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SCHUR FACTORIZATIONS FOR SOME CLASSES OF INFINITE MATRICES

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ABSTRACT. In this paper we obtain some Schur factorizations for some special classes of infinite matrices.

1. INTRODUCTION

First let us recall the definition of Schur product. If $A = (a_{jk})_{j,k\geq 1}$ and $B = (b_{jk})_{j,k\geq 1}$ are matrices of the same size (finite or infinite) their Schur product (or Hadamard product) is defined to be the matrix of elementwise products

$$A * B = (a_{jk}b_{jk})_{j,k\geq 1}.$$

There is, however, much justification for the term "Schur product" and we refer the reader to [3] and [7] for an historical discussion.

If X and Y are two spaces of infinite matrices and A is an infinite matrix such that

$$A * B \in Y \text{ for all } B \in X,$$

then A is called *Schur multiplier from X into Y*.

For an infinite matrix $A = (a_{ij})_{i,j\geq 1}$ and an integer k we denote by $A_k = (a'_{ij})_{i,j\geq 1}$, the kth-*diagonal matrix associated to A* (see e.g. [2]), where

$$a'_{ij} = \begin{cases} a_{ij} & \text{if } j - i = k \\ 0 & \text{otherwise,} \end{cases}$$

i.e. we have that

$$A_k = \begin{pmatrix} 0 & 0 & \dots & a_{1k} & 0 & \dots & 0 & \dots \\ 0 & 0 & \dots & 0 & a_{2\,k+1} & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots & a_{k\,2k-1} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

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In this paper we deal with infinite matrices A, whose entries a_k^l, for $k \in \mathbb{Z}$ and $l \geq 1$, are indexed with respect to the kth diagonal and with the lth place on this diagonal i.e.

$$a_k^l = \begin{cases} a_{l,l+k}, & \text{for } k \geq 0 \text{ and } l = 1, 2, \dots \\ a_{l-k,l}, & \text{for } k < 0 \text{ and } l = 1, 2, \dots \end{cases},$$

$$A = \begin{pmatrix} a_0^1 & a_1^1 & a_2^1 & a_3^1 & \ddots \\ a_{-1}^1 & a_0^2 & a_1^2 & a_2^2 & \ddots \\ a_{-2}^1 & a_{-1}^2 & a_0^3 & a_1^3 & \ddots \\ a_{-3}^1 & a_{-2}^2 & a_{-1}^3 & a_0^4 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

This notation was introduced in [1] and has since then arisen in several other papers e.g. [5], [6].

Using the terminology of [1], we say that a *scalar matrix* is an infinite matrix of the following form

$$[\alpha] = \begin{pmatrix} \alpha^1 & \alpha^1 & \alpha^1 & \dots \\ \alpha^1 & \alpha^2 & \alpha^2 & \dots \\ \alpha^1 & \alpha^2 & \alpha^3 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

Roughly speaking, Schur factorization for a space of infinite matrices X means that there exist Y and Z two spaces of infinite matrices such that

$$X = Y * Z.$$

More precisely, for every $A \in X$, there exist $B \in Y$ and $C \in Z$ such that

$$A = B * C.$$

In this paper we obtain some Schur factorizations for some quasi-Banach spaces of matrices, namely $d_M^q(\mathbf{a}, p)$, $g_M^q(\mathbf{a}, p)$, $ces_M^q(p)$ and $l_M^q(p)$.

For $0 < p \leq \infty$ the space $y_M^q(\mathbf{a}, p)$ consists of those upper triangular infinite matrices $A = \sum_{k=0}^{\infty} A_k$ with all the sequences on the diagonals belonging to $y(\mathbf{a}, p)$ and such that

$$\|A\|_{y_M^q(\mathbf{a},p)} = \left(\sum_{k=0}^{\infty} \|A_k\|_{y(\mathbf{a},p)}^q \right)^{\frac{1}{q}} < \infty,$$

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where $y^q_M(\mathbf{a}, p)$ is one of the spaces of matrices $d^q_M(\mathbf{a}, p)$, $g^q_M(\mathbf{a}, p)$, $ces^q_M(p)$ or $l^q_M(p)$ and where $y(\mathbf{a}, p)$ is the corresponding space of sequences i.e. $d(\mathbf{a}, p)$, $g(\mathbf{a}, p)$, $ces(p)$ or $l(p)$.

Recently, in [6], N. Popa has introduced and studied these spaces in connection with the general description of upper triangular Schur multipliers of scalar type for different quasi-Banach spaces of matrices. In what follows we will recall some definitions from [4], which we will use in this paper.

$$l(p) = \{x = \{x_k\}_{k=1}^{\infty} \ : \ \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}} < \infty\},$$

$$d(\mathbf{a}, p) = \{x = \{x_k\}_{k=1}^{\infty} \ : \ \left(\sum_{k=1}^{\infty} a_k \sup_{n \geq k} |x_n|^p\right)^{\frac{1}{p}} < \infty\},$$

$$g(\mathbf{a}, p) = \{x = \{x_k\}_{k=1}^{\infty} \ : \ \sup_{n \geq 1} \left(\frac{1}{A_n} \sum_{k=1}^{n} |x_k|\right)^{\frac{1}{p}} < \infty\}$$

and

$$ces(p) = \{x = \{x_k\}_{k=1}^{\infty} \ : \ \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k|\right)^{p} < \infty\},$$

where $\mathbf{a} = (a_1, a_2, \dots)$ is a fixed sequence of non-negative terms and suppose that $a_1 > 0$ so that the partial sums,

$$A_n = a_1 + a_2 + \cdots + a_n$$

never vanish. We refer to [4] for further details about these spaces.

The spaces $d(\mathbf{a}, p)$, $g(\mathbf{a}, p)$ and $ces(p)$ were introduced in G. Bennett's paper [4], where it also was described how they are connected to Hardy type inequalities. The main results in this paper are presented, proved and discussed in Section 2 below. One of the main results in Section 2 is a Schur factorization result (see Theorem 2.3).

2. Main results

Our first main result in this section is the following Schur factorization theorem:

Theorem 2.1. *Let* $0 < p, q \leq \infty$ *and* $\frac{1}{s} = \frac{1}{p} + \frac{1}{q}$. *Then*

$$l^{\infty}_M(p) * l^{\infty}_M(q) = l^{\infty}_M(s). \tag{2.1}$$

More precisely, given any matrix A*, we have that*

$$\|A\|_{l^{\infty}_M(s)} = \inf\{\|B\|_{l^{\infty}_M(p)} \cdot \|C\|_{l^{\infty}_M(q)} : B * C = A\}. \tag{2.2}$$

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Proof. Suppose that A admits a Schur factorization

$$A = B * C,$$

with $B \in l^{\infty}_{M}(p)$ and $C \in l^{\infty}_{M}(q)$. Then using Hölder's inequality, we obtain

$$\sup_k \|A_k\|_{l(s)} = \sup_k \|B_k * C_k\|_{l(s)} \leq \sup_k \|B_k\|_{l(p)} \cdot \sup_k \|C_k\|_{l(q)}.$$

We have thus shown that

$$l^{\infty}_{M}(p) * l^{\infty}_{M}(q) \subseteq l^{\infty}_{M}(s),$$

and that

$$\|A\|_{l^{\infty}_{M}(s)} \leq \inf\{\|B\|_{l^{\infty}_{M}(p)} \cdot \|C\|_{l^{\infty}_{M}(q)}\}.$$

To establish the converse, we assume that $A \in l^{\infty}_{M}(s)$. It follows that there exist $B = ((a^l_k)^{\frac{s}{p}})_{l\geq 1, k\geq 0}$ and $C = ((a^l_k)^{\frac{s}{q}})_{l\geq 1, k\geq 0}$ such that

$$\|A_k\|_{l(s)} = \|B_k\|_{l(p)} \cdot \|C_k\|_{l(q)}, \text{ for every } k$$

and

$$A = B * C.$$

It implies that

$$\sup_k \|A_k\|_{l(s)} = \sup_k \|B_k\|_{l(p)} \cdot \sup_k \|C_k\|_{l(q)}.$$

Therefore

$$l^{\infty}_{M}(s) \subseteq l^{\infty}_{M}(p) * l^{\infty}_{M}(q)$$

and

$$\inf\{\|B\|_{l^{\infty}_{M}(p)} \cdot \|C\|_{l^{\infty}_{M}(q)}\} \leq \|A\|_{l^{\infty}_{M}(s)}.$$

The proof is complete. □

Remark 2.2. Our proof shows that the infimum in Theorem 2.1 is actually attained, so that we may replace the "inf" in (2.2) by "min".

The second main result is the following product formulae which is very important when we come to study Schur multipliers for infinite matrices.

Theorem 2.3. *If* $0 < p, q \leq \infty$ *and* $\frac{1}{s} = \frac{1}{p} + \frac{1}{q}$. *Then*

$$d^{\infty}_{M}(\boldsymbol{a}, p) * d^{\infty}_{M}(\boldsymbol{a}, q) = d^{\infty}_{M}(\boldsymbol{a}, s) \tag{2.3}$$

and

$$g^{\infty}_{M}(\boldsymbol{a}, p) * g^{\infty}_{M}(\boldsymbol{a}, q) = g^{\infty}_{M}(\boldsymbol{a}, s). \tag{2.4}$$

The proofs are similar to that of the previous theorem and are omitted.

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Theorem 2.4. *If* $0 < p \leq \infty$ *and* $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_3}$, $0 < q_1, q_2 < \infty$ *then*

$$l_M^{q_1}(p) * l_M^{q_2}(p^*) \subseteq l_M^{q_3}(1). \tag{2.5}$$

More precisely, given any matrix A*, we have that*

$$\|A\|_{l_M^{q_3}(1)} \leq \inf\{\|B\|_{l_M^{q_1}(p)} \cdot \|C\|_{l_M^{q_2}(p^*)} : B * C = A\}. \tag{2.6}$$

Proof. If $A = B * C$ with $B \in l_M^{q_1}(p)$ and $C \in l_M^{q_2}(p^*)$ then applying Hölder's inequality we obtain

$$\begin{aligned}\left(\sum_k \|A_k\|_{l(1)}^{q_3}\right)^{\frac{1}{q_3}} &\leq \left(\sum_k \|B_k\|_{l(p)}^{q_3} \cdot \|C_k\|_{l(p^*)}^{q_3}\right)^{\frac{1}{q_3}} \\ &\leq \left(\sum_k \|B_k\|_{l(p)}^{q_1}\right)^{\frac{1}{q_1}} \cdot \left(\sum_k \|C_k\|_{l(p^*)}^{q_2}\right)^{\frac{1}{q_2}} \\ &= \|B\|_{l_M^{q_1}(p)} \cdot \|C\|_{l_M^{q_2}(p^*)}.\end{aligned}$$

It follows that

$$\|A\|_{l_M^{q_3}(1)} \leq \inf\{\|B\|_{l_M^{q_1}(p)} \cdot \|C\|_{l_M^{q_2}(p^*)} : B * C = A\}.$$

The proof is complete. □

Our next result shows in particular that matrices from $g_M^{q_2}(\mathbf{a}, p)$ are Schur multipliers from $d_M^{q_1}(\mathbf{a}, p)$ into $l_M^{q_3}(p)$, or equivalently that matrices from $d_M^{q_1}(\mathbf{a}, p)$ are Schur multipliers from $g_M^{q_2}(\mathbf{a}, p)$ into $l_M^{q_3}(p)$.

Theorem 2.5. *If* $0 < p \leq \infty$ *and* $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_3}$, $0 < q_1, q_2, q_3 < \infty$ *then*

$$d_M^{q_1}(\boldsymbol{a}, p) * g_M^{q_2}(\boldsymbol{a}, p) \subseteq l_M^{q_3}(p). \tag{2.7}$$

More precisely, given any matrix A*, we have that*

$$\|A\|_{l_M^{q_3}(p)} \leq \inf\{\|B\|_{d_M^{q_1}(\boldsymbol{a},p)} \cdot \|C\|_{g_M^{q_2}(\boldsymbol{a},p)} : B * C = A\}. \tag{2.8}$$

Proof. (The case $p = \infty$ is trivial and we do not consider it here).
Suppose, then, that A admits a Schur factorization

$$A = B * C,$$

with $B \in d_M^{q_1}(\mathbf{a}, p)$ and $C \in g_M^{q_2}(\mathbf{a}, p)$. Then A is an upper triangular matrix and applying G. Bennett's factorization technique (see e.g. [4])

and Hölder's inequality we obtain

$$\|A\|_{\ell_M^{q_3}(p)} = \left(\sum_{k=0}^{\infty} \|A_k\|_{\ell(p)}^{q_3}\right)^{\frac{1}{q_3}} = \left(\sum_{k=0}^{\infty} \|B_k * C_k\|_{\ell(p)}^{q_3}\right)^{\frac{1}{q_3}}$$
$$\leq \left(\sum_{k=0}^{\infty} \|B_k\|_{d(\mathbf{a},p)}^{q_3} \cdot \|C_k\|_{g(\mathbf{a},p)}^{q_3}\right)^{\frac{1}{q_3}}$$
$$\leq \left(\sum_{k=0}^{\infty} \|B_k\|_{d(\mathbf{a},p)}^{q_1}\right)^{\frac{1}{q_1}} \cdot \left(\sum_{k=0}^{\infty} \|C_k\|_{g(\mathbf{a},p)}^{q_2}\right)^{\frac{1}{q_2}}$$
$$= \|B\|_{d_M^{q_1}(\mathbf{a},p)} \cdot \|C\|_{g_M^{q_2}(\mathbf{a},p)}.$$

This implies that

$$\|A\|_{l_M^{q_3}(p)} \leq \inf\{\|B\|_{d_M^{q_1}(\mathbf{a},p)} \cdot \|C\|_{g_M^{q_2}(\mathbf{a},p)} : B * C = A\}.$$

The proof is complete. □

Another main result is the following:

Theorem 2.6. *Let $p > 1$ be fixed and $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_3}$. If a matrix A admits a Schur factorization*

$$A = B * C, \tag{2.9}$$

with

$$B \in l_M^{q_1}(p) \text{ and } C \in g_M^{q_2}(p^*) \tag{2.10}$$

then A belongs to $ces_M^{q_3}(p)$. Moreover, we have that

$$\|A\|_{ces_M^{q_3}(p)} \leq p^* !A!_p, \tag{2.11}$$

where $!A!_p = \inf\{\|B\|_{l_M^{q_1}(p)} \cdot \|C\|_{g_M^{q_2}(p^*)} : B * C = A\}$ and $\frac{1}{p} + \frac{1}{p^*} = 1$.

Proof. Suppose that A admits a factorization as described in (2.9), $B \in l_M^{q_1}(p)$ and $C \in g_M^{q_2}(p^*)$ with $B = (b_k^l)_{\substack{k\geq 0\\ l\geq 1}}$, $C = (c_k^l)_{\substack{k\geq 0\\ l\geq 1}}$. We observe that A is an upper triangular matrix and applying G. Bennett's factorization technique (see e.g. [4])

$$\left(\sum_{k=0}^{\infty} \|B_k * C_k\|_{ces(p)}^{q_3}\right)^{\frac{1}{q_3}} = \left(\sum_{k=0}^{\infty} \|(b_k^l \cdot c_k^l)_{l\geq 1}\|_{ces(p)}^{q_3}\right)^{\frac{1}{q_3}}$$
$$\leq p^* \left(\sum_{k=0}^{\infty} \|(b_k^l)_l\|_{l(p)}^{q_3} \cdot \|(c_k^l)_l\|_{g(p^*)}^{q_3}\right)^{\frac{1}{q_3}}.$$

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Hölder's inequality implies

$$\|A\|_{ces_M^{q_3}(p)} = \left(\sum_{k=0}^{\infty} \|B_k * C_k\|_{ces(p)}^{q_3} \right)^{\frac{1}{q_3}} \leq p^* \|B\|_{l_M^{q_1}} \cdot \|C\|_{g_M^{q_2}}.$$

Consequently

$$\|A\|_{ces_M^{q_3}(p)} \leq p^* !A!_p.$$

The proof is complete. ☐

Remark 2.7. We observe that in particular matrices from $l_M^{q_1}(p)$ are Schur multipliers from $g_M^{q_2}(p^*)$ into $ces_M^{q_3}(p)$ or equivalently that matrices from $g_M^{q_2}(p^*)$ are Schur multipliers from $l_M^{q_1}(p)$ into $ces_M^{q_3}(p)$.

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