii) $\rho_{p(\cdot)}(u_n - u) \to 0$ *as* $n \to \infty$.

(iii) $(u_n)_n$ *converges to u in measure and* $\rho_{p(\cdot)}(u_n) \to \rho_{p(\cdot)}(u)$ *as* $n \to \infty$.

2. The main result

The main result of this paper states sufficient conditions to ensure the Nemytskij operator that maps $\left[L^{p_1(\cdot)}(\Omega)\right]^M$ into $L^{p_2(\cdot)}(\Omega)$ is continuous and bounded.

On $\left[L^{p_1(\cdot)}(\Omega)\right]^M$ consider the norm

$$\|u\| := \left\|\sqrt{T[u,u]}\right\|_{p_1(\cdot)},$$

where $u = (u_1, u_2, \ldots, u_M)$, $T[u,u] := \sum_{i=1}^{M} u_i^2$.

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Theorem 2. *Let* $f:\Omega\times\mathbf{R}^M\to\mathbf{R}$ *be a Carathéodory function which satisfies the growth condition*

(3) $$|f(x,u)|\le c_1(x)+c(x)\sum_{i=1}^{M}|u_i|^{p_1(x)/p_2(x)},\ x\in\Omega,\ u\in\mathbf{R}^M,$$

where $c_1\in L^{p_2(\cdot)}(\Omega)$ *and* c *is a non-negative* $L^\infty(\Omega)$*-function. Then* N_f *is a well-defined, bounded, continuous operator from* $\left[L^{p_1(\cdot)}(\Omega)\right]^M$ *into* $L^{p_2(\cdot)}(\Omega)$.

Proof. First we prove that N_f is a well-defined and bounded operator from $\left[L^{p_1(\cdot)}(\Omega)\right]^M$ into $L^{p_2(\cdot)}(\Omega)$. Let $u=(u_1,u_2,\ldots,u_M)\in\left[L^{p_1(\cdot)}(\Omega)\right]^M$. From (3), by integrating over Ω and taking into account (2), it follows that

(4) $$\int_\Omega|N_f(u)(x)|^{p_2(x)}\mathrm{d}x\le$$

$$\le 2^{p_2^+}\left(\int_\Omega|c_1(x)|^{p_2(x)}\mathrm{d}x+C\int_\Omega\left(\sum_{i=1}^{M}|u_i(x)|^{p_1(x)/p_2(x)}\right)^{p_2(x)}\mathrm{d}x\right)\le$$

$$\le 2^{p_2^+}\left(\int_\Omega|c_1(x)|^{p_2(x)}\mathrm{d}x+2^{(M-1)p_2^+}C\sum_{i=1}^{M}\int_\Omega|u_i(x)|^{p_1(x)}\mathrm{d}x\right)<\infty,$$

where $C:=\max\left(\|c\|_{L^\infty(\Omega)}^{p_2^-},\|c\|_{L^\infty(\Omega)}^{p_2^+}\right)$. Consequently, $N_f\left(\left[L^{p_1(\cdot)}(\Omega)\right]^M\right)\subset L^{p_2(\cdot)}(\Omega)$.

To prove the operator N_f is bounded, let us consider $u=(u_1,u_2,\ldots,u_M)\in\left[L^{p_1(\cdot)}(\Omega)\right]^M$ such that $\|u\|\le C_2$. Since

(5) $$|u_i|\le\sqrt{T[u,u]},\ 1\le i\le M,$$

we deduce that $\|u_i\|_{p_1(\cdot)}\le C_2$. Therefore (Theorem 1 (c) and (d))

$$\rho_{p_1(\cdot)}(u_i)\le C_3:=\max\left(C_2^{p_1^+},C_2^{p_1^-}\right).$$

According to (4), it follows that N_f transforms norm bounded sets in $\left[L^{p_1(\cdot)}(\Omega)\right]^M$ into mean bounded sets in $L^{p_2(\cdot)}(\Omega)$, therefore in norm bounded sets in $L^{p_2(\cdot)}(\Omega)$ (Theorem 1 (c), (d)). Consequently N_f is bounded.

We now prove that the operator N_f is continuos.

Fix $u=(u_1,u_2,\ldots,u_M)\in\left[L^{p_1(\cdot)}(\Omega)\right]^M$. To establish the continuity of N_f, it is enough to show that every sequence $\left(u^{(n)}\right)_n\subset\left[L^{p_1(\cdot)}(\Omega)\right]^M$ such that

(6) $$\lim_{n\to\infty}\|u^{(n)}-u\|=0$$

has a subsequence $\left(u^{(n_k)}\right)_k$ such that $N_f\left(u^{(n_k)}\right)\to N_f(u)$ in $L^{p_2(\cdot)}(\Omega)$ as $k\to\infty$.

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Indeed, let $\left(u^{(n)}\right)_n$ be a sequence as above, $u^{(n)}=\left(u_1^{(n)},u_2^{(n)},\dots,u_M^{(n)}\right)$.

Taking into account (5), from (6) we infer that

$$\lim_{n\to\infty}\left\|u_i^{(n)}-u_i\right\|_{p_1(\cdot)}=0,\ 1\le i\le M,$$

therefore

$$\lim_{n\to\infty}\rho_{p_1(\cdot)}\left(u_i^{(n)}-u_i\right)=0,\ 1\le i\le M,$$

or

(7) $$\left(u_i^{(n)}-u_i\right)^{p_1(\cdot)}\to 0 \text{ in } L^1(\Omega) \text{ as } n\to\infty,\ 1\le i\le M.$$

By using the Brézis's Lemma ([1]), it follows that there exists a subsequence $\left(u_1^{(n_k)}\right)_k\subset\left(u_1^{(n)}\right)_n$ and $h_1\in L^1(\Omega)$ such that

$$\lim_{k\to\infty}\left(u_1^{(n_k)}(x)-u_1(x)\right)^{p_1(x)}=0 \text{ for almost all } x\in\Omega$$

and

$$\left|\left(u_1^{(n_k)}(x)-u_1(x)\right)^{p_1(x)}\right|\le\left|h_1(x)\right| \text{ for almost all } x\in\Omega,\ k\in\mathbf{N}.$$

By applying the Brézis's Lemma again, passing to a subsequence, there exists $h_2\in L^1(\Omega)$ such that

$$\lim_{k\to\infty}\left(u_2^{(n_k)}(x)-u_2(x)\right)^{p_1(x)}=0 \text{ for almost all } x\in\Omega,$$

and

$$\left|\left(u_2^{(n_k)}(x)-u_2(x)\right)^{p_1(x)}\right|\le\left|h_2(x)\right| \text{ for almost all } x\in\Omega,\ k\in\mathbf{N}.$$

The process continues. There exist a subsequence $\left(u^{(n_k)}\right)_k$ and $h_1,h_{2,}\dots,h_M\in L^1(\Omega)$ such that

(8) $$\lim_{k\to\infty}\left(u_i^{(n_k)}(x)-u_i(x)\right)^{p_1(x)}=0 \text{ for almost all } x\in\Omega,\ 1\le i\le M,$$

and

(9) $$\left|\left(u_i^{(n_k)}(x)-u_i(x)\right)^{p_1(x)}\right|\le\left|h_i(x)\right| \text{ for almost all } x\in\Omega,\ k\in\mathbf{N},\ 1\le i\le M.$$

Consequently

(10) $$\lim_{k\to\infty}u_i^{(n_k)}(x)=u_i(x) \text{ almost all } x\in\Omega,\ 1\le i\le M,$$

and

(11) $$\left|u_i^{(n_k)}(x)\right|\le\left|h_i(x)\right|^{1/p_1(x)}+\left|u_i(x)\right| \text{ almost all } x\in\Omega,\ k\in\mathbf{N},\ 1\le i\le M.$$

Since f is a Carathéodory function, it is clear that (see (10))

$$\lim_{k\to\infty}N_f\left(u^{(n_k)}\right)(x)=N_f(u)(x) \text{ for almost all } x\in\Omega,$$

therefore

(12) $$\lim_{k\to\infty}\left(N_f\left(u^{(n_k)}\right)(x)-N_f(u)(x)\right)^{p_2(x)}=0 \text{ for almost all } x\in\Omega.$$

On the other hand, from (3) it follows that

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$$\left|N_f\left(u^{(n_k)}\right)(x)\right|^{p_2(x)} = \left|f(x,u^{(n_k)}(x))\right|^{p_2(x)} \le 2^{p_2(x)-1}$$

$$\times\left(\left|c_1(x)\right|^{p_2(x)} + C\cdot 2^{(M-1)(p_2(x)-1)}\sum_{i=1}^{M}\left(\left|u_i^{(n_k)}(x)\right|\right)^{p_1(x)}\right) \text{ for almost all } x\in\Omega,\ k\in\mathbf{N}.$$

From (11) we deduce that

$$\left|N_f\left(u^{(n_k)}\right)(x)\right|^{p_2(x)} \le 2^{p_2^+-1}\left(\left|c_1(x)\right|^{p_2(x)} + C\cdot 2^{(M-1)\left(p_2^+-1\right)}\sum_{i=1}^{M}2^{p_1(x)-1}\left(\left|h_i(x)\right| + \left|u_i(x)\right|^{p_1(x)}\right)\right),$$

therefore

$$\left|N_f\left(u^{(n_k)}\right)(x) - N_f(u)(x)\right|^{p_2(x)} \le \left(\left|N_f\left(u^{(n_k)}\right)(x)\right| + \left|N_f(u)(x)\right|\right)^{p_2(x)} \le$$

$$\le 2^{p_2(x)-1}\left(\left|N_f\left(u^{(n_k)}\right)(x)\right|^{p_2(x)} + \left|N_f(u)(x)\right|^{p_2(x)}\right) \le 2^{p_2^+-1}g(x),$$

where

$$g(x) := 2^{p_2^+-1}\left(\left|c_1(x)\right|^{p_2(x)} + C\cdot 2^{(M-1)\left(p_2^+-1\right)+p_1^+-1}\sum_{i=1}^{M}\left(\left|h_i(x)\right| + \left|u_i(x)\right|^{p_1(x)}\right)\right) + \left|N_f(u)(x)\right|^{p_2(x)}.$$

Since the right term of this equality is in $L^1(\Omega)$ and (12) holds, by applying Lebesgue's dominated convergence theorem, it follows that

$$\lim_{k\to\infty}\int_\Omega\left(N_f\left(u^{(n_k)}\right)(x) - N_f(u)(x)\right)^{p_2(x)}\mathrm{d}x = 0,$$

that is the subsequence $\left(N_f(u^{(n_k)})\right)_k$ converges in mean to $N_f(u)$. It follows that the subsequence $\left(N_f(u_{n_k})\right)_k$ converges in norm to $N_f(u)$ (Theorem 1 (e)), therefore the operator N_f is continuous.

For $M=1$ we obtain:

Corollary 3. *Let $f:\Omega\times\mathbf{R}\to\mathbf{R}$ be a Carathéodory function which satisfies the growth condition*

$$\left|f(x,u)\right| \le c_1(x) + c(x)\left|u\right|^{p(x)-1},\ x\in\Omega,\ u\in\mathbf{R},$$

where $c_1\in L^{p'(\cdot)}(\Omega)$ and c is a non-negative $L^\infty(\Omega)$-function. Then N_f is a well-defined, bounded, continuous operator from $L^{p(\cdot)}(\Omega)$ into $L^{p'(\cdot)}(\Omega)$.

Note that this corollary is contained in Theorem 1.16, Fan and Zhao [3].

3. Fréchet differentiability of the gradient norm on a Sobolev space with a variable exponent

In this section, the above results are used to prove the Fréchet differentiability of a norm on a Sobolev space with a variable exponent.

Given a function $p(\cdot)\in L^\infty(\Omega)$ that satisfies $p^-\ge 1$, the *Sobolev space* $W^{1,p(\cdot)}(\Omega)$ *with variable exponent* $p(\cdot)$ is defined as

$$W^{1,p(\cdot)}(\Omega) := \left\{v\in L^{p(\cdot)}(\Omega);\ \partial_i v\in L^{p(\cdot)}(\Omega),\ 1\le i\le N\right\},$$

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where, for each $1 \le i \le N$, ∂_i denotes the distributional derivative operator with respect to the *i*-th variable. $W^{1,p(\cdot)}(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{1,p(\cdot),\nabla} := \|u\|_{0,p(\cdot)} + \sum_{i=1}^{N} \|\partial_i u\|_{0,p(\cdot)} .$$

Consider the space (see [2] for details)

$$U_{\Gamma_0} := \left\{ u \in W^{1,p(\cdot)}(\Omega);\ \mathrm{tr}\ u = 0 \text{ on } \Gamma_0 \right\},\ \Gamma_0 \subset \Gamma = \partial\Omega,\ \mathrm{d}\Gamma - meas\Gamma_0 > 0 .$$

The map

$$u \in U_{\Gamma_0} \to \|u\|_{0,p(\cdot),\nabla} := \| |\nabla u| \|_{p(\cdot)}$$

is a norm on U_{Γ_0}, equivalent to the norm $\|u\|_{1,p(\cdot),\nabla}$ ([2], Theorem 6 (b))

Moreover ([2], Lemma 1), the norm $\|u\|_{0,p(\cdot),\nabla}$ is Gâteaux-differentiable at any nonzero $u \in U_{\Gamma_0}$ and the Gâteaux-differential of this norm at any nonzero $u \in U_{\Gamma_0}$ is given for any $h \in U_{\Gamma_0}$ by

$$\left\langle \|\cdot\|'_{0,p(\cdot),\nabla}(u), h \right\rangle = \frac{\displaystyle\int_{\Omega\setminus\Omega_{0,u}} p(x) \frac{|\nabla u(x)|^{p(x)-2} \langle \nabla u(x), \nabla h(x) \rangle}{\|u\|_{0,p(\cdot),\nabla}^{p(x)-1}} \mathrm{d}x}{\displaystyle\int_{\Omega} p(x) \frac{|\nabla u(x)|^{p(x)}}{\|u\|_{0,p(\cdot),\nabla}^{p(x)}} \mathrm{d}x}, \tag{13}$$

where $\Omega_{0,u} := \left\{ x \in \Omega;\ |\nabla u(x)| = 0 \right\}$.

By using Theorem 2 and Corollary 3, we will prove:

Theorem 4. *The map*

$$u \in U_{\Gamma_0} \setminus \{0\} \to \|\cdot\|'_{p(\cdot)}$$

is continuous.

Proof. Another direct proof of this theorem can be found in [2], Lemma 2.

Let $\varphi : U_{\Gamma_0} \setminus \{0\} \to \left(U_{\Gamma_0}, \|\cdot\|_{p(\cdot)} \right)^*$ be defined by

$$\langle \varphi(u), h \rangle := \int_{\Omega\setminus\Omega_{0,u}} p(x) \frac{|\nabla u(x)|^{p(x)-2} \langle \nabla u(x), \nabla h(x) \rangle}{\|u\|_{0,p(\cdot),\nabla}^{p(x)-1}} \mathrm{d}x \text{ for each } h \in U_{\Gamma_0}$$

and let $q : U_{\Gamma_0} \setminus \{0\} \to \mathbb{R}$ be defined by

$$q(u) := \int_{\Omega} p(x) \frac{|\nabla u(x)|^{p(x)}}{\|u\|_{0,p(\cdot),\nabla}^{p(x)}} \mathrm{d}x.$$

Since

$$\left\langle \|\cdot\|'_{p(\cdot)}(u), \cdot \right\rangle = \frac{\langle \varphi(u), \cdot \rangle}{q(u)} \text{ for all } u \in U_{\Gamma_0} \setminus \{0\},$$

it is sufficient to prove that φ and q are continuous.

Fix $u \in U_{\Gamma_0} \setminus \{0\}$ and let $(u_n)_n \subset U_{\Gamma_0} \setminus \{0\}$ be such that $u_n \to u$ as $n \to \infty$ in the space $\left(U_{\Gamma_0}, \|\cdot\|_{0,p(\cdot),\nabla}\right)$. Since

$$\left| |\nabla u_n(x)| - |\nabla u(x)| \right| \le |\nabla(u_n - u)(x)|$$

and

$$\|\nabla(u_n - u)\|_{p(\cdot)} \to 0 \text{ as } n \to \infty,$$

it follows that

$$\left\| |\nabla u_n| - |\nabla u| \right\|_{p(\cdot)} \to 0 \text{ as } n \to \infty.$$

Consequently

$$\left\| \frac{|\nabla u_n|}{\|\nabla u_n\|_{p(\cdot)}} - \frac{|\nabla u|}{\|\nabla u\|_{p(\cdot)}} \right\|_{p(\cdot)} \to 0 \text{ as } n \to \infty. \tag{14}$$

For any i, $1 \le i \le N$, consider the function $f_i : \Omega \times \mathbf{R}^N \to \mathbf{R}$ given by:

$$f_i(x, s_1, s_2, \ldots, s_N) = \begin{cases} \left(\sqrt{\sum_{j=1}^N s_j^2} \right)^{p(x)-2} \cdot s_i & , \text{if } \sum_{j=1}^N s_j^2 > 0 \\ 0 & , \text{if } \sum_{j=1}^N s_j^2 = 0 \end{cases}.$$

We can write

$$\langle \varphi(u), h \rangle := \sum_{i=1}^N \int_\Omega p(x) f_i\left(x, \frac{\nabla u(x)}{\|u\|_{0,p(\cdot),\nabla}} \right) \partial_i h(x) \mathrm{d}x \text{ for each } h \in U_{\Gamma_0}$$

We have

$$\langle \varphi(u_n) - \varphi(u), h \rangle = \sum_{i=1}^N \int_\Omega p(x) w_n^i(x) \partial_i h(x) \mathrm{d}x, \tag{15}$$

where, for any i, $1 \le i \le N$,

$$w_n^i(x) := f_i\left(x, \frac{\nabla u_n(x)}{\|u_n\|_{0,p(\cdot),\nabla}} \right) - f_i\left(x, \frac{\nabla u(x)}{\|u\|_{0,p(\cdot),\nabla}} \right), \; x \in \Omega.$$

Since

$$|f_i(x, s_1, s_2, \ldots, s_N)| \le \left(\sqrt{\sum_{j=1}^N s_j^2} \right)^{p(x)-1} \text{ if } \sum_{j=1}^N s_j^2 > 0, \tag{16}$$

it follows that the functions f_i are continuous on $\mathbf{R}^N$. On the other hand,

$$\sqrt{\sum_{j=1}^N s_j^2} \le \sum_{j=1}^N |s_j| \tag{17}$$

and

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$$|s_j| = \left(|s_j|^{p(x)-1}\right)^{1/(p(x)-1)} \le \left(\sum_{j=1}^{N} |s_j|^{p(x)-1}\right)^{1/(p(x)-1)},$$

therefore

(18) $$\sum_{j=1}^{N} |s_j| \le N\left(\sum_{j=1}^{N} |s_j|^{p(x)-1}\right)^{1/(p(x)-1)}.$$

From (16), (17), and (18) it follows that

$$\left| f_i(x, s_1, s_2, \ldots, s_N)\right| \le N^{p(x)-1} \sum_{j=1}^{N} |s_j|^{p(x)-1} \le N^{p^+-1} \sum_{j=1}^{N} |s_j|^{p(x)/p'(x)}$$

that is (3) with $p_1(x) = p(x)$ and $p_2(x) = p'(x)$. By applying Theorem 2, it follows that $f_i\left(\cdot, \frac{\nabla u(\cdot)}{\|u\|_{0,p(\cdot),\nabla}}\right) \in L^{p'(\cdot)}(\Omega)$, therefore $w_n^i(x) \in L^{p'(\cdot)}(\Omega)$. But $\partial_i h \in L^{p(\cdot)}(\Omega)$. Therefore, taking (15) and (1) into account, we obtain

(19) $$\left|\langle \varphi(u_n) - \varphi(u), h\rangle\right| \le p^+ \sum_{i=1}^{N} \int_\Omega |w_n^i(x)| |\partial_i h(x)| \mathrm{d}x \le \underline{C} \sum_{i=1}^{N} \|w_n^i\|_{p'(\cdot)} \|\partial_i h\|_{p(\cdot)},$$

where $\underline{C} = p^+\left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)$.

Since

$$\|\partial_i h\|_{p(\cdot)} \le \|h\|_{0,p(\cdot),\nabla},$$

we deduce from (19) that

$$\left|\langle \varphi(u_n) - \varphi(u), h\rangle\right| \le \underline{C}\left(\sum_{i=1}^{N} \|w_n^i\|_{p'(\cdot)}\right) \|h\|_{0,p(\cdot),\nabla}.$$

Consequently,

$$\|\varphi(u_n) - \varphi(u)\| \le \underline{C} \sum_{i=1}^{N} \|w_n^i\|_{p'(\cdot)}.$$

It is now clear that in order to prove the continuity of φ, it suffices to show that $\|w_n^i\|_{p'(\cdot)} \to 0$ as $n \to \infty$, for any *i*, $1 \le i \le N$. Taking into account (14), that is a consequence of the continuity of Nemytskij operator (Theorem 2).

We now show that

$$q(u_n) \to q(u) \text{ as } n \to \infty.$$

Since

$$|q(u_n) - q(u)| = \left| \int_\Omega p(x)\left[\frac{|\nabla u_n(x)|^{p(x)}}{\|u_n\|_{0,p(\cdot),\nabla}^{p(x)}} - \frac{|\nabla u(x)|^{p(x)}}{\|u\|_{0,p(\cdot),\nabla}^{p(x)}}\right] \mathrm{d}x \right| \le$$

$$\le p^+ \int_\Omega \left| \frac{|\nabla u_n(x)|}{\|u_n\|_{0,p(\cdot),\nabla}} \left[\frac{|\nabla u_n(x)|^{p(x)-1}}{\|u_n\|_{0,p(\cdot),\nabla}^{p(x)-1}} - \frac{|\nabla u(x)|^{p(x)-1}}{\|u\|_{0,p(\cdot),\nabla}^{p(x)-1}}\right]\right| \mathrm{d}x$$

$$+p^+\int_\Omega \left|\frac{|\nabla u(x)|^{p(x)-1}}{\|u\|_{0,p(\cdot),\nabla}^{p(x)-1}}\left[\frac{|\nabla u_n(x)|}{\|u_n\|_{0,p(\cdot),\nabla}}-\frac{|\nabla u(x)|}{\|u\|_{0,p(\cdot),\nabla}}\right]\right|\mathrm{d}x.$$

it suffices to show that:

$$A_n := \int_\Omega \left|\frac{|\nabla u_n(x)|}{\|u_n\|_{0,p(\cdot),\nabla}}\left[\frac{|\nabla u_n(x)|^{p(x)-1}}{\|u_n\|_{0,p(\cdot),\nabla}^{p(x)-1}}-\frac{|\nabla u(x)|^{p(x)-1}}{\|u\|_{0,p(\cdot),\nabla}^{p(x)-1}}\right]\right|\mathrm{d}x \to 0 \text{ as } n\to\infty$$

and

$$B_n := \int_\Omega \left|\frac{|\nabla u(x)|^{p(x)-1}}{\|u\|_{0,p(\cdot),\nabla}^{p(x)-1}}\left[\frac{|\nabla u_n(x)|}{\|u_n\|_{0,p(\cdot),\nabla}}-\frac{|\nabla u(x)|}{\|u\|_{0,p(\cdot),\nabla}}\right]\right|\mathrm{d}x \to 0 \text{ as } n\to\infty.$$

Since $\frac{|\nabla u_n|}{\|u_n\|_{0,p(\cdot),\nabla}}\in L^{p(\cdot)}(\Omega)$, $\frac{|\nabla u_n|^{p(\cdot)-1}}{\|u_n\|_{0,p(\cdot),\nabla}^{p(\cdot)-1}}-\frac{|\nabla u|^{p(\cdot)-1}}{\|u\|_{0,p(\cdot),\nabla}^{p(\cdot)-1}}\in L^{p'(\cdot)}(\Omega)$, by using the inequality (1), we obtain that

$$A_n \le \bar{C}\left\|\frac{|\nabla u_n|^{p(\cdot)-1}}{\|u_n\|_{0,p(\cdot),\nabla}^{p(\cdot)-1}}-\frac{|\nabla u|^{p(\cdot)-1}}{\|u\|_{0,p(\cdot),\nabla}^{p(\cdot)-1}}\right\|_{p'(\cdot)},$$

where $\bar{C} := \frac{1}{p^-}+\frac{1}{(p')^-}$. It suffices to show that

$$\left\|\frac{|\nabla u_n|^{p(\cdot)-1}}{\|u_n\|_{0,p(\cdot),\nabla}^{p(\cdot)-1}}-\frac{|\nabla u|^{p(\cdot)-1}}{\|u\|_{0,p(\cdot),\nabla}^{p(\cdot)-1}}\right\|_{p'(\cdot)} \to 0 \text{ as } n\to\infty.$$

That follows from (14) and Corollary 3 with $f(x,u)=|u|^{p(x)-1}$.

Therefore $A_n \to 0$ as $n\to\infty$.

Similarly,

$$B_n \le \bar{C}\left\|\frac{|\nabla u_n|}{\|u_n\|_{0,p(\cdot),\nabla}}-\frac{|\nabla u|}{\|u\|_{0,p(\cdot),\nabla}}\right\|_{p(\cdot)}\left\|\frac{|\nabla u|^{p(\cdot)-1}}{\|u\|_{0,p(\cdot),\nabla}^{p(\cdot)-1}}\right\|_{p'(\cdot)},$$

Since

$$\rho_{p'(\cdot)}\left(\frac{|\nabla u|^{p(\cdot)-1}}{\|u\|_{0,p(\cdot),\nabla}^{p(\cdot)-1}}\right)=\rho_{p(\cdot)}\left(\frac{|\nabla u|}{\|u\|_{0,p(\cdot),\nabla}}\right)=1,$$

it follows (Theorem 1 (b)) that

$$\left\|\frac{|\nabla u|^{p(\cdot)-1}}{\|u\|_{0,p(\cdot),\nabla}^{p(\cdot)-1}}\right\|_{p'(\cdot)}=1.$$

Therefore

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$$B_n \leq \bar{C}\left\| \frac{|\nabla u_n|}{\|u_n\|_{0,p(\cdot),\nabla}} - \frac{|\nabla u|}{\|u\|_{0,p(\cdot),\nabla}} \right\|_{p(\cdot)}.$$

Taking into account (19), $B_n \to 0$ as $n \to \infty$.

Hence we conclude that

$$q(u_n) \to q(u) \text{ as } n \to \infty.$$

This completes the proof of Theorem 3.

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