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A nonlinear eigenvalue problem for the generalized Laplacian on Sobolev spaces with variable exponent

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Abstract

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a smooth bounded domain. Suppose that $p \in \mathcal{C}\left(\overline{\Omega}\right)$ and $p(x) > 1$, for any $x \in \overline{\Omega}$. Using a variational method, we will study the nonlinear eigenvalue problem involving the $(\varphi, p(\cdot))$ - Laplacian [6, p. 388] on the generalized Sobolev space with a variable exponent $\left(W_0^{1,p(\cdot)}(\Omega),\ \|\cdot\|_{1,p(\cdot)}\right)$($\varphi$ is a gauge function).

Keywords: Nonlinear eigenvalue problem; $(\varphi, p(\cdot))$-Laplacian; duality mapping; Sobolev space with variable exponent.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$,$N \geq 2$, be a bounded domain with a sufficiently smooth boundary $\partial\Omega$ and $p : \overline{\Omega} \to \mathbb{R}$ be a continuous function with $p(x) > 1$ for $x \in \overline{\Omega}$.

In this paper, we will consider the eigenvalues of the generalized - Laplacian Dirichlet problem

$$-\left\langle \Delta_{(\varphi,p(.))}(u), h\right\rangle + \langle g(\cdot, u), h\rangle = \lambda \left\langle \underline{J}_{\psi}(u), h\right\rangle, \tag{1}$$

where:

(i) $\lambda \in \mathbb{R}$ is a parameter;

(ii) $-\Delta_{(\varphi,p(.))} = J_\varphi : \left(W_0^{1,p(\cdot)}(\Omega),\ \|\cdot\|_{1,p(\cdot)}\right) \to \left(W_0^{1,p(\cdot)}(\Omega),\ \|\cdot\|_{1,p(\cdot)}\right)^*$ is the duality mapping corresponding to the gauge function φ (i.e. $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, strictly increasing, $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$);

(iii) $\underline{J}_\psi : \left(L^{q(\cdot)}(\Omega), \|\cdot\|_{q(\cdot)}\right) \to \left(L^{q(\cdot)}(\Omega), \|\cdot\|_{q(\cdot)}\right)^*$ is the duality mapping corresponding to the gauge functions ψ; here $q \in \mathcal{C}\left(\overline{\Omega}\right) \cap L_{+}^{\infty}(\Omega)$ satisfies

$$q(x) < p^{*}(x) = \begin{cases} \dfrac{Np(x)}{N - p(x)} & \text{if } p(x) < N \\ +\infty & \text{if } p(x) \geq N \end{cases} ; \tag{2}$$

(iii) the nonlinear term $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Moreover, $u \to g(\cdot, u)$ is a strictly increasing odd function with $\lim\limits_{t\to\infty} g(x,t) = \infty$, which satisfies the growth condition:

$$|g(x,s)| \leq C\,|s|^{p(x)/p'(x)} + a(x) \text{ for a.e. } x \in \Omega \text{ and for all } s \in \mathbb{R}, \tag{3}$$

where $C = \mathit{const.} > 0$, $a \in L^{p'(\cdot)}(\Omega)$, $a(x) \geq 0$ a.e. $x \in \Omega$, and

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1 \text{ for a.e. } x \in \Omega. \tag{4}$$

Let $\lambda \in \mathbb{R}$ and $u \in W_0^{1,p(\cdot)}(\Omega)$ which satisfies (1). The pair (u, λ) is called a *solution* of the problem (1). If, additionally, $u \neq 0$, then λ is called an *eigenvalue* of problem (1) and u an *eigenfunction* corresponding to λ.

The case $g = 0$ is studied in [7]. We mention that the $(\varphi, p(\cdot))$ - Laplacian is a natural generalization of the classical p-Laplacian appropriate from the standpoint of duality maps for the case of variable p ([6], [7, p. 208]). Being inhomogeneous, the $(\varphi, p(\cdot))$ - Laplacian possesses more complicated nonlinearity than the p-Laplacian.

2 Duality mappings on Sobolev spaces with variable exponents

In order to deal with the problem (1), we need some theory of the generalized Lebesgue-Sobolev spaces (see Fan and Zhao [8]). For convenience, we give a simple description here.

2.1 Lebesgue and Sobolev spaces with variable exponents

Given a function $p \in L^{\infty}(\Omega)$ that satisfies

$$1 \leq p^{-} := \operatorname{ess}\inf_{x\in\Omega} p(x),$$

the Lebesgue space $L^{p(\cdot)}(\Omega)$ *with variable exponent* $p(\cdot)$ is defined as

$$L^{p(\cdot)}(\Omega) := \{v : \Omega \to \mathbb{R} \mid v \text{ is } \mathrm{d}x\text{-measurable and } \rho_{p(\cdot)}(v) < \infty\},$$

where

$$\rho_{p(\cdot)}(v) := \int_{\Omega} |v(x)|^{p(x)}\,\mathrm{d}x.$$

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Equipped with the norm

$$v \in L^{p(\cdot)}(\Omega) \to \|v\|_{p(\cdot)} := \inf\{\lambda > 0 \mid \int_{\Omega} \left|\frac{v(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \le 1\},$$

the space $L^{p(\cdot)}(\Omega)$ is a separable Banach space. In addition, if $p^- > 1$ then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive. Also for any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$, one has

$$\int_{\Omega} |u(x)v(x)|\, dx \le \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right) \|u\|_{p(\cdot)} \cdot \|v\|_{p'(\cdot)}. \tag{5}$$

Remark 1 *If $u \in L^{p(\cdot)}(\Omega)$, then $\|u\|_{p(\cdot)} = 1$ if and only if $\rho_{p(\cdot)}(u) = 1$.*

It follows from [8, Theorem 1.16]

Proposition 2 *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function which satisfies the growth condition (3). Then the Nemytskij operator*

$$N_g : L^{p(\cdot)}(\Omega) \to L^{p'(\cdot)}(\Omega),\ (N_g u)(x) = g(x, u(x)),\ a.e.\ x \in \Omega,$$

is well defined, continuous and bounded.

Given a function $p(\cdot) \in L^{\infty}(\Omega)$ that satisfies $p^- \ge 1$, the Sobolev space $W^{1,p(\cdot)}(\Omega)$ *with variable exponent* $p(\cdot)$ is defined as:

$$W^{1,p(\cdot)}(\Omega) := \{u \in L^{p(\cdot)}(\Omega) \mid |\nabla u| \in L^{p(\cdot)}(\Omega)\},\ |\nabla u|^2 = \sum_{i=1}^{N} \left(\frac{\partial u}{\partial x_i}\right)^2,$$

and it is endowed with the norm

$$\|u\| := \|u\|_{p(\cdot)} + \||\nabla u|\|_{p(\cdot)},\ u \in W^{1,p(\cdot)}(\Omega).$$

The space $\left(W^{1,p(\cdot)}(\Omega), \|\cdot\|\right)$ is a separable Banach space. Also $W^{1,p(\cdot)}(\Omega)$ is uniformly convex and thus reflexive.

Let $p,\ q \in \mathcal{C}\left(\overline{\Omega}\right) \cap L_{+}^{\infty}(\Omega)$. If

$$q(x) < p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N \\ +\infty & \text{if } p(x) \ge N \end{cases},$$

then $W^{1,p(\cdot)}(\Omega)$ is compactly imbedded in $L^{q(\cdot)}(\Omega)$.

If $p \in L_{+}^{\infty}(\Omega)$, we define $W_0^{1,p(\cdot)}(\Omega)$ as the closure of $\mathcal{C}_0^{\infty}(\Omega)$ in $\left(W^{1,p(\cdot)}(\Omega), \|\cdot\|\right)$.

Theorem 3 (a) *If $p \in L_{+}^{\infty}(\Omega)$, then $\left(W_0^{1,p(\cdot)}(\Omega), \|\cdot\|\right)$ is a separable Banach space;*

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(b) *If* $p \in L_{+}^{\infty}(\Omega)$ *and* $1 < p^{-}$, *then* $\left(W_0^{1,p(\cdot)}(\Omega), \|\cdot\|\right)$ *is uniformly convex and thus reflexive;*

(c) *If* $p \in \mathcal{C}\left(\overline{\Omega}\right) \cap L_{+}^{\infty}(\Omega)$, *then* $\left(W_0^{1,p(\cdot)}(\Omega), \|\cdot\|\right)$ *is compactly imbedded in* $L^{q(\cdot)}(\Omega)$, *for any* $q \in \mathcal{C}\left(\overline{\Omega}\right) \cap L_{+}^{\infty}(\Omega)$ *satisfying* $q(x) < p^{*}(x)$, $x \in \overline{\Omega}$;

(d) *(Poincaré inequality) If* $p \in \mathcal{C}\left(\overline{\Omega}\right) \cap L_{+}^{\infty}(\Omega)$, *then there is a constant* $c > 0$ *such that*

$$\|u\|_{p(\cdot)} \leq c \, \||\nabla u|\|_{p(\cdot)}, \text{ for any } u \in W_0^{1,p(\cdot)}(\Omega).$$

Using (d) of Theorem 3, it follows that $\|u\|$ and

$$\|u\|_{1,p(\cdot)} := \||\nabla u|\|_{p(\cdot)}$$

are equivalent norms on $W_0^{1,p(\cdot)}(\Omega)$.

In what follows, $W_0^{1,p(\cdot)}(\Omega)$ will be considered as endowed with the norm $\|\cdot\|_{1,p(\cdot)}$ and we will often write $W_0^{1,p(\cdot)}(\Omega)$ instead of $\left(W_0^{1,p(\cdot)}(\Omega), \|\cdot\|_{1,p(\cdot)}\right)$.

2.2 Duality mappings on $\left(W_0^{1,p(\cdot)}(\Omega), \|\cdot\|_{1,p(\cdot)}\right)$

We recall that a real Banach space X is said to be *smooth* if it has the following property: for any $x \in X$, $x \neq 0$, there exists a unique $u^{*}(x) \in X^{*}$ such that $\langle u^{*}(x), x\rangle = \|x\|$ and $\|u^{*}(x)\|_{X^{*}} = 1$. It is well known (see, for instance, Diestel [3], Zeidler [10]) that the smoothness of X is equivalent to the Gâteaux differentiability of the norm. Consequently, if $(X, \|\cdot\|)$ is smooth, then, for any $x \in X$, $x \neq 0$, the only element $u^{*}(x) \in X^{*}$ with the properties $\langle u^{*}(x), x\rangle = \|x\|$ and $\|u^{*}(x)\| = 1$ is $u^{*}(x) = \|\cdot\|'(x)$ (where $\|\cdot\|'(x)$ denotes the Gâteaux gradient of the $\|\cdot\|$-norm at x).

We have

Theorem 4 1) *If* $q \in L_{+}^{\infty}(\Omega)$ *and* $1 < q^{-}$, *then* $\left(L^{q(\cdot)}(\Omega), \ \|\cdot\|_{q(\cdot)}\right)$ *is smooth. The norm* $\|u\|_{q(\cdot)}$ *is Fréchet-differentiable at any nonzero* $u \in L^{q(\cdot)}(\Omega)$ *and the Fréchet-differential of this norm at any nonzero* $u \in L^{q(\cdot)}(\Omega)$ *is given for any* $h \in L^{q(\cdot)}(\Omega)$ *by*

$$\left\langle \|\cdot\|_{q(\cdot)}'(u), h\right\rangle = \frac{\int_{\Omega} q(x) \dfrac{|u(x)|^{q(x)-1} \, sgn \ u(x)}{\|u\|_{q(\cdot)}^{q(x)-1}} h(x) \, dx}{\int_{\Omega} q(x) \dfrac{|u(x)|^{q(x)}}{\|u\|_{q(\cdot)}^{q(x)}} \, dx}. \tag{6}$$

2) ([2]) *The space* $\left(W_0^{1,p(\cdot)}(\Omega), \ \|\cdot\|_{1,p(\cdot)}\right)$ *is smooth. The norm* $\|u\|_{1,p(\cdot)}$ *is Fréchet-differentiable at any nonzero* $u \in W_0^{1,p(\cdot)}(\Omega)$ *and the Fréchet-differential*

of this norm at any nonzero $u \in W_0^{1,p(\cdot)}(\Omega)$ *is given for any* $h \in W_0^{1,p(\cdot)}(\Omega)$ *by*

$$\left\langle \|\cdot\|'_{1,p(\cdot)}(u), h \right\rangle = \frac{\int_{\Omega \setminus \Omega_{0,u}} p(x) \frac{|\nabla u(x)|^{p(x)-2} \quad \langle \nabla u(x), \nabla h(x) \rangle}{\|u\|_{1,p(\cdot)}^{p(x)-1}} dx}{\int_\Omega p(x) \frac{|\nabla u(x)|^{p(x)}}{\|u\|_{1,p(\cdot)}^{p(x)}} dx},$$

where $\Omega_{0,u} := \{x \in \Omega \mid |\nabla u(x)| = 0\}$.

Proof. 1) It follows from [5], [6, Lemma 1, p. 378] that the norm $\|u\|_{q(\cdot)}$ is Gâteaux-differentiable at any nonzero $u \in L^{q(\cdot)}(\Omega)$ and the Gâteaux-differential of this norm at any nonzero $u \in L^{q(\cdot)}(\Omega)$ is given for any $h \in L^{q(\cdot)}(\Omega)$ by (6). To prove the Fréchet-differentiability of the map $u \in L^{q(\cdot)}(\Omega) \setminus \{0\} \to \|u\|_{q(\cdot)}$ it suffices to show that the map $u \in L^{q(\cdot)}(\Omega) \setminus \{0\} \to \|u\|'_{q(\cdot)}$ is continuous.

Let $\phi : L^{q(\cdot)}(\Omega) \setminus \{0\} \to \left(L^{q(\cdot)}(\Omega), \|\cdot\|_{q(\cdot)}\right)^*$ be defined by

$$\langle \phi(u), h \rangle := \int_\Omega q(x) \frac{|u(x)|^{q(x)-1} \, sgn \; u(x)}{\|u\|_{q(\cdot)}^{q(x)-1}} h(x) \mathrm{d}x \text{ for each } h \in L^{q(\cdot)}(\Omega)$$

and let $\omega : L^{q(\cdot)}(\Omega) \setminus \{0\} \to \mathbb{R}$ be defined by

$$\omega(u) := \int_\Omega q(x) \frac{|u(x)|^{q(x)}}{\|u\|_{q(\cdot)}^{q(x)}} \mathrm{d}x.$$

Since

$$\left\langle \|\cdot\|'_{q(\cdot)}(u), \cdot \right\rangle = \frac{\langle \phi(u), \cdot \rangle}{\omega(u)}, \text{ for all } u \in L^{q(\cdot)}(\Omega) \setminus \{0\},$$

it is sufficient to prove that ϕ and ω are continuous.

Fix $u \in L^{q(\cdot)}(\Omega) \setminus \{0\}$ and let $(u_n)_n \subset L^{q(\cdot)}(\Omega) \setminus \{0\}$ be such that $u_n \to u$ as $n \to \infty$ in the space $\left(L^{q(\cdot)}(\Omega), \|\cdot\|_{q(\cdot)}\right)$.

We now show that

$$\omega(u_n) \to \omega(u) \text{ as } n \to \infty.$$

We have

$$|\omega(u_n) - \omega(u)| \le q^+ \int_\Omega \left| \frac{|u_n(x)|^{q(x)}}{\|u_n\|_{q(\cdot)}^{q(x)}} - \frac{|u(x)|^{q(x)}}{\|u\|_{q(\cdot)}^{q(x)}} \right| \mathrm{d}x =$$

$$= q^+ \int_\Omega \left| \frac{|u_n(x)|^{q(x)}}{\|u_n\|_{q(\cdot)}^{q(x)}} - \frac{|u(x)|^{q(x)-1}}{\|u\|_{q(\cdot)}^{q(x)-1}} \frac{|u_n(x)|}{\|u_n\|_{q(\cdot)}} + \right.$$

$$+ \frac{|u(x)|^{q(x)-1}}{\|u\|_{q(\cdot)}^{q(x)-1}} \frac{|u_n(x)|}{\|u_n\|_{q(\cdot)}} - \frac{|u(x)|^{q(x)}}{\|u\|_{q(\cdot)}^{q(x)}} \Bigg| \mathrm{d}x \le$$

$$\le q^+ \int_\Omega \left| \frac{|u_n(x)|}{\|u_n\|_{q(\cdot)}} \left[\frac{|u_n(x)|^{q(x)-1}}{\|u_n\|_{q(\cdot)}^{q(x)-1}} - \frac{|u(x)|^{q(x)-1}}{\|u\|_{q(\cdot)}^{q(x)-1}} \right] \right| \mathrm{d}x +$$

$$+ q^+ \int_\Omega \left| \frac{|u(x)|^{q(x)-1}}{\|u\|_{q(\cdot)}^{q(x)-1}} \left[\frac{|u_n(x)|}{\|u_n\|_{q(\cdot)}} - \frac{|u(x)|}{\|u\|_{q(\cdot)}} \right] \right| \mathrm{d}x.$$

Denote:

$$A_n := \int_\Omega \left| \frac{|u_n(x)|}{\|u_n\|_{q(\cdot)}} \left[\frac{|u_n(x)|^{q(x)-1}}{\|u_n\|_{q(\cdot)}^{q(x)-1}} - \frac{|u(x)|^{q(x)-1}}{\|u\|_{q(\cdot)}^{q(x)-1}} \right] \right| \mathrm{d}x,$$

$$B_n := \int_\Omega \left| \frac{|u(x)|^{q(x)-1}}{\|u\|_{q(\cdot)}^{q(x)-1}} \left[\frac{|u_n(x)|}{\|u_n\|_{q(\cdot)}} - \frac{|u(x)|}{\|u\|_{q(\cdot)}} \right] \right| \mathrm{d}x.$$

Since $\frac{|u_n|}{\|u_n\|_{q(\cdot)}} \in L^{q(\cdot)}(\Omega)$, and $\frac{|u_n|^{q(\cdot)-1}}{\|u_n\|_{q(\cdot)}^{q(\cdot)-1}} - \frac{|u|^{q(\cdot)-1}}{\|u\|_{q(\cdot)}^{q(\cdot)-1}} \in L^{q'(\cdot)}(\Omega)$, by using Hölder's inequality, we obtain

$$A_n \le M \left\| \frac{|u_n|^{q(\cdot)-1}}{\|u_n\|_{q(\cdot)}^{q(\cdot)-1}} - \frac{|u|^{q(\cdot)-1}}{\|u\|_{q(\cdot)}^{q(\cdot)-1}} \right\|_{q'(\cdot)},$$

where $M := \frac{1}{q^-} + \frac{1}{(q')^-}$.

But

$$\left\| \frac{|u_n|}{\|u_n\|_{q(\cdot)}} - \frac{|u|}{\|u\|_{q(\cdot)}} \right\|_{q(\cdot)} \to 0 \text{ as } n \to \infty.$$

By applying Proposition 2 with $g(x,u) = |u|^{q(x)-1}$, it follows that

$$\left\| \frac{|u_n|^{q(\cdot)-1}}{\|u_n\|_{q(\cdot)}^{q(\cdot)-1}} - \frac{|u|^{q(\cdot)-1}}{\|u\|_{q(\cdot)}^{q(\cdot)-1}} \right\|_{q'(\cdot)} \to 0 \text{ as } n \to \infty,$$

therefore $A_n \to 0$ as $n \to \infty$.

In a similar manner, since $\frac{|u_n|}{\|u_n\|_{q(\cdot)}} - \frac{|u|}{\|u\|_{q(\cdot)}} \in L^{q(\cdot)}(\Omega)$, $\frac{|u|^{q(\cdot)-1}}{\|u\|_{q(\cdot)}^{q(\cdot)-1}} \in L^{q'(\cdot)}(\Omega)$, we derive that

$$B_n \le M \left\| \frac{|u_n|}{\|u_n\|_{q(\cdot)}} - \frac{|u|}{\|u\|_{q(\cdot)}} \right\|_{q(\cdot)} \left\| \frac{|u|^{q(\cdot)-1}}{\|u\|_{q(\cdot)}^{q(\cdot)-1}} \right\|_{q'(\cdot)} =$$

$$= M \left\| \frac{|u_n|}{\|u_n\|_{q(\cdot)}} - \frac{|u|}{\|u\|_{q(\cdot)}} \right\|_{q(\cdot)}, \tag{7}$$

because

$$\rho_{q'(\cdot)} \left(\frac{|u|^{q(\cdot)-1}}{\|u\|_{q(\cdot)}^{q(\cdot)-1}} \right) = \rho_{q(\cdot)} \left(\frac{|u|}{\|u\|_{q(\cdot)}} \right) = 1,$$

therefore

$$\left\| \frac{|u|^{q(\cdot)-1}}{\|u\|_{q(\cdot)}^{q(\cdot)-1}} \right\|_{q'(\cdot)} = 1.$$

It follows from (7) that $B_n \to 0$ as $n \to \infty$.

Consequently

$$\omega(u_n) \to \omega(u) \text{ as } n \to \infty.$$

Now, we will prove the continuity of ϕ.

Fix $u \in L^{q(\cdot)}(\Omega) \setminus \{0\}$ and let $(u_n)_n \subset L^{q(\cdot)}(\Omega) \setminus \{0\}$ be such that $u_n \to u$ as $n \to \infty$ in the space $\left(L^{q(\cdot)}(\Omega),\ \|\cdot\|_{q(\cdot)} \right)$.

We now show that

$$\phi(u_n) \to \phi(u) \text{ in } \left(L^{q(\cdot)}(\Omega) \right)^* \text{ as } n \to \infty. \tag{8}$$

We have

$$|\langle \phi(u_n) - \phi(u), h \rangle| \leq$$

$$\leq q^+ \int_\Omega \left| \frac{|u_n(x)|^{q(x)-1}\, sgn\ u_n(x)}{\|u_n\|_{q(\cdot)}^{q(x)-1}} - \frac{|u(x)|^{q(x)-1}\, sgn\ u(x)}{\|u\|_{q(\cdot)}^{q(x)-1}} \right| |h(x)|\, \mathrm{d}x$$

Clearly, $\frac{|u_n|^{q(\cdot)-1}\, sgn\ u_n(\cdot)}{\|u_n\|_{q(\cdot)}^{q(\cdot)-1}} - \frac{|u|^{q(\cdot)-1}\, sgn\ u(\cdot)}{\|u\|_{q(\cdot)}^{q(\cdot)-1}} \in L^{q'(\cdot)}(\Omega)$. But $h \in L^{q(\cdot)}(\Omega)$. Therefore, taking (5) into account we obtain

$$|\langle \varphi(u_n) - \varphi(u), h \rangle| \leq$$

$$\leq q^+ M \left\| \frac{|u_n(x)|^{q(x)-1}\, sgn\ u_n(x)}{\|u_n\|_{q(\cdot)}^{q(x)-1}} - \frac{|u(x)|^{q(x)-1}\, sgn\ u(x)}{\|u\|_{q(\cdot)}^{q(x)-1}} \right\|_{q'(\cdot)} \|h\|_{q(\cdot)}.$$

Consequently,

$$\|\varphi(u_n) - \varphi(u)\| \leq q^+ M \left\| \frac{|u_n(x)|^{q(x)-1}\, sgn\ u_n(x)}{\|u_n\|_{q(\cdot)}^{q(x)-1}} - \frac{|u(x)|^{q(x)-1}\, sgn\ u(x)}{\|u\|_{q(\cdot)}^{q(x)-1}} \right\|_{q'(\cdot)}.$$

By applying Proposition 2 with

$$g(x,u) = \begin{cases} |u|^{q(x)-2}\, u & \text{, if } u \neq 0 \\ 0 & \text{, if } u = 0 \end{cases},$$

it follows that

$$\left\| \frac{|u_n|^{q(\cdot)-1}}{\|u_n\|_{q(\cdot)}^{q(\cdot)-1}} - \frac{|u|^{q(\cdot)-1}}{\|u\|_{q(\cdot)}^{q(\cdot)-1}} \right\|_{q'(\cdot)} \to 0 \text{ as } n \to \infty,$$

therefore (8) holds. ■

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a *gauge function*, i.e. φ is continuous, strictly increasing, $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$.

By *duality mapping corresponding to the gauge function* φ we understand the multivalued mapping $J_\varphi : X \to \mathcal{P}(X^*)$, defined as follows:

$$J_\varphi 0 := \{0\},$$

$$J_\varphi x := \varphi(\|x\|) \{u^* \in X^* \mid \|u^*\| = 1, \langle u^*, x\rangle = \|x\|\}, \text{ if } x \neq 0.$$

According to the Hahn-Banach theorem, it is easy to see that the domain of J_φ is the whole space:

$$D(J_\varphi) := \{x \in X \mid J_\varphi x \neq \varnothing\} = X.$$

Due to Asplund's result ([1]),

$$J_\varphi = \partial\Phi,\ \Phi(x) = \int_0^{\|x\|} \varphi(t)dt, \tag{9}$$

for any $x \in X$. $\partial\Phi$ stands for the subdifferential of Φ in the sense of convex analysis.

By the preceding definition, it follows that J_φ is single valued if and only if X is smooth. Since, at any $x \neq 0$, the gradient of the norm satisfies

$$\left\| \|\cdot\|'(x) \right\| = 1$$

$$\left\langle \|\cdot\|'(x), x \right\rangle = \|x\|,$$

and it is the unique element in the dual space having these properties, we immediately derive that: if X is a smooth real Banach space, then the duality mapping corresponding to a gauge function φ is the single valued mapping $J_\varphi : X \to X^*$ defined by:

$$J_\varphi 0 = 0$$

$$J_\varphi x = \varphi(\|x\|)\, \|\cdot\|'(x), \text{ if } x \neq 0. \tag{10}$$

Remark 5 *By coupling (10) with Asplund's result quoted in above, we get: if X is smooth, then*

$$J_\varphi x = \Phi'(x) = \begin{cases} 0, & \text{if } x = 0, \\ \varphi(\|x\|)\, \|\cdot\|'(x), & \text{if } x \neq 0, \end{cases} \tag{11}$$

Φ being given by (9).

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3 The main result

The following theorem represents the main result of this paper.

Theorem 6 *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a smooth bounded domain. Suppose that $p \in \mathcal{C}\left(\overline{\Omega}\right)$ and $p(x) > 1$, for any $x \in \overline{\Omega}$. Also let $q \in \mathcal{C}\left(\overline{\Omega}\right) \cap L_+^\infty(\Omega)$ be such that $1 < q^-$, and satisfying $q(x) < p^*(x)$, $x \in \overline{\Omega}$, where is given by (2). Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that $u \to g(\cdot, u)$ is a strictly increasing odd function with $\lim_{t \to +\infty} g(x,t) = +\infty$, which satisfies the growth condition (3). Then, for any $\alpha > 0$ there exist $u = u_\alpha \neq 0$ and $\lambda = \lambda_\alpha$ such that (1) holds.*

The basic result we need for proving Theorem 6 is the following classical Lagrange multiplier rule (see, for example, [11, 292], [4]):

Theorem 7 *Let X be a real Banach space. Let $\mathcal{F}$ and Ψ be real $\mathcal{C}^1$-functionals on X. If u minimizes $\mathcal{F}$ under the constraint $\Psi(v) = 0$ and if $\Psi'(u) \neq 0$, then there exist $\lambda \in \mathbb{R}$ such that*

$$\mathcal{F}'(u) = \lambda \Psi'(u).$$

Now we are ready for the

Proof of Theorem 6. The idea is as follows: the hypotheses of Theorem 6 entail the fulfillment of those of Theorem 7.

We set $X := W_0^{1,p(\cdot)}(\Omega)$, $\mathcal{F} : W_0^{1,p(\cdot)}(\Omega) \to \mathbb{R}$,

$$\mathcal{F}(u) := \Phi(u) + \mathcal{G}(u),$$

with

$$\Phi(u) := \int_0^{\|u\|_{1,p(\cdot)}} \varphi(s)\, \mathrm{d}s,$$

and

$$\mathcal{G}(u) := \int_\Omega G(x, u(x))\, \mathrm{d}x,$$

$$G(x,t) := \int_0^t g(x,s)\, \mathrm{d}s.$$

Remark that the oddness of the function $u \to g(\cdot, u)$ means

$$G(x,t) = \int_0^{|t|} g(x,s)\, \mathrm{d}s$$

and $g(x,0) = 0$. Being a strictly increasing function, it follows that $g(x,s) \geq 0$ for $s > 0$, therefore $G(x,t) > 0$ for $t > 0$. Consequently $\mathcal{G}(u) \geq 0$ for any $u \in W_0^{1,p(\cdot)}(\Omega)$.

Also we set $\Psi : W_0^{1,p(\cdot)}(\Omega) \to \mathbb{R}$,

$$\Psi(u) := \int_0^{\|u\|_{q(\cdot)}} \psi(s)\, \mathrm{d}s.$$

First, according to Remark 5 and to Theorem 4, 2), Φ is $\mathcal{C}^1$ on $W_0^{1,p(\cdot)}(\Omega)$ and $\Phi'(u) = J_\varphi u$, where $J_\varphi 0 = 0$ and, at any nonzero $u \in W_0^{1,p(\cdot)}(\Omega)$,

$$\langle J_\varphi u, h\rangle = \varphi\left(\|u\|_{1,p(\cdot)}\right) \cdot \left\langle \|\cdot\|'_{1,p(\cdot)}(u), h\right\rangle =$$

$$= \frac{\varphi\left(\|u\|_{1.p(\cdot)}\right) \cdot \int_{\Omega\setminus\Omega_{0,u}} p(x) \dfrac{|\nabla u|^{p(x)-2}\ \nabla u \cdot \nabla h}{\|u\|_{1,p(\cdot)}^{p(x)-1}} \mathrm{d}x}{\int_\Omega p(x) \dfrac{|\nabla u|^{p(x)}}{\|u\|_{1,p(\cdot)}^{p(x)}} \mathrm{d}x}, \text{ for any } h \in W_0^{1,p(\cdot)}(\Omega).$$

Secondly, we will prove that $\mathcal{G}$ is $\mathcal{C}^1$ on $W_0^{1,p(\cdot)}(\Omega)$ and

$$\langle \mathcal{G}'(u), h\rangle = \int_\Omega g(x, u(x))\, h(x)\mathrm{d}x,\ u,\ h \in W_0^{1,p(\cdot)}(\Omega). \tag{12}$$

Indeed, let u, $h \in W_0^{1,p(\cdot)}(\Omega)$. One has

$$|\mathcal{G}(u+h) - \mathcal{G}(u) - \langle \mathcal{G}'(u), h\rangle| =$$

$$= \left|\int_\Omega \left[G(x, u(x)+h(x)) - G(x, u(x)) - g(x,u(x))h(x)\right] dx\right| =$$

$$= \left|\int_\Omega \left[g(x, u(x)+\theta_h(x)\cdot h(x))\, h(x) - g(x,u(x))h(x)\right] dx\right| \le$$

$$\le \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) \|g(x, u(x)+\theta_h(x)\cdot h(x)) - g(x,u(x))\|_{p'(\cdot)} \|h\|_{p(\cdot)},$$

where $0 \le \theta_h(x) \le 1$ ([9, Lemma 18.1]) and Hölder's type inequality (5) was used.

Consequently,

$$\frac{|\mathcal{G}(u+h) - \mathcal{G}(u) - \langle \mathcal{G}'(u), h\rangle|}{\|h\|_{p(\cdot)}} \le$$

$$\le \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) \|g(x, u(x)+\theta_h(x)\cdot h(x)) - g(x,u(x))\|_{p'(\cdot)}.$$

Suppose $\|h\|_{1,p(\cdot)} \to 0$. It follows that $\|h\|_{1,p(\cdot)} \to 0$. Taking into account the continuity of Nemytskij operators (see Proposition 2), it follows that $\mathcal{G}$ is Fréchet differentiable on $W_0^{1,p(\cdot)}(\Omega)$ and $\mathcal{G}'$ is given by (12).

Thirdly, taking into account Remark 5 and to Theorem 4, 1), it follows that Ψ is $\mathcal{C}^1$ on $L^{q(\cdot)}(\Omega)$ and $\Psi'(u) = \underline{J}_\psi u$, where $\underline{J}_\psi 0 = 0$ and, at any nonzero $u \in W_0^{1,p(\cdot)}(\Omega)$,

$$\left\langle \underline{J}_\psi u, h\right\rangle = \psi\left(\|u\|_{q(\cdot)}\right) \left\langle \|\cdot\|'_{q(\cdot)}(u), h\right\rangle =$$

$$= \frac{\psi\left(\|u\|_{q(\cdot)}\right)\int_\Omega q(x)\frac{|u(x)|^{q(x)-1}\, sgn\; u(x)}{\|u\|_{q(\cdot)}^{q(x)-1}}h(x)\mathrm{d}x}{\int_\Omega q(x)\frac{|u(x)|^{q(x)}}{\|u\|_{q(\cdot)}^{q(x)}}\mathrm{d}x}, \text{ for any } h \in L^{q(\cdot)}(\Omega)\,. \quad (13)$$

Theorem 3, c) ensures that Ψ is $\mathcal{C}^1$ on $W_0^{1,p(\cdot)}(\Omega)$ and Ψ' is given by (13). So $\mathcal{F}$ and Ψ are $\mathcal{C}^1$ on $W_0^{1,p(\cdot)}(\Omega)$.

Now, for $\alpha > 0$ denote

$$M_\alpha = \left\{u \in W_0^{1,p(\cdot)}(\Omega) \mid \Psi\left(\|u\|_{q(\cdot)}\right) = \alpha\right\}.$$

Put

$$C_\alpha := \inf_{u\in M_\alpha} \mathcal{F}(u)\,.$$

We will show the existence of a minimizer for $\mathcal{F}$ from M_α. Remark that $\mathcal{F}(u) \geq 0$ for all $u \in W_0^{1,p(\cdot)}(\Omega)$.

Let $(u_n)_n \subset M_\alpha$ be a minimizing sequence:

$$\Psi\left(\|u_n\|_{q(\cdot)}\right) = \alpha,$$

$$\lim_{n\to\infty} \mathcal{F}(u_n) = C_\alpha.$$

Then $(\mathcal{F}(u_n))_n$ is bounded. Since $\mathcal{F}(u) \geq 0$ and $\mathcal{G}(u) \geq 0$ for all $u \in W_0^{1,p(\cdot)}(\Omega)$, it follows that the sequence $(\Phi(u_n))_n$ is bounded, therefore the sequence $(u_n)_n$ is bounded in the reflexive Banach space $W_0^{1,p(\cdot)}(\Omega)$. So there exists a subsequence, again denoted by $(u_n)_n$ for convenience, that converges weakly in $W_0^{1,p(\cdot)}(\Omega)$, to, say, u. Since $W_0^{1,p(\cdot)}(\Omega)$ is compactly embedded in $L^{q(\cdot)}(\Omega)$, it follows that $(u_n)_n$ strongly converges to u in $L^{q(\cdot)}(\Omega)$, therefore $\Psi(u) = \alpha > 0$ (that is $u \in M_\alpha$) and so $u \neq 0$.

Because $u \in M_\alpha$, we infer that

$$C_\alpha \leq \mathcal{F}(u)\,. \quad (14)$$

On the other hand, the functional Φ is convex and continuous, therefore is weakly lower semicontinuous. Consequently

$$\Phi(u) \leq \liminf_{n\to\infty} \Phi(u_n)\,.$$

But $u_n \to u$ as $n \to \infty$ in $L^{q(\cdot)}(\Omega)$, therefore, passing to a subsequence also denoted $(u_n)_n$, one has

$$u_n(x) \to u(x) \text{ as } n \to \infty \text{ for a.e. } x \in \Omega.$$

Consequently

$$G(x, u_n(x)) \to G(x, u(x)) \text{ as } n \to \infty \text{ for a.e. } x \in \Omega.$$

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From Fatou's Lemma we derive that

$$\mathcal{G}(u) = \int_\Omega G(x, u(x))\,\mathrm{d}x \leq \liminf_{n\to\infty} \int_\Omega G(x, u_n(x))\,\mathrm{d}x = \liminf_{n\to\infty} \mathcal{G}(u_n),$$

therefore

$$\mathcal{F}(u) = \Phi(u) + \mathcal{G}(u) \leq \liminf_{n\to\infty} \Phi(u_n) + \liminf_{n\to\infty} \mathcal{G}(u_n)$$
$$= \liminf_{n\to\infty} (\Phi(u_n) + \mathcal{G}(u_n)) = \lim_{n\to\infty} \mathcal{F}(u_n) = C_\alpha. \quad (15)$$

We then conclude from (14) and (15) that $C_\alpha = \mathcal{F}(u)$. Since $\langle \Psi'(u), u\rangle = \psi\left(\|u\|_{q(\cdot)}\right)\|u\|_{q(\cdot)} \neq 0$, Theorem 7 applies.

It follows that there exists $u \in M_\alpha$ and $\lambda = \lambda(\alpha)$ such that (1) holds and the proof is complete. ■

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