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MULTILINEAR SINGULAR INTEGRAL OPERATORS AND THEIR COMMUTATORS ON GENERALIZED LOCAL MORREY TYPE SPACES

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Abstract. In this paper, the authors study the multilinear singular integral operators with generalized kernels, their multilinear commutators and multilinear iterated commutators generated by BMO functions. The boundedness on product of Morrey spaces, product of generalized Morrey spaces and product of generalized local Morrey spaces are obtained, respectively. Indeed, the results in this paper are extensions of some known results.

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1. Introduction

The singular integral operator theory, which plays a significant part in many respects of harmonic analysis. In the last century, the theory of the Calderón-Zygmund operator is a crucial part of accomplishment in the classical analysis, and it has been fully applied in Fourier analysis, complex analysis, operator theory and so on. The generalized Calderón-Zygmund operator originated in the classical Calderón-Zygmund operator has attracted many researchers to explore it. Lin established the sharp maximal function pointwise estimates for generalized Calderón-Zygmund operators and their commutators with BMO functions in [15]. In [18], the authors obtained not only the boundedness of the generalized Calderón-Zygmund operator but also boundedness of its commutators with weighted BMO functions on weighted Morrey spaces. Lin considered the boundedness of strongly singular Calderón-Zygmund operators and their commutators with Lipschitz functions on the classical Morrey space and the generalized Morrey space in [16]. In [2], the author established the boundedness of some sublinear operators and their commutators on generalized local Morrey spaces.

On the other hand, more and more researchers study the multilinear singular integrals as the generalizations of the theory of linear ones, natural appearance and important applications. For example, Grafakos and Torres established the multilinear Calderón-Zygmund theory in [1]. The authors in [25], proved the boundedness of commutators generated by multilinear Calderón-Zygmund operators and BMO functions on generalized Morrey spaces.

In [19], Lin and Xiao gave some useful conclusions about multilinear singular integral operators with generalized kernels. They established the sharp maximal estimates of the multilinear singular integral operators and their multilinear commutators with BMO functions. Moreover, not only did they acquire the boundedness of the multilinear commutators with BMO functions on the product of weighted Lebesgue spaces but also they drew a conclusion about the multilinear commutators on the product of variable exponent Lesbegue spaces in [19]. These results can improve the corresponding known results of classical multilinear Calderón-Zygmund operators and multilinear Calderón-Zygmund operators with Dini type kernels.

In [23], Perez, Pradolini, Torres and Trujillo-Gonzaólez first established the sharp maximal estimates about multilinear iterated commutators of multilinear Calderón-Zygmund operators and then the end-point estimates were acquired. Lin and Zhang established the sharp maximal estimates for the multilinear iterated commutators generated by BMO functions and multilinear singular integral operators with generalized kernels in [20]. And they applied it to get the boundedness of this kind of multilinear iterated commutators on the product of variable exponent Lebesgue spaces and on the product of weighted Lebesgue spaces. Other related results of multilinear singular integral operators and multilinear commutators can be seen in [14, 21, 22] and so on.

The author in [6] got the boundedness of certain multi-sublinear operators and the multilinear commutators generated by local Campanato functions and multilinear fractional operators on the product of generalized local Morrey spaces under generic size conditions which are satisfied by most of the operators in harmonic analysis.

Inspired by the above studies, we will focus on the boundedness of the multilinear singular integral operators with generalized kernels and their multilinear commutators with BMO functions on the product of classical Morrey spaces, the product of generalized Morrey spaces and the product of generalized local Morrey spaces, respectively. Here and hereafter, C's represent positive constants independent of appropriate quantities.

Before beginning our main results, we need review some necessary definitions and notations as follows.

Let $m \in \mathbb{N}_+$ and $K(y_0, y_1, \cdots, y_m)$ be a function away from the diagonal in $(\mathbb{R}^n)^{m+1}$. T stands for an m-linear singular integral operator defined by

$$T(f_1, \cdots, f_m)(x) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} K(x, y_1, \cdots, y_m) \prod_{j=1}^{m} f_j(y_j) dy_1 \cdots dy_m, \tag{1.1}$$

where $f_j (j = 1, \cdots, m)$ are smooth functions with compact support, and $x \notin \bigcap_{j=1}^{m} \text{supp} f_j$.

If the kernel K satisfies the following two conditions:

For some $C > 0$ and all $(y_0, y_1, \cdots, y_m) \in (\mathbb{R}^n)^{m+1}$ defined away from the diagonal,

$$|K(y_0, y_1, \cdots, y_m)| \leq \frac{C}{(\sum_{k,l=0}^{m} |y_k - y_l|)^{mn}}; \tag{1.2}$$

For some $\varepsilon > 0$, where $0 \leqslant j \leqslant m$ and $|y_j - y_j'| \leqslant \frac{1}{2} \max_{0 \leqslant k \leqslant m} |y_j - y_k|$,

$$|K(y_0, \cdots, y_j, \cdots, y_m) - K(y_0, \cdots, y_j', \cdots, y_m)| \leqslant \frac{C|y_j - y_j'|^{\varepsilon}}{(\sum_{k,l=0}^{m} |y_k - y_l|)^{mn+\varepsilon}}, \tag{1.3}$$

then we call T an standard m-linear Calderón-Zygmund operator.

If $K(y_0, y_1, \cdots, y_m)$ is defined away from the diagonal $y_0 = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$ and satisfies the condition (1.2) and the following condition:

$$\begin{aligned} &|K(y_0, \cdots, y_j, \cdots, y_m) - K(y_0, \cdots, y_j', \cdots, y_m)| \\ &\leq \frac{C}{(|y_0 - y_1| + \cdots + |y_0 - y_m|)^{mn}} \rho\Big(\frac{|y_j - y_j'|}{|y_0 - y_1| + \cdots + |y_0 - y_m|}\Big), \end{aligned} \tag{1.4}$$

where $|y_j - y_j'| \leq \frac{1}{2} \max_{1 \leq k \leq m} |y_0 - y_k|$ whenever $0 \leq j \leq m$ and $\rho(t)$ is a non-negative and non-decreasing function on $\mathbb{R}^+$, then we call it an m-linear Calderón-Zygmund kernel of type ρ. It is obvious that condition (1.4) implies condition (1.3) while $\rho(t) = t^{\varepsilon}$ for some $\varepsilon > 0$. So the special case of m-linear Calderón-Zygmund kernel of type ρ is the standard m-linear Calderón-Zygmund kernel.

If the operator T defined by (1.1) with the kernel K satisfying the condition (1.2) and a weaker condition: Whenever $i = 1, \dots, m$, C_{k_i} are positive constants for any $(k_1, \cdots, k_m) \in \mathbb{N}_+$,

$$
\begin{aligned}
&\Big(\int_{2^{k_m}|y_0-y_0'|\le|y_m-y_0|<2^{k_m+1}|y_0-y_0'|} \cdots \int_{2^{k_1}|y_0-y_0'|\le|y_1-y_0|<2^{k_1+1}|y_0-y_0'|} \\
&|K(y_0, y_1, \cdots, y_m) - K(y_0', y_1, \cdots, y_m)|^s dy_1 \cdots dy_m\Big)^{\frac{1}{s}} \qquad (1.5)\\
&\le C|y_0 - y_0'|^{-\frac{mn}{s'}} \prod_{i=1}^{m} C_{k_i} 2^{-\frac{nk_i}{s'}},
\end{aligned}
$$

where (s, s') is a fixed pair of positive numbers satisfying $\frac{1}{s} + \frac{1}{s'} = 1$ and $1 < s < \infty$, then we call T an m-linear singular integral operator with generalized kernel.

It is easy to see that the condition (1.3) implies the condition (1.5) with $C_{k_i} = 2^{-\frac{k_i \varepsilon}{m}}, i = 1, \dots, m$, for any $1 < s < \infty$. And the condition (1.4) implies the condition (1.5) by putting $C_{k_i} = \rho(2^{-k_i})^{\frac{1}{m}}, i = 1, \dots, m$, for any $1 < s < \infty$. Therefore, the m-linear singular integral operator with generalized kernel can generalize the m-linear Calderón-Zygmund operator of type ρ and the standard m-linear Calderón-Zygmund operator. This fact illustrates that our results obtained in this paper will improve most of the earlier conclusions by weekening the conditions of the kernel.

Definition 1.1. A function $f \in L^p_{loc}(\mathbb{R}^n)$ belongs to the classical Morrey space $M^q_p(\mathbb{R}^n)$, $1 \le p \le q < \infty$, if

$$\|f\|_{M^q_p(\mathbb{R}^n)} = \sup_{B \subset \mathbb{R}^n} |B|^{\frac{1}{q}-\frac{1}{p}} \left(\int_B |f(x)|^p \mathrm{d}x\right)^{\frac{1}{p}} < \infty.$$

Here, we would like to mention that in many research papers, such as in [3, 4, 5, 7], the Morrey space is defined in another way.

Recall that the generalized Morrey and local Morrey spaces have been defined by Gürbüz in [2, 3, 5, 6, 7, 10, 11].

Definition 1.2. For a general positive function φ on $\mathbb{R}^n \times \mathbb{R}^+$, the generalized Morrey space $L^{p,\varphi}$ with $1 \le p < \infty$ is defined as follows:

$$L^{p,\varphi} = \left\{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n), \|f\|_{L^{p,\varphi}} < +\infty\right\},$$

where

$$\|f\|_{L^{p,\varphi}} = \sup_{x \in \mathbb{R}^n, r>0} \left(\frac{1}{\varphi(x,r)} \int_{B(x,r)} |f(y)|^p \mathrm{d}y\right)^{1/p}.$$

Definition 1.3. Let $1 \le p < \infty$, $\varphi(x, r)$ be a positive measurable function on $\mathbb{R}^n \times \mathbb{R}^+$. Then, for any fixed $x_0 \in \mathbb{R}^n$, the generalized local Morrey space $\mathrm{LM}^{\{x_0\}}_{p,\varphi}$ is defined by

$$
\begin{aligned}
&\mathrm{LM}^{\{x_0\}}_{p,\varphi} \equiv \mathrm{LM}^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n) \\
&= \Big\{f \in L^{\mathrm{loc}}_p(\mathbb{R}^n) : \|f\|_{\mathrm{LM}^{\{x_0\}}_{p,\varphi}} = \sup_{r>0} \varphi(x_0, r)^{-1} |B(x_0, r)|^{-\frac{1}{p}} \|f\|_{L^p(B(x_0,r))} < \infty\Big\}.
\end{aligned}
$$

Definition 1.4. For any $1 \le p < \infty$, the function in the BMO space can be described by means of the condition

$$\|f\|_{BMO} = \sup_{B \subset \mathbb{R}^n} \left(\frac{1}{|B|} \int_B |f(x) - f_B|^p \mathrm{d}x\right)^{\frac{1}{p}} < \infty,$$

where B denotes an arbitrary ball on $\mathbb{R}^n$, $f \in L^p_{loc}(\mathbb{R}^n)$, and $f_B = \frac{1}{|B|} \int_B f(y)\mathrm{d}y$.

Remark 1.5. Functions of BMO were introduced by John and Nirenberg in [13]. For more properties of the BMO space, one can read the references [7, 12, 13] and so on.

Definition 1.6. Let T be an m-linear operator defined by (1.1). Given a collection of locally integrable functions $\vec{b} = (b_1, \cdots, b_m)$, the m-linear commutator with $\vec{b}$ is defined by

$$T_{\vec{b}}(f_1, \cdots, f_m) = \sum_{j=1}^{\infty} T_{b_j}^{j}(\vec{f}),$$

where

$$T_{b_j}^{j}(\vec{f})(x) = b_j(x)T(f_1, \cdots, f_m)(x) - T(f_1, \cdots, f_{j-1}, b_j f_j, f_{j+1}, \cdots, f_m)(x).$$

The notation $\vec{b} \in BMO^m$ will stand for $b_j \in BMO(\mathbb{R}^n)$ for $j = 1, \ldots, m$, and we use the notation

$$\|\vec{b}\|_{BMO^m} = \max_{1 \le j \le m} \|b_j\|_{BMO(\mathbb{R}^n)}.$$

Definition 1.7. Let T be an m-linear operator, $\vec{b} = (b_1, \cdots, b_m)$ is a group of locally integrable functions and $\vec{f} = (f_1, \cdots, f_m)$. Then the m-linear iterated commutator generated by T and $\vec{b}$ is defined to be

$$T_{\Pi\vec{b}}(f_1, \cdots, f_m) = [b_1, [b_2, \cdots [b_{m-1}, [b_m, T]_m]_{m-1} \cdots]_2]_1(\vec{f}).$$

If K is the kernel of T, then we can write

$$T_{\Pi\vec{b}}(f_1, \cdots, f_m)(x) = \int_{(\mathbb{R}^n)^m} \prod_{j=1}^{m} (b_j(x) - b_j(y_j))K(x, y_1, \cdots, y_m)f_1(y_1) \cdots f_m(y_m)dy_1 \cdots dy_m.$$

In order to illustrate the relationship of the Lebesgue space, the classical Morrey space and the generalized Morrey space, we give the following remarks.

Remark 1.8. Obviously, the Morrey space is the generalization of the Lebesgue space that can be seen from the special case $M_p^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ with $1 \le p < \infty$.

Remark 1.9. If $\varphi(x, r) = r^{n(1-p/q)}$, we have $L^{p,\varphi} = M_p^q(\mathbb{R}^n), 1 \le p \le q < \infty$. So it is obvious that the generalized Morrey space is the generalization of the classical Morrey space.

2. Necessary Lemmas

In this paper, we need some necessary lemmas as follows.

Lemma 2.1. [19] *Let $m \ge 2$, T be an m-linear singular integral operator with generalized kernel and $\sum_{k_i=1}^{\infty} C_{k_i} < \infty$, $i = 1, \cdots, m$. Suppose for fixed $1 \le r_1, \cdots, r_m \le s'$ with $1/r = 1/r_1 + \cdots + 1/r_m$, T is bounded from $L^{r_1} \times \cdots \times L^{r_m}$ into $L^{r,\infty}$. Then for any $s' < p_1, \cdots, p_m < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, T is bounded from $L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m)$ into $L^p(\omega)$, where $(\omega_1, \cdots, \omega_m) \in (A_{p_1/s'}, \cdots, A_{p_m/s'})$ and $\omega = \prod_{j=1}^{m} \omega_j^{p/p_j}$.*

Lemma 2.2. [19] *Let $m \ge 2$, T be an m-linear singular integral operator with generalized kernel and $\sum_{k_i=1}^{\infty} k_i C_{k_i} < \infty$, $i = 1, \cdots, m$. Suppose for fixed $1 \le r_1, \cdots, r_m \le s'$ with $1/r = 1/r_1 + \cdots + 1/r_m$, T is bounded from $L^{r_1} \times \cdots \times L^{r_m}$ into $L^{r,\infty}$. If $\vec{b} \in BMO^m$, then for any $s' < p_1, \cdots, p_m < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $T_{\vec{b}}$ is bounded from $L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m)$ into $L^p(\omega)$, where $(\omega_1, \cdots, \omega_m) \in (A_{p_1/s'}, \cdots, A_{p_m/s'})$ and $\omega = \prod_{j=1}^{m} \omega_j^{p/p_j}$.*

Lemma 2.3. [17] *Let f be a function in* BMO. *Suppose $1 \le p < \infty$, $x \in \mathbb{R}^n$, and $r_1, r_2 > 0$. Then*

$$\left(\frac{1}{|B(x, r_1)|} \int_{B(x, r_1)} |f(y) - f_{B(x, r_2)}|^p dy \right)^{1/p} \le C \left(1 + \left| \ln \frac{r_1}{r_2} \right| \right) \|f\|_{BMO},$$

where $C > 0$ is a positive constant independent of f, x, r_1 and r_2.

Lemma 2.4. [20] *Let $m \geq 2$, T be an m-linear singular integral operator with generalized kernel and $\sum_{k_i=1}^{\infty} k_i C_{k_i} < \infty$, $i = 1, \cdots, m$. Suppose for fixed $1 \leq r_1, \cdots, r_m \leq s'$ with $1/r = 1/r_1 + \cdots + 1/r_m$, T is bounded from $L^{r_1} \times \cdots \times L^{r_m}$ into $L^{r,\infty}$. If $\vec{b} \in BMO^m$, then for any $s' < p_1, \cdots, p_m < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $T_{\Pi\vec{b}}$ is bounded from $L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m)$ into $L^p(\omega)$, where $(\omega_1, \cdots, \omega_m) \in (A_{p_1/s'}, \cdots, A_{p_m/s'})$ and $\omega = \prod_{j=1}^m \omega_j^{p/p_j}$.*

Lemma 2.5. [24] *If f is any real-valued nonnegative function and measurable on E, then*

$$\left(\operatorname*{ess\,inf}_{x\in E} f(x)\right)^{-1} = \operatorname*{ess\,sup}_{x\in E} \frac{1}{f(x)}.$$

3. Main Results

In this section, we will state the main results. And inspired by the proof technique about generalized Morrey type spaces in [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12], we will prove the theorems as follows.

Theorem 3.1. *Let $m \geq 2$, T be an m-linear singular integral operator with generalized kernel and $\sum_{k_i=1}^{\infty} C_{k_i} < \infty$, $i = 1, \cdots, m$. Suppose for fixed $1 \leq r_1, \cdots, r_m \leq s'$ with $1/r = 1/r_1 + \cdots + 1/r_m$, T is bounded from $L^{r_1} \times \cdots \times L^{r_m}$ into $L^{r,\infty}$. Let φ, φ_j be a positive function on $\mathbb{R}^n \times \mathbb{R}^+$ and suppose there exist $0 < D_j < 2^n$ such that $\varphi_j(x, 2r) \leq D_j \varphi_j(x, r)$ $(j = 1, \cdots, m)$ with $\varphi(x, r)^{\frac{1}{p}} = \prod_{j=1}^m \varphi_j(x, r)^{\frac{1}{p_j}}$. Then for any $s' < p_1, \cdots, p_m < \infty$ with $1/p = \sum_{i=1}^m 1/p_i$ and $1 < p < \infty$, T is bounded from $L^{p_1,\varphi_1}(\mathbb{R}^n) \times \cdots \times L^{p_m,\varphi_m}(\mathbb{R}^n)$ into $L^{p,\varphi}(\mathbb{R}^n)$.*

Proof. For the sake of simplicity, we will only consider the situation $m = 2$. Actually, the proof of other cases is similar.

Let f_1, f_2 be bounded measurable functions with compact support, then for any ball $B = B(x_0, r)$ centered at x_0 with radius $r > 0$, we decompose f_1 and f_2 as follows:

$$f_1 = f_1\chi_{2B} + f_1\chi_{(2B)^c} := f_1^1 + f_1^2, \quad f_2 = f_2\chi_{2B} + f_2\chi_{(2B)^c} := f_2^1 + f_2^2.$$

Thus, we have

$$\begin{aligned}
&\left(\int_{B(x_0,r)} |T(f_1, f_2)(x)|^p dx\right)^{\frac{1}{p}} \\
&\leq \left(\int_{B(x_0,r)} |T(f_1^1, f_2^1)(x)|^p dx\right)^{\frac{1}{p}} + \left(\int_{B(x_0,r)} |T(f_1^1, f_2^2)(x)|^p dx\right)^{\frac{1}{p}} \\
&\quad + \left(\int_{B(x_0,r)} |T(f_1^2, f_2^1)(x)|^p dx\right)^{\frac{1}{p}} + \left(\int_{B(x_0,r)} |T(f_1^2, f_2^2)(x)|^p dx\right)^{\frac{1}{p}} \\
&:= \sum_{j=1}^4 G_j.
\end{aligned}$$

By Lemma 2.1,

$$\begin{aligned}
G_1 &\leq C\left(\int_{2B} |f_1(y_1)|^{p_1} dy_1\right)^{\frac{1}{p_1}} \left(\int_{2B} |f_2(y_2)|^{p_2} dy_2\right)^{\frac{1}{p_2}} \\
&\leq C\varphi_1(x_0, 2r)^{\frac{1}{p_1}} \varphi_2(x_0, 2r)^{\frac{1}{p_2}} \|f_1\|_{L^{p_1,\varphi_1}} \|f_2\|_{L^{p_2,\varphi_2}} \\
&\leq C\varphi(x_0, r)^{\frac{1}{p}} \|f_1\|_{L^{p_1,\varphi_1}} \|f_2\|_{L^{p_2,\varphi_2}}.
\end{aligned}$$

In order to estimate G_2, we first have to prove the below inequality:

$$|T(f_1^1, f_2^2)(x)| \leq C|B|^{-\frac{1}{p}} \varphi(x_0, r)^{\frac{1}{p}} \|f_1\|_{L^{p_1,\varphi_1}} \|f_2\|_{L^{p_2,\varphi_2}}, \text{ for any } x \in B. \tag{3.1}$$

Using the condition (1.2), the following inequality

$$|T(f_1^1,f_2^2)(x)|\le C\int_{2B}|f_1(y_1)|dy_1\int_{(2B)^c}\frac{|f_2(y_2)|}{|x_0-y_2|^{2n}}dy_2$$

holds for any $x\in B$. By using Hölder's inequality, it is obvious that

$$\tag{3.2}\begin{aligned}\int_{2B}|f_1(y_1)|dy_1&\le|2B|^{1-\frac{1}{p_1}}\|f_1\|_{L^{p_1}(2B)}\\&\le C|B|^{1-\frac{1}{p_1}}\varphi_1(x_0,r)^{\frac{1}{p_1}}\|f_1\|_{L^{p_1,\varphi_1}}.\end{aligned}$$

And we have

$$\tag{3.3}\begin{aligned}\int_{(2B)^c}\frac{|f_2(y_2)|}{|x_0-y_2|^{2n}}dy_2&\le C|B|^{-2}\sum_{k=1}^{\infty}2^{-2kn}\int_{2^{k+1}B}|f_2(y_2)|dy_2\\&\le C|B|^{-2}\sum_{k=1}^{\infty}2^{-2kn}\left(\int_{2^{k+1}B}|f_2(y_2)|^{p_2}dy_2\right)^{\frac{1}{p_2}}|2^{k+1}B|^{1-\frac{1}{p_2}}\\&\le C|B|^{-1-\frac{1}{p_2}}\|f_2\|_{L^{p_2,\varphi_2}}\sum_{k=1}^{\infty}2^{-kn(1+\frac{1}{p_2})}\varphi_2(x_0,2^{k+1}r)^{\frac{1}{p_2}}\\&\le C|B|^{-1-\frac{1}{p_2}}\varphi_2(x_0,r)^{\frac{1}{p_2}}\|f_2\|_{L^{p_2,\varphi_2}}\sum_{k=1}^{\infty}2^{-kn(1+\frac{1}{p_2})}\ D_2^{\frac{k+1}{p_2}}\\&\le C|B|^{-1-\frac{1}{p_2}}\varphi_2(x_0,r)^{\frac{1}{p_2}}\|f_2\|_{L^{p_2,\varphi_2}}\sum_{k=1}^{\infty}2^{-kn}\\&\le C|B|^{-1-\frac{1}{p_2}}\varphi_2(x_0,r)^{\frac{1}{p_2}}\|f_2\|_{L^{p_2,\varphi_2}}.\end{aligned}$$

Hence, by (3.2) and (3.3), it follows that

$$\begin{aligned}|T(f_1^1,f_2^2)(x)|&\le C|B|^{1-\frac{1}{p_1}}\varphi_1(x_0,r)^{\frac{1}{p_1}}\|f_1\|_{L^{p_1,\varphi_1}}|B|^{-1-\frac{1}{p_2}}\varphi_2(x_0,r)^{\frac{1}{p_2}}\|f_2\|_{L^{p_2,\varphi_2}}\\&\le C|B|^{-\frac{1}{p}}\varphi(x_0,r)^{\frac{1}{p}}\|f_1\|_{L^{p_1,\varphi_1}}\|f_2\|_{L^{p_2,\varphi_2}}.\end{aligned}$$

This completes the proof of (3.1). Thus,

$$G_2\le C\varphi(x_0,r)^{\frac{1}{p}}\|f_1\|_{L^{p_1,\varphi_1}}\|f_2\|_{L^{p_2,\varphi_2}}.$$

Similarly, we have

$$G_3\le C\varphi(x_0,r)^{\frac{1}{p}}\|f_1\|_{L^{p_1,\varphi_1}}\|f_2\|_{L^{p_2,\varphi_2}}.$$

In order to estimate G_4, we first have to prove the below inequality:

$$\tag{3.4}|T(f_1^2,f_2^2)(x)|\le C|B|^{-\frac{1}{p}}\varphi(x_0,r)^{\frac{1}{p}}\|f_1\|_{L^{p_1,\varphi_1}}\|f_2\|_{L^{p_2,\varphi_2}},\ \text{for any } x\in B.$$

Using the condition (1.2), the following inequality

$$|T(f_1^2,f_2^2)(x)|\le C\int_{(2B)^c}\frac{|f_1(y_1)|}{|x_0-y_1|^{n}}dy_1\int_{(2B)^c}\frac{|f_2(y_2)|}{|x_0-y_2|^{n}}dy_2$$

holds for any $x\in B$. Similarly to (3.3), we have

$$\tag{3.5}\int_{(2B)^c}\frac{|f_i(y_i)|}{|x_0-y_i|^{n}}dy_i\le C|B|^{-\frac{1}{p_i}}\varphi_i(x_0,r)^{\frac{1}{p_i}}\|f_i\|_{L^{p_i,\varphi_i}},\ i=1,2.$$

Hence, by (3.5), it follows that

$$\begin{aligned}|T(f_1^2,f_2^2)(x)|&\le C|B|^{-\frac{1}{p_1}}\varphi_1(x_0,r)^{\frac{1}{p_1}}\|f_1\|_{L^{p_1,\varphi_1}}|B|^{-\frac{1}{p_2}}\varphi_2(x_0,r)^{\frac{1}{p_2}}\|f_2\|_{L^{p_2,\varphi_2}}\\&\le C|B|^{-\frac{1}{p}}\varphi(x_0,r)^{\frac{1}{p}}\|f_1\|_{L^{p_1,\varphi_1}}\|f_2\|_{L^{p_2,\varphi_2}}.\end{aligned}$$

Thus,

$$G_4 \le C\varphi(x_0,r)^{\frac{1}{p}}\|f_1\|_{L^{p_1,\varphi_1}}\|f_2\|_{L^{p_2,\varphi_2}}.$$

By combining the above inequalities for $G_1 \sim G_4$, we have

$$\left(\int_{B(x_0,r)}|T(f_1,f_2)(x)|^p dx\right)^{\frac{1}{p}} \le C\varphi(x_0,r)^{\frac{1}{p}}\|f_1\|_{L^{p_1,\varphi_1}}\|f_2\|_{L^{p_2,\varphi_2}}.$$

Therefore,

$$\begin{aligned}\|T(f_1,f_2)\|_{L^{p,\varphi}} &= \sup_{x_0\in\mathbb{R}^n,r>0}\varphi(x_0,r)^{-\frac{1}{p}}\left(\int_{B(x_0,r)}|T(f_1,f_2)(x)|^p dx\right)^{\frac{1}{p}}\\ &\le C\|f_1\|_{L^{p_1,\varphi_1}}\|f_2\|_{L^{p_2,\varphi_2}},\end{aligned}$$

which completes the proof of Theorem 3.1. □

Theorem 3.2. *Let $m \ge 2$, T be an m-linear singular integral operator with generalized kernel and $\sum_{k_i=1}^{\infty} k_i C_{k_i} < \infty$, $i = 1, \cdots, m$. Suppose for fixed $1 \le r_1, \cdots, r_m \le s'$ with $1/r = 1/r_1 + \cdots + 1/r_m$, T is bounded from $L^{r_1} \times \cdots \times L^{r_m}$ into $L^{r,\infty}$. Let φ, φ_j be a positive function on $\mathbb{R}^n \times \mathbb{R}^+$ and suppose there exist $0 < D_j < 2^n$ such that $\varphi_j(x,2r) \le D_j\varphi_j(x,r)$ $(j = 1, \cdots, m)$ with $\varphi(x,r)^{\frac{1}{p}} = \prod_{j=1}^{m}\varphi_j(x,r)^{\frac{1}{p_j}}$. If $\vec{b} \in BMO^m$, then for any $s' < p_1, \cdots, p_m < \infty$ with $1/p = \sum_{i=1}^{m} 1/p_i$ and $1 < p < \infty$, $T_{\vec{b}}$ and $T_{\Pi\vec{b}}$ are bounded from $L^{p_1,\varphi_1}(\mathbb{R}^n) \times \cdots \times L^{p_m,\varphi_m}(\mathbb{R}^n)$ into $L^{p,\varphi}(\mathbb{R}^n)$.*

Proof. As in the proof of Theorem 3.1, we will only consider the situation $m = 2$. Actually, the proof of other cases is similar. Let f_1, f_2 be bounded measurable functions with compact support, $b_1, b_2 \in BMO$. Then for any ball $B = B(x_0,r)$ centered at x_0 with radius $r > 0$, we decompose f_1 and f_2 as follows:

$$f_1 = f_1\chi_{2B} + f_1\chi_{(2B)^c} := f_1^1 + f_1^2, \quad f_2 = f_2\chi_{2B} + f_2\chi_{(2B)^c} := f_2^1 + f_2^2.$$

Firstly, we will prove the boundedness of $T_{\vec{b}}$. Thus, we have

$$\begin{aligned}&\left(\int_{B(x_0,r)}|T_{\vec{b}}^1(f_1,f_2)(x)|^p dx\right)^{\frac{1}{p}}\\ &\le \left(\int_{B(x_0,r)}|T_{\vec{b}}^1(f_1^1,f_2^1)(x)|^p dx\right)^{\frac{1}{p}} + \left(\int_{B(x_0,r)}|T_{\vec{b}}^1(f_1^1,f_2^2)(x)|^p dx\right)^{\frac{1}{p}}\\ &\quad + \left(\int_{B(x_0,r)}|T_{\vec{b}}^1(f_1^2,f_2^1)(x)|^p dx\right)^{\frac{1}{p}} + \left(\int_{B(x_0,r)}|T_{\vec{b}}^1(f_1^2,f_2^2)(x)|^p dx\right)^{\frac{1}{p}}\\ &:= \sum_{j=1}^{4} I_j.\end{aligned}$$

By Lemma 2.2, we get

$$\begin{aligned}I_1 &\le C\|\vec{b}\|_{BMO^2}\left(\int_{2B}|f_1(y_1)|^{p_1}dy_1\right)^{\frac{1}{p_1}}\left(\int_{2B}|f_2(y_2)|^{p_2}dy_2\right)^{\frac{1}{p_2}}\\ &\le C\|\vec{b}\|_{BMO^2}\varphi_1(x_0,2r)^{\frac{1}{p_1}}\varphi_2(x_0,2r)^{\frac{1}{p_2}}\|f_1\|_{L^{p_1,\varphi_1}}\|f_2\|_{L^{p_2,\varphi_2}}\\ &\le C\|\vec{b}\|_{BMO^2}\varphi(x_0,r)^{\frac{1}{p}}\|f_1\|_{L^{p_1,\varphi_1}}\|f_2\|_{L^{p_2,\varphi_2}}.\end{aligned}$$

For I_2, we have

$$\begin{aligned}I_2 &\le \left(\int_B|(b_1(x)-b_B^1)T(f_1^1,f_2^2)(x)|^p dx\right)^{\frac{1}{p}} + \left(\int_B|T((b_1-b_B^1)f_1^1,f_2^2)(x)|^p dx\right)^{\frac{1}{p}}\\ &:= I_{21} + I_{22}.\end{aligned}$$

By using (3.1), we have

$$\begin{aligned} I_{21} &\le C\left(\int_{B(x_0,r)} |b_1(x)-b_B^1|^p dx\right)^{\frac{1}{p}} |B|^{-\frac{1}{p}}\varphi(x_0,r)^{\frac{1}{p}}\|f_1\|_{L^{p_1,\varphi_1}}\|f_2\|_{L^{p_2,\varphi_2}} \\ &\le C\varphi(x_0,r)^{\frac{1}{p}}\|b_1\|_{BMO}\|f_1\|_{L^{p_1,\varphi_1}}\|f_2\|_{L^{p_2,\varphi_2}}. \end{aligned}$$

In order to estimate I_{22}, we first have to prove the below inequality:

$$\left|T\big((b_1-b_B^1)f_1^1,f_2^2\big)(x)\right| \le C|B|^{-\frac{1}{p}}\|b_1\|_{BMO}\varphi(x_0,r)^{\frac{1}{p}}\|f_1\|_{L^{p_1,\varphi_1}}\|f_2\|_{L^{p_2,\varphi_2}}, \tag{3.6}$$

for any $x\in B$. Using the condition (1.2), the following inequality

$$\left|T\big((b_1-b_B^1)f_1^1,f_2^2\big)(x)\right| \le C\int_{2B}|b_1(y_1)-b_B^1||f_1(y_1)|dy_1\int_{(2B)^c}\frac{|f_2(y_2)|}{|x_0-y_2|^{2n}}dy_2$$

holds for any $x\in B$. By using Hölder's inequality, it is obvious that

$$\begin{aligned} &\int_{2B}|b_1(y_1)-b_B^1||f_1(y_1)|dy_1 \\ &\le |2B|^{1-\frac{1}{p_1}}\left(\frac{1}{|2B|}\int_{2B}|b_1(x)-b_B^1|^{p_1'}dx\right)^{\frac{1}{p_1'}}\left(\int_{2B}|f_1(y_1)|^{p_1}dy_1\right)^{\frac{1}{p_1}} \\ &\le C\|b_1\|_{BMO}|2B|^{1-\frac{1}{p_1}}\left(\int_{2B}|f_1(y_1)|^{p_1}dy_1\right)^{\frac{1}{p_1}} \\ &\le C|B|^{1-\frac{1}{p_1}}\|b_1\|_{BMO}\varphi_1(x_0,r)^{\frac{1}{p_1}}\|f_1\|_{L^{p_1,\varphi_1}}. \end{aligned} \tag{3.7}$$

Hence, by (3.3) and (3.7), it follows that

$$\left|T\big((b_1-b_B^1)f_1^1,f_2^2\big)(x)\right| \le C|B|^{-\frac{1}{p}}\|b_1\|_{BMO}\varphi(x_0,r)^{\frac{1}{p}}\|f_1\|_{L^{p_1,\varphi_1}}\|f_2\|_{L^{p_2,\varphi_2}}.$$

Thus,

$$I_{22}\le C\varphi(x_0,r)^{\frac{1}{p}}\|b_1\|_{BMO}\|f_1\|_{L^{p_1,\varphi_1}}\|f_2\|_{L^{p_2,\varphi_2}}.$$

By combining the above inequalities for I_{21}, I_{22}, we have

$$I_2\le C\varphi(x_0,r)^{\frac{1}{p}}\|\vec{b}\|_{BMO^2}\|f_1\|_{L^{p_1,\varphi_1}}\|f_2\|_{L^{p_2,\varphi_2}}.$$

For I_3, we have

$$\begin{aligned} I_3 &\le \left(\int_B |(b_1(x)-b_B^1)T(f_1^2,f_2^1)(x)|^p dx\right)^{\frac{1}{p}} + \left(\int_B |T((b_1-b_B^1)f_1^2,f_2^1)(x)|^p dx\right)^{\frac{1}{p}} \\ &:= I_{31}+I_{32}. \end{aligned}$$

Similarly to (3.1), we have

$$\begin{aligned} I_{31} &\le C\left(\int_{B(x_0,r)} |b_1(x)-b_B^1|^p dx\right)^{\frac{1}{p}} |B|^{-\frac{1}{p}}\varphi(x_0,r)^{\frac{1}{p}}\|f_1\|_{L^{p_1,\varphi_1}}\|f_2\|_{L^{p_2,\varphi_2}} \\ &\le C\varphi(x_0,r)^{\frac{1}{p}}\|b_1\|_{BMO}\|f_1\|_{L^{p_1,\varphi_1}}\|f_2\|_{L^{p_2,\varphi_2}}. \end{aligned}$$

In order to estimate I_{32}, we first have to prove the below inequality:

$$\left|T\big((b_1-b_B^1)f_1^2,f_2^1\big)(x)\right| \le C|B|^{-\frac{1}{p}}\|b_1\|_{BMO}\varphi(x_0,r)^{\frac{1}{p}}\|f_1\|_{L^{p_1,\varphi_1}}\|f_2\|_{L^{p_2,\varphi_2}}, \tag{3.8}$$

for any $x\in B$. Using the condition (1.2), the following inequality

$$\left|T\big((b_1-b_B^1)f_1^2,f_2^1\big)(x)\right| \le C\int_{(2B)^c}\frac{|b_1(y_1)-b_B^1||f_1(y_1)|}{|x_0-y_1|^{2n}}dy_1\int_{2B}|f_2(y_2)|dy_2$$

holds for any $x \in B$. By using Hölder's inequality and Lemma 2.3, we have

$$
\begin{aligned}
&\int_{(2B)^c} \frac{|b_1(y_1)-b_B^1||f_1(y_1)|}{|x_0-y_1|^{2n}} dy_1 \\
&\le C|B|^{-2} \sum_{k=1}^{\infty} 2^{-2kn} \Big(\int_{2^{k+1}B} |b_1(y_1)-b_B^1|^{p_1'} dy_1 \Big)^{\frac{1}{p_1'}} \Big(\int_{2^{k+1}B} |f_1(y_1)|^{p_1} dy_1 \Big)^{\frac{1}{p_1}} \\
&\le C|B|^{-1-\frac{1}{p_1}} \|b_1\|_{BMO} \|f_1\|_{L^{p_1,\varphi_1}} \sum_{k=1}^{\infty} (1+|\ln 2^{k+1}|) 2^{-kn(1+\frac{1}{p_1})} \varphi_1(x_0, 2^{k+1}r)^{\frac{1}{p_1}} \\
&\le C|B|^{-1-\frac{1}{p_1}} \|b_1\|_{BMO} \varphi_1(x_0,r)^{\frac{1}{p_1}} \|f_1\|_{L^{p_1,\varphi_1}} \sum_{k=1}^{\infty} k 2^{-kn(1+\frac{1}{p_1})} \ D_1^{\frac{k+1}{p_1}} \\
&\le C|B|^{-1-\frac{1}{p_1}} \|b_1\|_{BMO} \varphi_1(x_0,r)^{\frac{1}{p_1}} \|f_1\|_{L^{p_1,\varphi_1}} \sum_{k=1}^{\infty} k 2^{-kn} \\
&\le C|B|^{-1-\frac{1}{p_1}} \|b_1\|_{BMO} \varphi_1(x_0,r)^{\frac{1}{p_1}} \|f_1\|_{L^{p_1,\varphi_1}}.
\end{aligned} \tag{3.9}
$$

Hence, by (3.9) and the similarity of (3.2), it follows that

$$\big|T\big((b_1-b_B^1)f_1^2, f_2^1\big)(x)\big| \le C|B|^{-\frac{1}{p}} \|b_1\|_{BMO} \varphi(x_0,r)^{\frac{1}{p}} \|f_1\|_{L^{p_1,\varphi_1}} \|f_2\|_{L^{p_2,\varphi_2}}.$$

Thus,

$$I_{32} \le C\varphi(x_0,r)^{\frac{1}{p}} \|b_1\|_{BMO} \|f_1\|_{L^{p_1,\varphi_1}} \|f_2\|_{L^{p_2,\varphi_2}}.$$

By combining the above inequalities for I_{31}, I_{32}, we have

$$I_3 \le C\varphi(x_0,r)^{\frac{1}{p}} \|\vec{b}\|_{BMO^2} \|f_1\|_{L^{p_1,\varphi_1}} \|f_2\|_{L^{p_2,\varphi_2}}.$$

For I_4, we have

$$
\begin{aligned}
I_4 &\le \Big(\int_B |(b_1(x)-b_B^1) T(f_1^2, f_2^2)(x)|^p dx \Big)^{\frac{1}{p}} + \Big(\int_B |T((b_1-b_B^1)f_1^2, f_2^2)(x)|^p dx \Big)^{\frac{1}{p}} \\
&:= I_{41} + I_{42}.
\end{aligned}
$$

By (3.4), we have

$$
\begin{aligned}
I_{41} &\le C \left(\int_{B(x_0,r)} |b_1(x)-b_B^1|^p dx \right)^{\frac{1}{p}} |B|^{-\frac{1}{p}} \varphi(x_0,r)^{\frac{1}{p}} \|f_1\|_{L^{p_1,\varphi_1}} \|f_2\|_{L^{p_2,\varphi_2}} \\
&\le C \ \varphi(x_0,r)^{\frac{1}{p}} \|b_1\|_{BMO} \|f_1\|_{L^{p_1,\varphi_1}} \|f_2\|_{L^{p_2,\varphi_2}}.
\end{aligned}
$$

By the condition (1.2), similarly to (3.3) and (3.9), we can obtain for any $x \in B$,

$$
\begin{aligned}
&\big|T\big((b_1-b_B^1)f_1^2, f_2^2\big)(x)\big| \\
&\le C \int_{(2B)^c} \frac{|b_1(y_1)-b_B^1||f_1(y_1)|}{|x_0-y_1|^n} dy_1 \int_{(2B)^c} \frac{|f_2(y_2)|}{|x_0-y_2|^n} dy_2 \\
&\le C|B|^{-\frac{1}{p_1}} \|b_1\|_{BMO} \varphi(x_0,r)^{\frac{1}{p_1}} \|f_1\|_{L^{p_1,\varphi_1}} |B|^{-\frac{1}{p_2}} \varphi_2(x_0,r)^{\frac{1}{p_2}} \|f_2\|_{L^{p_2,\varphi_2}} \\
&\le C|B|^{-\frac{1}{p}} \varphi(x_0,r)^{\frac{1}{p}} \|b_1\|_{BMO} \|f_1\|_{L^{p_1,\varphi_1}} \|f_2\|_{L^{p_2,\varphi_2}}.
\end{aligned} \tag{3.10}
$$

Thus,

$$I_{42} \le C\varphi(x_0,r)^{\frac{1}{p}} \|b_1\|_{BMO} \|f_1\|_{L^{p_1,\varphi_1}} \|f_2\|_{L^{p_2,\varphi_2}}.$$

By combining the above inequalities for I_{41} and I_{42}, we have

$$I_4 \le C\varphi(x_0,r)^{\frac{1}{p}} \|\vec{b}\|_{BMO^2} \|f_1\|_{L^{p_1,\varphi_1}} \|f_2\|_{L^{p_2,\varphi_2}}.$$

Combining the estimates of $I_j (j = 1, \cdots, 4)$, we get

$$\left(\int_{B(x_0,r)} |T^1_{\vec{b}}(f_1, f_2)(x)|^p dx \right)^{\frac{1}{p}} \leq C \|\vec{b}\|_{BMO^2} \varphi(x_0, r)^{\frac{1}{p}} \|f_1\|_{L^{p_1,\varphi_1}} \|f_2\|_{L^{p_2,\varphi_2}}.$$

Similarly,

$$\left(\int_{B(x_0,r)} |T^2_{\vec{b}}(f_1, f_2)(x)|^p dx \right)^{\frac{1}{p}} \leq C \|\vec{b}\|_{BMO^2} \varphi(x_0, r)^{\frac{1}{p}} \|f_1\|_{L^{p_1,\varphi_1}} \|f_2\|_{L^{p_2,\varphi_2}}.$$

Thus,

$$\begin{aligned} \|T_{\vec{b}}(f_1, f_2)\|_{L^{p,\varphi}} &= \sup_{x_0 \in \mathbb{R}^n, r>0} \varphi(x_0, r)^{-\frac{1}{p}} \left(\int_{B(x_0,r)} |T_{\vec{b}}(f_1, f_2)(x)|^p dx \right)^{\frac{1}{p}} \\ &\leq C \|\vec{b}\|_{BMO^2} \|f_1\|_{L^{p_1,\varphi_1}} \|f_2\|_{L^{p_2,\varphi_2}}. \end{aligned}$$

And then, we prove the boundedness of $T_{\Pi\vec{b}}$. We have

$$\begin{aligned} &\left(\int_{B(x_0,r)} |T_{\Pi\vec{b}}(f_1, f_2)(x)|^p dx \right)^{\frac{1}{p}} \\ &\leq \left(\int_{B(x_0,r)} |T_{\Pi\vec{b}}(f_1^1, f_2^1)(x)|^p dx \right)^{\frac{1}{p}} + \left(\int_{B(x_0,r)} |T_{\Pi\vec{b}}(f_1^1, f_2^2)(x)|^p dx \right)^{\frac{1}{p}} \\ &\quad + \left(\int_{B(x_0,r)} |T_{\Pi\vec{b}}(f_1^2, f_2^1)(x)|^p dx \right)^{\frac{1}{p}} + \left(\int_{B(x_0,r)} |T_{\Pi\vec{b}}(f_1^2, f_2^2)(x)|^p dx \right)^{\frac{1}{p}} \\ &:= \sum_{j=1}^4 H_j. \end{aligned}$$

By Lemma 2.4, we get

$$\begin{aligned} H_1 &\leq C \|b_1\|_{BMO} \|b_2\|_{BMO} \left(\int_{2B} |f_1(y_1)|^{p_1} dy_1 \right)^{\frac{1}{p_1}} \left(\int_{2B} |f_2(y_2)|^{p_2} dy_2 \right)^{\frac{1}{p_2}} \\ &\leq C \|b_1\|_{BMO} \|b_2\|_{BMO} \varphi(x_0, r)^{\frac{1}{p}} \|f_1\|_{L^{p_1,\varphi_1}} \|f_2\|_{L^{p_2,\varphi_2}}. \end{aligned}$$

And we have the following decomposition,

$$\begin{aligned} T_{\Pi\vec{b}}(f_1^1, f_2^2)(x) =& (b_1(x) - b_B^1)(b_2(x) - b_B^2) T(f_1^1, f_2^2)(x) \\ & - (b_1(x) - b_B^1) T\left(f_1^1, (b_2 - b_B^2) f_2^2\right)(x) \\ & - (b_2(x) - b_B^2) T\left((b_1 - b_B^1) f_1^1, f_2^2\right)(x) \\ & + T\left((b_1 - b_B^1) f_1^1, (b_2 - b_B^2) f_2^2\right)(x). \end{aligned} \tag{3.11}$$

Thus,

$$
\begin{aligned}
H_2 \leq & \left(\int_{B(x_0,r)} |(b_1(x)-b_B^1)(b_2(x)-b_B^2)T(f_1^1,f_2^2)(x)|^p dx \right)^{\frac{1}{p}} \\
& + \left(\int_{B(x_0,r)} \left|(b_1(x)-b_B^1)T\big(f_1^1,(b_2-b_B^2)f_2^2\big)(x)\right|^p dx \right)^{\frac{1}{p}} \\
& + \left(\int_{B(x_0,r)} \left|(b_2(x)-b_B^2)T\big((b_1-b_B^1)f_1^1,f_2^2\big)(x)\right|^p dx \right)^{\frac{1}{p}} \\
& + \left(\int_{B(x_0,r)} \left|T\big((b_1-b_B^1)f_1^1,(b_2-b_B^2)f_2^2\big)(x)\right|^p dx \right)^{\frac{1}{p}} \\
& := \sum_{j=1}^{4} H_{2j}.
\end{aligned}
$$

By (3.1) and Hölder's inequality, we have

$$
\begin{aligned}
H_{21} \leq & \, C \left(\int_{B(x_0,r)} |(b_1(x)-b_B^1)(b_2(x)-b_B^2)|^p dx \right)^{\frac{1}{p}} |B|^{-\frac{1}{p}} \varphi(x_0,r)^{\frac{1}{p}} \|f_1\|_{L^{p_1,\varphi_1}} \|f_2\|_{L^{p_2,\varphi_2}} \\
\leq & \, C \left(\int_{B(x_0,r)} |b_1(x)-b_B^1|^{p_1} dx \right)^{\frac{1}{p_1}} \left(\int_{B(x_0,r)} |b_2(x)-b_B^2|^{p_2} dx \right)^{\frac{1}{p_2}} \\
& \times |B|^{-\frac{1}{p}} \varphi(x_0,r)^{\frac{1}{p}} \|f_1\|_{L^{p_1,\varphi_1}} \|f_2\|_{L^{p_2,\varphi_2}} \\
\leq & \, C\varphi(x_0,r)^{\frac{1}{p}} \|b_1\|_{BMO} \|b_2\|_{BMO} \|f_1\|_{L^{p_1,\varphi_1}} \|f_2\|_{L^{p_2,\varphi_2}}.
\end{aligned}
$$

By the similarity of (3.8), we have

$$
\begin{aligned}
H_{22} \leq & \, C \left(\int_{B(x_0,r)} |b_1(x)-b_B^1|^p dx \right)^{\frac{1}{p}} |B|^{-\frac{1}{p}} \|b_2\|_{BMO} \varphi(x_0,r)^{\frac{1}{p}} \|f_1\|_{L^{p_1,\varphi_1}} \|f_2\|_{L^{p_2,\varphi_2}} \\
\leq & \, C\varphi(x_0,r)^{\frac{1}{p}} \|b_1\|_{BMO} \|b_2\|_{BMO} \|f_1\|_{L^{p_1,\varphi_1}} \|f_2\|_{L^{p_2,\varphi_2}}.
\end{aligned}
$$

By (3.6), we have

$$
\begin{aligned}
H_{23} \leq & \, C \left(\int_{B(x_0,r)} |b_2(x)-b_B^2|^p dx \right)^{\frac{1}{p}} |B|^{-\frac{1}{p}} \|b_1\|_{BMO} \varphi(x_0,r)^{\frac{1}{p}} \|f_1\|_{L^{p_1,\varphi_1}} \|f_2\|_{L^{p_2,\varphi_2}} \\
\leq & \, C\varphi(x_0,r)^{\frac{1}{p}} \|b_1\|_{BMO} \|b_2\|_{BMO} \|f_1\|_{L^{p_1,\varphi_1}} \|f_2\|_{L^{p_2,\varphi_2}}.
\end{aligned}
$$

By the condition (1.2), Hölder's inequality, (3.7) and the similarity of (3.9), we have

$$
\begin{aligned}
& \left|T\big((b_1-b_B^1)f_1^1,(b_2-b_B^2)f_2^2\big)(x)\right| \\
& \leq C \int_{2B} |b_1(y_1)-b_B^1||f_1(y_1)| dy_1 \int_{(2B)^c} \frac{|(b_2(y_2)-b_B^2)f_2(y_2)|}{|x_0-y_2|^{2n}} dy_2 \\
& \leq C|B|^{1-\frac{1}{p_1}} \|b_1\|_{BMO} \varphi_1(x_0,r)^{\frac{1}{p_1}} \|f_1\|_{L^{p_1,\varphi_1}} |B|^{-1-\frac{1}{p_2}} \|b_2\|_{BMO} \varphi_2(x_0,r)^{\frac{1}{p_2}} \|f_2\|_{L^{p_2,\varphi_2}} \\
& \leq C|B|^{-\frac{1}{p}} \varphi(x_0,r)^{\frac{1}{p}} \|b_1\|_{BMO} \|b_2\|_{BMO} \|f_1\|_{L^{p_1,\varphi_1}} \|f_2\|_{L^{p_2,\varphi_2}}.
\end{aligned}
$$

Thus,

$$
H_{24} \leq C\varphi(x_0,r)^{\frac{1}{p}} \|b_1\|_{BMO} \|b_2\|_{BMO} \|f_1\|_{L^{p_1,\varphi_1}} \|f_2\|_{L^{p_2,\varphi_2}}.
$$

By combining the above inequalities for $H_{21} \sim H_{24}$, we have

$$H_2 \leq C\varphi(x_0, r)^{\frac{1}{p}} \|b_1\|_{BMO} \|b_2\|_{BMO} \|f_1\|_{L^{p_1,\varphi_1}} \|f_2\|_{L^{p_2,\varphi_2}}.$$

Similarly, the inequality

$$H_3 \leq C\varphi(x_0, r)^{\frac{1}{p}} \|b_1\|_{BMO} \|b_2\|_{BMO} \|f_1\|_{L^{p_1,\varphi_1}} \|f_2\|_{L^{p_2,\varphi_2}}$$

is valid.

Then similar to the decomposition of (3.11), we have

$$\begin{aligned} H_4 \leq & \left(\int_{B(x_0,r)} |(b_1(x) - b_B^1)(b_2(x) - b_B^2)T(f_1^2, f_2^2)(x)|^p dx \right)^{\frac{1}{p}} \\ & + \left(\int_{B(x_0,r)} \big|(b_1(x) - b_B^1)T\big(f_1^2, (b_2 - b_B^2)f_2^2\big)(x)\big|^p dx \right)^{\frac{1}{p}} \\ & + \left(\int_{B(x_0,r)} \big|(b_2(x) - b_B^2)T\big((b_1 - b_B^1)f_1^2, f_2^2\big)(x)\big|^p dx \right)^{\frac{1}{p}} \\ & + \left(\int_{B(x_0,r)} \big|T\big((b_1 - b_B^1)f_1^2, (b_2 - b_B^2)f_2^2\big)(x)\big|^p dx \right)^{\frac{1}{p}} \\ := & \sum_{j=1}^{4} H_{4j}. \end{aligned}$$

By (3.4) and Hölder's inequality, we have

$$\begin{aligned} H_{41} \leq & C \left(\int_{B(x_0,r)} |b_1(x) - b_B^1)(b_2(x) - b_B^2)|^p dx \right)^{\frac{1}{p}} |B|^{-\frac{1}{p}} \varphi(x_0, r)^{\frac{1}{p}} \|f_1\|_{L^{p_1,\varphi_1}} \|f_2\|_{L^{p_2,\varphi_2}} \\ \leq & C \left(\int_{B(x_0,r)} |b_1(x) - b_B^1|^{p_1} dx \right)^{\frac{1}{p_1}} \left(\int_{B(x_0,r)} |b_2(x) - b_B^2|^{p_2} dx \right)^{\frac{1}{p_2}} \\ & \times |B|^{-\frac{1}{p}} \varphi(x_0, r)^{\frac{1}{p}} \|f_1\|_{L^{p_1,\varphi_1}} \|f_2\|_{L^{p_2,\varphi_2}} \\ \leq & C\varphi(x_0, r)^{\frac{1}{p}} \|b_1\|_{BMO} \|b_2\|_{BMO} \|f_1\|_{L^{p_1,\varphi_1}} \|f_2\|_{L^{p_2,\varphi_2}}. \end{aligned}$$

By the similarity of (3.10), we have

$$\begin{aligned} H_{42} \leq & C \left(\int_{B(x_0,r)} |b_1(x) - b_B^1|^p dx \right)^{\frac{1}{p}} |B|^{-\frac{1}{p}} \|b_2\|_{BMO} \varphi(x_0, r)^{\frac{1}{p}} \|f_1\|_{L^{p_1,\varphi_1}} \|f_2\|_{L^{p_2,\varphi_2}} \\ \leq & C\varphi(x_0, r)^{\frac{1}{p}} \|b_1\|_{BMO} \|b_2\|_{BMO} \|f_1\|_{L^{p_1,\varphi_1}} \|f_2\|_{L^{p_2,\varphi_2}}. \end{aligned}$$

By (3.10), we have

$$\begin{aligned} H_{43} \leq & C \left(\int_{B(x_0,r)} |b_2(x) - b_B^2|^p dx \right)^{\frac{1}{p}} |B|^{-\frac{1}{p}} \|b_1\|_{BMO} \varphi(x_0, r)^{\frac{1}{p}} \|f_1\|_{L^{p_1,\varphi_1}} \|f_2\|_{L^{p_2,\varphi_2}} \\ \leq & C\varphi(x_0, r)^{\frac{1}{p}} \|b_1\|_{BMO} \|b_2\|_{BMO} \|f_1\|_{L^{p_1,\varphi_1}} \|f_2\|_{L^{p_2,\varphi_2}}. \end{aligned}$$

By the condition (1.2), Hölder's inequality and the similarity of (3.9), we have

$$\begin{aligned}
&\left|T\left((b_1-b_B^1)f_1^2,(b_2-b_B^2)f_2^2\right)(x)\right|\\
&\le C\int_{(2B)^c}\frac{|(b_1(y_1)-b_B^1)f_1(y_1)|}{|x_0-y_1|^n}dy_1\int_{(2B)^c}\frac{|(b_2(y_2)-b_B^2)f_2(y_2)|}{|x_0-y_2|^n}dy_2\\
&\le C|B|^{-\frac{1}{p_1}}\|b_1\|_{BMO}\varphi_1(x_0,r)^{\frac{1}{p_1}}\|f_1\|_{L^{p_1,\varphi_1}}|B|^{-\frac{1}{p_2}}\|b_2\|_{BMO}\varphi_2(x_0,r)^{\frac{1}{p_2}}\|f_2\|_{L^{p_2,\varphi_2}}\\
&\le C|B|^{-\frac{1}{p}}\varphi(x_0,r)^{\frac{1}{p}}\|b_1\|_{BMO}\|b_2\|_{BMO}\|f_1\|_{L^{p_1,\varphi_1}}\|f_2\|_{L^{p_2,\varphi_2}}.
\end{aligned}$$

Thus,

$$H_{44}\le C\varphi(x_0,r)^{\frac{1}{p}}\|b_1\|_{BMO}\|b_2\|_{BMO}\|f_1\|_{L^{p_1,\varphi_1}}\|f_2\|_{L^{p_2,\varphi_2}}.$$

By combining the above inequalities for $H_{41}\sim H_{44}$, we have

$$H_4\le C\varphi(x_0,r)^{\frac{1}{p}}\|b_1\|_{BMO}\|b_2\|_{BMO}\|f_1\|_{L^{p_1,\varphi_1}}\|f_2\|_{L^{p_2,\varphi_2}}.$$

Combining the estimates of $H_j(j=1,\cdots,4)$, we get

$$\left(\int_{B(x_0,r)}|T_{\Pi\vec{b}}(f_1,f_2)(x)|^pdx\right)^{\frac{1}{p}}\le C\varphi(x_0,r)^{\frac{1}{p}}\|b_1\|_{BMO}\|b_2\|_{BMO}\|f_1\|_{L^{p_1,\varphi_1}}\|f_2\|_{L^{p_2,\varphi_2}}.$$

Therefore,

$$\begin{aligned}
\|T_{\Pi\vec{b}}(f_1,f_2)\|_{L^{p,\varphi}}&=\sup_{x_0\in\mathbb{R}^n,r>0}\varphi(x_0,r)^{-\frac{1}{p}}\left(\int_{B(x_0,r)}|T_{\Pi\vec{b}}(f_1,f_2)(x)|^pdx\right)^{\frac{1}{p}}\\
&\le C\|b_1\|_{BMO}\|b_2\|_{BMO}\|f_1\|_{L^{p_1,\varphi_1}}\|f_2\|_{L^{p_2,\varphi_2}},
\end{aligned}$$

which completes the proof of Theorem 3.2. □

Theorem 3.3. *Let $m\ge 2$, T be an m-linear singular integral operator with generalized kernel and $\sum_{k_i=1}^{\infty}C_{k_i}<\infty$, $i=1,\cdots,m$. Suppose for fixed $1\le r_1,\cdots,r_m\le s'$ with $1/r=1/r_1+\cdots+1/r_m$, T is bounded from $L^{r_1}\times\cdots\times L^{r_m}$ into $L^{r,\infty}$. Let $x_0\in\mathbb{R}^n$, then for any $s'<p_1,\cdots,p_m<\infty$ with $1/p=\sum_{i=1}^m 1/p_i$ and $1<p<\infty$, the inequality*

$$\|T(\vec{f})\|_{L^p(B_r)}\le Cr^{\frac{n}{p}}\prod_{i=1}^m\int_{2r}^{\infty}\|f_i\|_{L^{p_i}(B_{t_i})}\,t_i^{-\frac{n}{p_i}-1}\mathrm{d}t_i$$

holds for any ball $B_r=B(x_0,r)$, $B_t=B(x_0,t)$ and for all $\vec{f}\in L_{p_1}^{\mathrm{loc}}(\mathbb{R}^n)\times L_{p_2}^{\mathrm{loc}}(\mathbb{R}^n)\times\cdots\times L_{p_m}^{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. For the sake of simplicity, we will only consider the situation $m=2$. Let f_1,f_2 be bounded measurable functions with compact support, then for any ball $B_r=B(x_0,r)$ centered at x_0 with radius $r>0$, we decompose f_1 and f_2 as follows:

$$f_1=f_1\chi_{B_{2r}}+f_1\chi_{(B_{2r})^c}:=f_1^1+f_1^2,\quad f_2=f_2\chi_{B_{2r}}+f_2\chi_{(B_{2r})^c}:=f_2^1+f_2^2.$$

Thus, we have

$$\begin{aligned}
&\|T(f_1,f_2)\|_{L^p(B_r)}\\
&\le\|T(f_1^1,f_2^1)\|_{L^p(B_r)}+\|T(f_1^1,f_2^2)\|_{L^p(B_r)}+\|T(f_1^2,f_2^1)\|_{L^p(B_r)}+\|T(f_1^2,f_2^2)\|_{L^p(B_r)}\\
&:=\sum_{j=1}^4\widetilde{G}_j.
\end{aligned}$$

By Lemma 2.1,

$$\begin{aligned}\widetilde{G}_1 &\le C\|f_1\|_{L^{p_1}(B_{2r})}\|f_2\|_{L^{p_2}(B_{2r})}\\ &\le Cr^{\frac{n}{p}}\|f_1\|_{L^{p_1}(B_{2r})}\|f_2\|_{L^{p_2}(B_{2r})}\int_{2r}^{\infty} t_1^{-\frac{n}{p_1}-1}dt_1\int_{2r}^{\infty} t_2^{-\frac{n}{p_2}-1}dt_2\\ &\le Cr^{\frac{n}{p}}\prod_{i=1}^{2}\int_{2r}^{\infty}\|f_i\|_{L^{p_i}(B_{t_i})}\, t_i^{-\frac{n}{p_i}-1}dt_i.\end{aligned}$$

In order to estimate $\widetilde{G}_2$, we first have to prove the below inequality:

$$|T(f_1^1,f_2^2)(x)|\le C\prod_{i=1}^{2}\int_{2r}^{\infty}\|f_i\|_{L^{p_i}(B_{t_i})}\, t_i^{-\frac{n}{p_i}-1}dt_i,\ \text{for any } x\in B_r. \tag{3.12}$$

Using the condition (1.2), the following inequality

$$|T(f_1^1,f_2^2)(x)|\le C(2r)^{-n}\int_{B_{2r}}|f_1(y_1)|dy_1\int_{(B_{2r})^c}\frac{|f_2(y_2)|}{|x_0-y_2|^n}dy_2$$

holds for any $x\in B_r$. By using Hölder's inequality, it is obvious that

$$\begin{aligned}(2r)^{-n}\int_{B_{2r}}|f_1(y_1)|dy_1 &\le (2r)^{-n}\Big(\int_{B_{2r}}|f_1(y_1)|^{p_1}dy_1\Big)^{\frac{1}{p_1}}|B_{2r}|^{1-\frac{1}{p_1}}\\ &\le \|f_1\|_{L^{p_1}(B_{2r})}\int_{2r}^{\infty} t_1^{-\frac{n}{p_1}-1}dt_1\\ &\le \int_{2r}^{\infty}\|f_1\|_{L^{p_1}(B_{t_1})}t_1^{-\frac{n}{p_1}-1}dt_1,\end{aligned} \tag{3.13}$$

and we have

$$\begin{aligned}\int_{(B_{2r})^c}\frac{|f_2(y_2)|}{|x_0-y_2|^n}dy_2 &= C\int_{(B_{2r})^c}|f_2(y_2)|\int_{|x_0-y_2|}^{\infty}\frac{dt_2}{t_2^{n+1}}dy_2\\ &\le C\int_{2r}^{\infty}\int_{B_{t_2}}|f_2(y_2)|\frac{1}{t_2^{n+1}}dy_2dt_2\\ &\le C\int_{2r}^{\infty}\Big(\int_{B_{t_2}}|f_2(y_2)|^{p_2}dy_2\Big)^{\frac{1}{p_2}}|B_{t_2}|^{1-\frac{1}{p_2}}\frac{dt_2}{t_2^{n+1}}\\ &= C\int_{2r}^{\infty}\|f_2\|_{L^{p_2}(B_{t_2})}t_2^{-\frac{n}{p_2}-1}dt_2\end{aligned} \tag{3.14}$$

Hence, by (3.13) and (3.14), it follows that

$$|T(f_1^1,f_2^2)(x)|\le C\prod_{i=1}^{2}\int_{2r}^{\infty}\|f_i\|_{L^{p_i}(B_{t_i})}\, t_i^{-\frac{n}{p_i}-1}dt_i.$$

This completes the proof of (3.12). Thus,

$$\widetilde{G}_2\le Cr^{\frac{n}{p}}\prod_{i=1}^{2}\int_{2r}^{\infty}\|f_i\|_{L^{p_i}(B_{t_i})}\, t_i^{-\frac{n}{p_i}-1}dt_i.$$

Similarly, we have

$$\widetilde{G}_3\le Cr^{\frac{n}{p}}\prod_{i=1}^{2}\int_{2r}^{\infty}\|f_i\|_{L^{p_i}(B_{t_i})}\, t_i^{-\frac{n}{p_i}-1}dt_i.$$

In order to estimate $\widetilde{G}_4$, we first have to prove the below inequality:

$$|T(f_1^2, f_2^2)(x)| \le C \prod_{i=1}^{2} \int_{2r}^{\infty} \|f_i\|_{L^{p_i}(B_{t_i})}\ t_i^{-\frac{n}{p_i}-1} dt_i, \text{ for any } x \in B_r. \tag{3.15}$$

Using the condition (1.2) and (3.14), the following inequality

$$\begin{aligned} |T(f_1^2, f_2^2)(x)| &\le C \int_{(B_{2r})^c} \frac{|f_1(y_1)|}{|x_0 - y_1|^n} dy_1 \int_{(B_{2r})^c} \frac{|f_2(y_2)|}{|x_0 - y_2|^n} dy_2 \\ &\le C \prod_{i=1}^{2} \int_{2r}^{\infty} \|f_i\|_{L^{p_i}(B_{t_i})}\ t_i^{-\frac{n}{p_i}-1} dt_i \end{aligned}$$

is satisfied for any $x \in B_r$. Thus,

$$\widetilde{G}_4 \le C r^{\frac{n}{p}} \prod_{i=1}^{2} \int_{2r}^{\infty} \|f_i\|_{L^{p_i}(B_{t_i})}\ t_i^{-\frac{n}{p_i}-1} dt_i.$$

By combining the above inequalities for $\widetilde{G}_1 \sim \widetilde{G}_4$, we have

$$\|T(f_1, f_2)\|_{L^p(B_r)} \le C r^{\frac{n}{p}} \prod_{i=1}^{2} \int_{2r}^{\infty} \|f_i\|_{L^{p_i}(B_{t_i})} t_i^{-\frac{n}{p_i}-1} dt_i,$$

which completes the proof of Theorem 3.3. □

Theorem 3.4. *Let $m \ge 2$, T be an m-linear singular integral operator with generalized kernel and $\sum_{k_i=1}^{\infty} C_{k_i} < \infty$, $i = 1, \cdots, m$. Suppose for fixed $1 \le r_1, \cdots, r_m \le s'$ with $1/r = 1/r_1 + \cdots + 1/r_m$, T is bounded from $L^{r_1} \times \cdots \times L^{r_m}$ into $L^{r,\infty}$. Let $x_0 \in \mathbb{R}^n$, the functions φ, $\varphi_j : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ $(i = 1 \cdots m)$ and $(\varphi_1, \varphi_2, \cdots, \varphi_m, \varphi)$ satisfy the condition*

$$\prod_{i=1}^{m} \int_{2r}^{\infty} \frac{\underset{t_i<\tau_i<\infty}{\text{ess inf}}\ \varphi_i(x_0, \tau_i)\, \tau_i^{\frac{n}{p_i}}}{t_i^{\frac{n}{p_i}+1}} \mathrm{d}t_i \le C\varphi(x_0, r), \tag{3.16}$$

where C does not depend on r. Then for any $s' < p_1, \cdots, p_m < \infty$ with $1/p = \sum_{i=1}^{m} 1/p_i$ and $1 < p < \infty$, the operator T is bounded from $\mathrm{LM}^{\{x_0\}}_{p_1,\varphi_1} \times \cdots \times \mathrm{LM}^{\{x_0\}}_{p_m,\varphi_m}$ into $\mathrm{LM}^{\{x_0\}}_{p,\varphi}$.

Proof. Since $\vec{f} \in \mathrm{LM}^{\{x_0\}}_{p_1,\varphi_1} \times \cdots \times \mathrm{LM}^{\{x_0\}}_{p_m,\varphi_m}$, by Lemma 2.5 and the non-decreasing, with respect to t, of the norm $\prod_{i=1}^{m} \|f_i\|_{L^{p_i}(B_t)}$, we get

$$\begin{aligned} \frac{\prod_{i=1}^{m} \|f_i\|_{L^{p_i}(B_{t_i})}}{\prod_{i=1}^{m} \underset{0<t_i<\tau_i<\infty}{\text{ess inf}}\ \varphi_i(x_0, \tau_i)\tau_i^{\frac{n}{p_i}}} &\le \prod_{i=1}^{m} \underset{0<t_i<\tau_i<\infty}{\text{ess sup}} \frac{\|f_i\|_{L^{p_i}(B_{t_i})}}{\varphi_i(x_0, \tau_i)\, \tau_i^{\frac{n}{p_i}}} \\ &\le \prod_{i=1}^{m} \underset{0<\tau_i<\infty}{\text{ess sup}} \frac{\|f_i\|_{L^{p_i}(B_{\tau_i})}}{\varphi_i(x_0, \tau_i)\, \tau_i^{\frac{n}{p_i}}} \\ &\le \prod_{i=1}^{m} \|f_i\|_{\mathrm{LM}^{\{x_0\}}_{p_i,\varphi_i}}. \end{aligned} \tag{3.17}$$

By the condition (3.16) and (3.17), we have

$$
\begin{aligned}
&\prod_{i=1}^{m}\int_{2r}^{\infty}\|f_i\|_{L^{p_i}(B_{t_i})}\ t_i^{-\frac{n}{p_i}-1}dt_i\\
&=\int_{2r}^{\infty}\cdots\int_{2r}^{\infty}\frac{\prod_{i=1}^{m}\|f_i\|_{L^{p_i}(B_{t_i})}}{\prod_{i=1}^{m}\underset{0<t_i<\tau_i<\infty}{\operatorname{ess\,inf}}\varphi_i(x_0,\tau_i)\tau_i^{\frac{n}{p_i}}}\frac{\prod_{i=1}^{m}\underset{0<t_i<\tau_i<\infty}{\operatorname{ess\,inf}}\varphi_i(x_0,\tau_i)\tau_i^{\frac{n}{p_i}}}{\prod_{i=1}^{m}t_i^{\frac{n}{p_i}+1}}dt_1\cdots dt_m\\
&\leq\prod_{i=1}^{m}\|f_i\|_{LM^{\{x_0\}}_{p_i,\varphi_i}}\prod_{i=1}^{m}\int_{2r}^{\infty}\frac{\underset{t_i<\tau_i<\infty}{\operatorname{ess\,inf}}\varphi_i\left(x_0,\tau_i\right)\tau_i^{\frac{n}{p_i}}}{t_i^{\frac{n}{p_i}+1}}dt_i\\
&\leq C\varphi(x_0,r)\prod_{i=1}^{m}\|f_i\|_{LM^{\{x_0\}}_{p_i,\varphi_i}}.
\end{aligned}
$$

Then by Theorem 3.3, we have

$$
\begin{aligned}
\|T(\vec{f})\|_{LM^{\{x_0\}}_{p_i,\varphi_i}}&=\sup_{r>0}\varphi(x_0,r)^{-1}|B_r|^{-\frac{1}{p}}\|T(\vec{f})\|_{L^p(B_r)}\\
&\leq C\sup_{r>0}\varphi(x_0,r)^{-1}|B_r|^{-\frac{1}{p}}r^{\frac{n}{p}}\prod_{i=1}^{m}\int_{2r}^{\infty}\|f_i\|_{L^{p_i}(B_{t_i})}t_i^{-\frac{n}{p_i}-1}dt_i\\
&\leq C\prod_{i=1}^{m}\|f_i\|_{LM^{\{x_0\}}_{p_i,\varphi_i}},
\end{aligned}
$$

which completes the proof of Theorem 3.4. □

Theorem 3.5. *Let $m\geq 2$, T be an m-linear singular integral operator with generalized kernel and $\sum_{k_i=1}^{\infty}k_iC_{k_i}<\infty$, $i=1,\cdots,m$. Suppose for fixed $1\leq r_1,\cdots,r_m\leq s'$ with $1/r=1/r_1+\cdots+1/r_m$, T is bounded from $L^{r_1}\times\cdots\times L^{r_m}$ into $L^{r,\infty}$. Let $x_0\in\mathbb{R}^n$. If $\vec{b}\in BMO^m$, then for any $s'<p_1,\cdots,p_m<\infty$ with $1/p=\sum_{i=1}^{m}1/p_i$ and $1<p<\infty$, the inequalities*

$$\|T_{\vec{b}}(\vec{f})\|_{L^p(B_r)}\leq Cr^{\frac{n}{p}}\|\vec{b}\|_{BMO^m}\prod_{i=1}^{m}\int_{2r}^{\infty}(1+\ln\frac{t_i}{r})\|f_i\|_{L^{p_i}(B_{t_i})}t_i^{-\frac{n}{p_i}-1}\mathrm{d}t_i$$

and

$$\|T_{\Pi\vec{b}}(\vec{f})\|_{L^p(B_r)}\leq Cr^{\frac{n}{p}}\prod_{j=1}^{m}\|b_j\|_{BMO}\prod_{i=1}^{m}\int_{2r}^{\infty}(1+\ln\frac{t_i}{r})\|f_i\|_{L^{p_i}(B_{t_i})}t_i^{-\frac{n}{p_i}-1}\mathrm{d}t_i$$

hold for any ball $B_r=B(x_0,r)$, $B_t=B(x_0,t)$ and for all $\vec{f}\in L^{\rm loc}_{p_1}(\mathbb{R}^n)\times L^{\rm loc}_{p_2}(\mathbb{R}^n)\times\cdots\times L^{\rm loc}_{p_m}(\mathbb{R}^n)$.

Proof. As in the proof of Theorem 3.5, we will only consider the situation $m=2$. Let f_1,f_2 be bounded measurable functions with compact support, $b_1,b_2\in BMO$. Then for any ball $B_r=B(x_0,r)$, we decompose f_1 and f_2 as follows:

$$f_1=f_1\chi_{B_{2r}}+f_1\chi_{(B_{2r})^c}:=f_1^1+f_1^2,\quad f_2=f_2\chi_{B_{2r}}+f_2\chi_{(B_{2r})^c}:=f_2^1+f_2^2.$$

Firstly, we estimate the following inequality

$$\|T_{\vec{b}}(\vec{f})\|_{L^p(B_r)}\leq Cr^{\frac{n}{p}}\|\vec{b}\|_{BMO^2}\prod_{i=1}^{2}\int_{2r}^{\infty}(1+\ln\frac{t_i}{r})\|f_i\|_{L^{p_i}(B_{t_i})}\ t^{-\frac{n}{p_i}-1}\mathrm{d}t_i.$$

We have

$$\begin{aligned}\|T^1_{\vec{b}}(f_1,f_2)\|_{L^p(B_r)} \le&\|T^1_{\vec{b}}(f_1^1,f_2^1)\|_{L^p(B_r)}+\|T^1_{\vec{b}}(f_1^1,f_2^2)\|_{L^p(B_r)}\\&+\|T^1_{\vec{b}}(f_1^2,f_2^1)\|_{L^p(B_r)}+\|T^1_{\vec{b}}(f_1^2,f_2^2)\|_{L^p(B_r)}\\:=&\sum_{j=1}^4 \widetilde{I}_j.\end{aligned}$$

By Lemma 2.2, we get

$$\begin{aligned}\widetilde{I}_1 &\le C\|\vec{b}\|_{BMO^2}\|f_1\|_{L^{p_1}(B_{2r})}\|f_2\|_{L^{p_2}(B_{2r})}\\&\le Cr^{\frac{n}{p}}\|\vec{b}\|_{BMO^2}\|f_1\|_{L^{p_1}(B_{2r})}\|f_2\|_{L^{p_2}(B_{2r})}\int_{2r}^\infty t_1^{-\frac{n}{p_1}-1}dt_1\int_{2r}^\infty t_2^{-\frac{n}{p_2}-1}dt_2\\&\le Cr^{\frac{n}{p}}\|\vec{b}\|_{BMO^2}\prod_{i=1}^2\int_{2r}^\infty \|f_i\|_{L^{p_i}(B_{t_i})}\, t_i^{-\frac{n}{p_i}-1}dt_i.\end{aligned}$$

For $\widetilde{I}_2$, we have

$$\begin{aligned}\widetilde{I}_2 &= \|(b_1-b^1_{B_r})T(f_1^1,f_2^2)-T((b_1-b^1_{B_r})f_1^1,f_2^2)\|_{L^p(B_r)}\\&\le \|(b_1-b^1_{B_r})T(f_1^1,f_2^2)\|_{L^p(B_r)}+\|T((b_1-b^1_{B_r})f_1^1,f_2^2)\|_{L^p(B_r)}\\&:=\widetilde{I}_{21}+\widetilde{I}_{22}.\end{aligned}$$

By using (3.12), we have

$$\begin{aligned}\widetilde{I}_{21} &\le C\|b_1-b^1_{B_r}\|_{L^p(B_r)}\prod_{i=1}^2\int_{2r}^\infty \|f_i\|_{L^{p_i}(B_{t_i})}\, t_i^{-\frac{n}{p_i}-1}dt_i\\&\le Cr^{\frac{n}{p}}\|\vec{b}\|_{BMO^2}\prod_{i=1}^2\int_{2r}^\infty \|f_i\|_{L^{p_i}(B_{t_i})}\, t_i^{-\frac{n}{p_i}-1}dt_i.\end{aligned}$$

In order to estimate $\widetilde{I}_{22}$, we first have to prove the below inequality:

$$\left|T\big((b_1-b^1_{B_r})f_1^1,f_2^2\big)(x)\right| \le Cr^{\frac{n}{p}}\|\vec{b}\|_{BMO^2}\prod_{i=1}^2\int_{2r}^\infty \|f_i\|_{L^{p_i}(B_{t_i})}\, t_i^{-\frac{n}{p_i}-1}dt_i, \tag{3.18}$$

for any $x\in B_r$. Using the condition (1.2), the following inequality

$$\left|T\big((b_1-b^1_{B_r})f_1^1,f_2^2\big)(x)\right| \le C(2r)^{-n}\int_{B_{2r}}|b_1(y_1)-b^1_{B_r}||f_1(y_1)|dy_1\int_{(B_{2r})^c}\frac{|f_2(y_2)|}{|x_0-y_2|^n}dy_2$$

is satisfied for any $x\in B_r$. By using Hölder's inequality, it is obvious that

$$\begin{aligned}&(2r)^{-n}\int_{B_{2r}}|b_1(y_1)-b^1_{B_r}||f_1(y_1)|dy_1\\&\le (2r)^{-n}\left(\int_{B_{2r}}|f_1(y_1)|^{p_1}dy_1\right)^{\frac{1}{p_1}}\left(\int_{B_{2r}}|b_1(y_1)-b^1_{B_r}|^{p_1'}dy_1\right)^{\frac{1}{p_1'}}\\&\le (2r)^{-n}\|b_1\|_{BMO}\|f_1\|_{L^{p_1}(B_{2r})}|B_{2r}|^{1-\frac{1}{p_1}}\\&\le \|b_1\|_{BMO}\int_{2r}^\infty \|f_1\|_{L^{p_1}(B_{t_1})}t_1^{-\frac{n}{p_1}-1}dt_1.\end{aligned} \tag{3.19}$$

Hence, by (3.14) and (3.19), it follows that

$$\begin{aligned}&\big|T\big((b_1-b^1_{B_r})f^1_1,f^2_2\big)(x)\big|\\&\le C\|b_1\|_{BMO}\int_{2r}^{\infty}\|f_1\|_{L^{p_1}(B_{t_1})}t_1^{-\frac{n}{p_1}-1}dt_1\int_{2r}^{\infty}\|f_2\|_{L^{p_2}(B_{t_2})}t_2^{-\frac{n}{p_2}-1}dt_2\\&=C\|b_1\|_{BMO}\prod_{i=1}^{2}\int_{2r}^{\infty}\|f_i\|_{L^{p_i}(B_{t_i})}\ t_i^{-\frac{n}{p_i}-1}dt_i.\end{aligned}$$

Thus,

$$\widetilde{I}_{22}\le Cr^{\frac{n}{p}}\|\vec{b}\|_{BMO^2}\prod_{i=1}^{2}\int_{2r}^{\infty}\|f_i\|_{L^{p_i}(B_{t_i})}\ t_i^{-\frac{n}{p_i}-1}dt_i.$$

Combining the above inequalities for $\widetilde{I}_{21}$ and $\widetilde{I}_{22}$, we have

$$\widetilde{I}_{2}\le Cr^{\frac{n}{p}}\|\vec{b}\|_{BMO^2}\prod_{i=1}^{2}\int_{2r}^{\infty}\|f_i\|_{L^{p_i}(B_{t_i})}\ t_i^{-\frac{n}{p_i}-1}dt_i.$$

For $\widetilde{I}_3$, we have

$$\begin{aligned}\widetilde{I}_3&=\|(b_1-b^1_{B_r})T(f^2_1,f^1_2)-T((b_1-b^1_{B_r})f^2_1,f^1_2)\|_{L^p(B_r)}\\&\le\|(b_1-b^1_{B_r})T(f^2_1,f^1_2)\|_{L^p(B_r)}+\|T((b_1-b^1_{B_r})f^2_1,f^1_2)\|_{L^p(B_r)}\\&:=\widetilde{I}_{31}+\widetilde{I}_{32}.\end{aligned}$$

Similarly to (3.12), we have

$$\begin{aligned}\widetilde{I}_{31}&\le C\|(b_1-b^1_{B_r})\|_{L^p(B_r)}\prod_{i=1}^{2}\int_{2r}^{\infty}\|f_i\|_{L^{p_i}(B_{t_i})}\ t_i^{-\frac{n}{p_i}-1}dt_i\\&\le Cr^{\frac{n}{p}}\|\vec{b}\|_{BMO^2}\prod_{i=1}^{2}\int_{2r}^{\infty}\|f_i\|_{L^{p_i}(B_{t_i})}\ t_i^{-\frac{n}{p_i}-1}dt_i.\end{aligned}$$

In order to estimate $\widetilde{I}_{32}$, we first have to prove the below inequality:

$$\big|T\big((b_1-b^1_{B_r})f^2_1,f^1_2\big)(x)\big|\le C\|b_1\|_{BMO}\prod_{i=1}^{2}\int_{2r}^{\infty}(1+\ln\frac{t_i}{r})\|f_i\|_{L^{p_i}(B_{t_i})}\ t_i^{-\frac{n}{p_i}-1}dt_i, \tag{3.20}$$

for any $x\in B_r$. Using the condition (1.2), the following inequality

$$\big|T\big((b_1-b^1_{B_r})f^2_1,f^1_2\big)(x)\big|\le C\bigg(\int_{(B_{2r})^c}\frac{|b_1(y_1)-b^1_{B_r}||f_1(y_1)|}{|x_0-y_1|^n}dy_1\bigg)\bigg(\frac{1}{(2r)^n}\int_{B_{2r}}|f_2(y_2)|dy_2\bigg)$$

is satisfied for any $x \in B_r$. By using Hölder's inequality and Lemma 2.3, we have

$$\begin{aligned}
&\int_{(B_{2r})^c} \frac{|b_1(y_1)-b^1_{B_r}||f_1(y_1)|}{|x_0-y_1|^n}dy_1\\
&= C\int_{(B_{2r})^c}|b_1(y_1)-b^1_{B_r}||f_1(y_1)|\int_{|x_0-y_1|}^{\infty}\frac{dt_1}{t_1^{n+1}}dy_1\\
&\le C\int_{2r}^{\infty}\int_{B_{t_1}}|b_1(y_1)-b^1_{B_r}||f_1(y_1)|\frac{1}{t_1^{n+1}}dy_1dt_1\\
&\le C\int_{2r}^{\infty}\Big(\int_{B_{t_1}}|f_1(y_1)|^{p_1}dy_1\Big)^{\frac{1}{p_1}}\Big(\int_{B_{t_1}}|b_1(y_1)-b^1_{B_r}|^{p_1'}dy_1\Big)^{\frac{1}{p_1'}}\frac{dt_1}{t_1^{n+1}}\\
&\le C\|b_1\|_{BMO}\int_{2r}^{\infty}(1+\ln\frac{t_1}{r})\|f_1\|_{L^{p_1}(B_{t_1})}t_1^{-\frac{n}{p_1}-1}dt_1.
\end{aligned}\tag{3.21}$$

Hence, by (3.21) and the simlilarity of (3.13), it follows that

$$\begin{aligned}
&\big|T\big((b_1-b^1_{B_r})f_1^2,f_2^1\big)(x)\big|\\
&\le C\|b_1\|_{BMO}\int_{2r}^{\infty}(1+\ln\frac{t_1}{r})\|f_1\|_{L^{p_1}(B_{t_1})}t_1^{-\frac{n}{p_1}-1}dt_1\int_{2r}^{\infty}\|f_2\|_{L^{p_2}(B_{t_2})}t_2^{-\frac{n}{p_2}-1}dt_2\\
&\le C\|b_1\|_{BMO}\prod_{i=1}^{2}\int_{2r}^{\infty}(1+\ln\frac{t_i}{r})\|f_i\|_{L^{p_i}(B_{t_i})}\ t_i^{-\frac{n}{p_i}-1}dt_i.
\end{aligned}$$

Thus,

$$\widetilde{I}_{32}\le Cr^{\frac{n}{p}}\|\vec{b}\|_{BMO^2}\prod_{i=1}^{2}\int_{2r}^{\infty}(1+\ln\frac{t_i}{r})\|f_i\|_{L^{p_i}(B_{t_i})}\ t_i^{-\frac{n}{p_i}-1}dt_i.$$

By combining the above inequalities for $\widetilde{I}_{31}$, $\widetilde{I}_{32}$, we have

$$\widetilde{I}_{3}\le Cr^{\frac{n}{p}}\|\vec{b}\|_{BMO^2}\prod_{i=1}^{2}\int_{2r}^{\infty}(1+\ln\frac{t_i}{r})\|f_i\|_{L^{p_i}(B_{t_i})}\ t_i^{-\frac{n}{p_i}-1}dt_i.$$

For $\widetilde{I}_4$, we have

$$\begin{aligned}
\widetilde{I}_4 &= \|(b_1-b^1_{B_r})T(f_1^2,f_2^2)-T((b_1-b^1_{B_r})f_1^2,f_2^2)\|_{L^p(B_r)}\\
&\le \|(b_1-b^1_{B_r})T(f_1^2,f_2^2)\|_{L^p(B_r)}+\|T((b_1-b^1_{B_r})f_1^2,f_2^2)\|_{L^p(B_r)}\\
&:= \widetilde{I}_{41}+\widetilde{I}_{42}.
\end{aligned}$$

By (3.15), we have

$$\begin{aligned}
\widetilde{I}_{41} &\le C\|b_1-b^1_{B_r}\|_{L^p(B_r)}\prod_{i=1}^{2}\int_{2r}^{\infty}\|f_i\|_{L^{p_i}(B_{t_i})}\ t_i^{-\frac{n}{p_i}-1}dt_i\\
&\le Cr^{\frac{n}{p}}\|\vec{b}\|_{BMO^2}\prod_{i=1}^{2}\int_{2r}^{\infty}\|f_i\|_{L^{p_i}(B_{t_i})}\ t_i^{-\frac{n}{p_i}-1}dt_i.
\end{aligned}$$

By the condition (1.2), (3.14) and (3.21), we can obtain for any $x \in B_r$,

$$\begin{aligned}
&\big|T\big((b_1 - b^1_{B_r})f^2_1, f^2_2\big)(x)\big| \\
&\le C\int_{(B_{2r})^c} \frac{|b_1(y_1) - b^1_{B_r}||f_1(y_1)|}{|x_0 - y_1|^n} dy_1 \int_{(B_{2r})^c} \frac{|f_2(y_2)|}{|x_0 - y_2|^n} dy_2 \\
&\le C\|b_1\|_{BMO} \int_{2r}^{\infty} (1 + \ln \frac{t_1}{r})\|f_1\|_{L^{p_1}(B_{t_1})} t_1^{-\frac{n}{p_1}-1} dt_1 \int_{2r}^{\infty} \|f_2\|_{L^{p_2}(B_{t_2})} t_2^{-\frac{n}{p_2}-1} dt_2 \qquad (3.22)\\
&\le C\|b_1\|_{BMO} \prod_{i=1}^{2} \int_{2r}^{\infty} (1 + \ln \frac{t_i}{r})\|f_i\|_{L^{p_i}(B_{t_i})} t_i^{-\frac{n}{p_i}-1} dt_i.
\end{aligned}$$

Thus,

$$\widetilde{I}_{42} \le Cr^{\frac{n}{p}} \|\vec{b}\|_{BMO^2} \prod_{i=1}^{2} \int_{2r}^{\infty} (1 + \ln \frac{t_i}{r})\|f_i\|_{L^{p_i}(B_{t_i})}\ t_i^{-\frac{n}{p_i}-1} dt_i.$$

By combining the above inequalities for $\widetilde{I}_{41}$, $\widetilde{I}_{42}$, we have

$$\widetilde{I}_{4} \le Cr^{\frac{n}{p}} \|\vec{b}\|_{BMO^2} \prod_{i=1}^{2} \int_{2r}^{\infty} (1 + \ln \frac{t_i}{r})\|f_i\|_{L^{p_i}(B_{t_i})}\ t_i^{-\frac{n}{p_i}-1} dt_i.$$

Combining the estimates of $\widetilde{I}_j (j = 1, \cdots, 4)$, we get

$$\|T^1_{\vec{b}}(f_1, f_2)\|_{L^p(B_r)} \le Cr^{\frac{n}{p}} \|\vec{b}\|_{BMO^2} \prod_{i=1}^{2} \int_{2r}^{\infty} (1 + \ln \frac{t_i}{r})\|f_i\|_{L^{p_i}(B_{t_i})}\ t_i^{-\frac{n}{p_i}-1} dt_i.$$

Similarly,

$$\|T^2_{\vec{b}}(f_1, f_2)\|_{L^p(B_r)} \le Cr^{\frac{n}{p}} \|\vec{b}\|_{BMO^2} \prod_{i=1}^{2} \int_{2r}^{\infty} (1 + \ln \frac{t_i}{r})\|f_i\|_{L^{p_i}(B_{t_i})}\ t_i^{-\frac{n}{p_i}-1} dt_i.$$

Thus,

$$\|T_{\vec{b}}(f_1, f_2)\|_{L^p(B_r)} \le Cr^{\frac{n}{p}} \|\vec{b}\|_{BMO^2} \prod_{i=1}^{2} \int_{2r}^{\infty} (1 + \ln \frac{t_i}{r})\|f_i\|_{L^{p_i}(B_{t_i})}\ t_i^{-\frac{n}{p_i}-1} dt_i.$$

And then, we estimate the following inequality

$$\|T_{\Pi\vec{b}}(\vec{f})\|_{L^p(B_r)} \le Cr^{\frac{n}{p}} \|b_1\|_{BMO}\|b_2\|_{BMO} \prod_{i=1}^{2} \int_{2r}^{\infty} (1 + \ln \frac{t_i}{r})\|f_i\|_{L^{p_i}(B_{t_i})} t_i^{-\frac{n}{p_i}-1} dt_i.$$

We have

$$\begin{aligned}
\|T_{\Pi\vec{b}}(f_1, f_2)\|_{L^p(B_r)} &\le \|T_{\Pi\vec{b}}(f^1_1, f^1_2)\|_{L^p(B_r)} + \|T_{\Pi\vec{b}}(f^1_1, f^2_2)\|_{L^p(B_r)} \\
&+ \|T_{\Pi\vec{b}}(f^2_1, f^1_2)\|_{L^p(B_r)} + \|T_{\Pi\vec{b}}(f^2_1, f^2_2)\|_{L^p(B_r)} \\
&:= \sum_{j=1}^{4} \widetilde{H}_j.
\end{aligned}$$

By Lemma 2.4, we get

$$\begin{aligned}
\widetilde{H}_1 &\le C\|b_1\|_{BMO}\|b_2\|_{BMO}\|f_1\|_{L^{p_1}(B_{2r})}\|f_2\|_{L^{p_2}(B_{2r})} \\
&\le C\|b_1\|_{BMO}\|b_2\|_{BMO} r^{\frac{n}{p}} \|f_1\|_{L^{p_1}(B_{2r})}\|f_2\|_{L^{p_2}(B_{2r})} \int_{2r}^{\infty} t_1^{-\frac{n}{p_1}-1} dt_1 \int_{2r}^{\infty} t_2^{-\frac{n}{p_2}-1} dt_2 \\
&\le Cr^{\frac{n}{p}} \|b_1\|_{BMO}\|b_2\|_{BMO} \prod_{i=1}^{2} \int_{2r}^{\infty} (1 + \ln \frac{t_i}{r})\|f_i\|_{L^{p_i}(B_{t_i})} t_i^{-\frac{n}{p_i}-1} dt_i.
\end{aligned}$$

And we have the following decomposition,

$$
\begin{aligned}
T_{\Pi\vec{b}}(f_1^1,f_2^2)(x)=&(b_1(x)-b^1_{B_r})(b_2(x)-b^2_{B_r})T(f_1^1,f_2^2)(x)\\
&-(b_1(x)-b^1_{B_r})T\left(f_1^1,(b_2-b^2_{B_r})f_2^2\right)(x)\\
&-(b_2(x)-b^2_{B_r})T\left((b_1-b^1_{B_r})f_1^1,f_2^2\right)(x)\\
&+T\left((b_1-b^1_{B_r})f_1^1,(b_2-b^2_{B_r})f_2^2\right)(x).
\end{aligned}\tag{3.23}
$$

Thus,

$$
\begin{aligned}
\widetilde{H}_2\le&\|(b_1-b^1_{B_r})(b_2-b^2_{B_r})T(f_1^1,f_2^2)\|_{L^p(B_r)}\\
&+\|(b_1-b^1_{B_r})T\left(f_1^1,(b_2-b^2_{B_r})f_2^2\right)\|_{L^p(B_r)}\\
&+\|(b_2-b^2_{B_r})T\left((b_1-b^1_{B_r})f_1^1,f_2^2\right)\|_{L^p(B_r)}\\
&+\|T\left((b_1-b^1_{B_r})f_1^1,(b_2-b^2_{B_r})f_2^2\right)\|_{L^p(B_r)}\\
:=&\sum_{j=1}^4\widetilde{H}_{2j}.
\end{aligned}
$$

By (3.12) and Hölder's inequality, we have

$$
\begin{aligned}
\widetilde{H}_{21}&\le C\|(b_1-b^1_{B_r})(b_2-b^2_{B_r})\|_{L^p(B_r)}\prod_{i=1}^2\int_{2r}^\infty\|f_i\|_{L^{p_i}(B_{t_i})}\,t_i^{-\frac{n}{p_i}-1}dt_i\\
&\le Cr^{\frac{n}{p}}\|b_1\|_{BMO}\|b_2\|_{BMO}\prod_{i=1}^2\int_{2r}^\infty(1+\ln\frac{t_i}{r})\|f_i\|_{L^{p_i}(B_{t_i})}\,t_i^{-\frac{n}{p_i}-1}dt_i.
\end{aligned}
$$

By the similarity of (3.20), we have

$$
\begin{aligned}
\widetilde{H}_{22}&\le C\|b_1-b^1_{B_r}\|_{L^p(B_r)}\|b_2\|_{BMO}\prod_{i=1}^2\int_{2r}^\infty(1+\ln\frac{t_i}{r})\|f_i\|_{L^{p_i}(B_{t_i})}\,t_i^{-\frac{n}{p_i}-1}dt_i\\
&\le Cr^{\frac{n}{p}}\|b_1\|_{BMO}\|b_2\|_{BMO}\prod_{i=1}^2\int_{2r}^\infty(1+\ln\frac{t_i}{r})\|f_i\|_{L^{p_i}(B_{t_i})}\,t_i^{-\frac{n}{p_i}-1}dt_i.
\end{aligned}
$$

By (3.18), we have

$$
\begin{aligned}
\widetilde{H}_{23}&\le C\|b_2-b^2_{B_r}\|_{L^p(B_r)}\|b_1\|_{BMO}\prod_{i=1}^2\int_{2r}^\infty\|f_i\|_{L^{p_i}(B_{t_i})}\,t_i^{-\frac{n}{p_i}-1}dt_i\\
&\le Cr^{\frac{n}{p}}\|b_1\|_{BMO}\|b_2\|_{BMO}\prod_{i=1}^2\int_{2r}^\infty(1+\ln\frac{t_i}{r})\|f_i\|_{L^{p_i}(B_{t_i})}\,t_i^{-\frac{n}{p_i}-1}dt_i.
\end{aligned}
$$

By the condition (1.2), Hölder's inequality, (3.19) and the similarity of (3.21), we have

$$
\begin{aligned}
&\left|T\big((b_1-b^1_{B_r})f_1^1,(b_2-b^2_{B_r})f_2^2\big)(x)\right|\\
&\le C(2r)^{-n}\int_{B_{2r}}|b_1(y_1)-b^1_{B_r}||f_1(y_1)|dy_1\int_{(B_{2r})^c}\frac{|(b_2(y_2)-b^2_{B_r})f_2(y_2)|}{|x_0-y_2|^n}dy_2\\
&\le C\|b_1\|_{BMO}\left(\int_{2r}^\infty\|f_1\|_{L^{p_1}(B_{t_1})}t_1^{-\frac{n}{p_1}-1}dt_1\right)\|b_2\|_{BMO}\\
&\quad\times\left(\int_{2r}^\infty(1+\ln\frac{t_2}{r})\|f_2\|_{L^{p_2}(B_{t_2})}t_2^{-\frac{n}{p_2}-1}dt_2\right)\\
&\le C\|b_1\|_{BMO}\|b_2\|_{BMO}\prod_{i=1}^2\int_{2r}^\infty(1+\ln\frac{t_i}{r})\|f_i\|_{L^{p_i}(B_{t_i})}t_i^{-\frac{n}{p_i}-1}dt_i.
\end{aligned}
$$

Thus,

$$\widetilde{H}_{24} \le Cr^{\frac{n}{p}} \|b_1\|_{BMO} \|b_2\|_{BMO} \prod_{i=1}^{2} \int_{2r}^{\infty} (1+\ln \frac{t_i}{r}) \|f_i\|_{L^{p_i}(B_{t_i})} \, t_i^{-\frac{n}{p_i}-1} dt_i.$$

By combining the above inequalities for $\widetilde{H}_{21} \sim \widetilde{H}_{24}$, we have

$$\widetilde{H}_2 \le Cr^{\frac{n}{p}} \|b_1\|_{BMO} \|b_2\|_{BMO} \prod_{i=1}^{2} \int_{2r}^{\infty} (1+\ln \frac{t_i}{r}) \|f_i\|_{L^{p_i}(B_{t_i})} \, t_i^{-\frac{n}{p_i}-1} dt_i.$$

Similarly, $\widetilde{H}_3$ has the same estimate as above, thus the inequality

$$\widetilde{H}_3 \le Cr^{\frac{n}{p}} \|b_1\|_{BMO} \|b_2\|_{BMO} \prod_{i=1}^{2} \int_{2r}^{\infty} (1+\ln \frac{t_i}{r}) \|f_i\|_{L^{p_i}(B_{t_i})} \, t_i^{-\frac{n}{p_i}-1} dt_i$$

is valid.

Then similar to the decomposition of (3.23), we have

$$\begin{aligned}
\widetilde{H}_4 \le & \|(b_1 - b_{B_r}^1)(b_2 - b_{B_r}^2) T(f_1^2, f_2^2)\|_{L^p(B_r)} \\
& + \|(b_1 - b_{B_r}^1) T\left(f_1^2, (b_2 - b_{B_r}^2) f_2^2\right)\|_{L^p(B_r)} \\
& + \|(b_2 - b_{B_r}^2) T\left((b_1 - b_{B_r}^1) f_1^2, f_2^2\right)\|_{L^p(B_r)} \\
& + \|T\left((b_1 - b_{B_r}^1) f_1^2, (b_2 - b_{B_r}^2) f_2^2\right)\|_{L^p(B_r)} \\
:= & \sum_{j=1}^{4} \widetilde{H}_{4j}.
\end{aligned}$$

By (3.12) and Hölder's inequality, we have

$$\begin{aligned}
\widetilde{H}_{41} & \le C \|(b_1 - b_{B_r}^1)(b_2 - b_{B_r}^2)\|_{L^p(B_r)} \prod_{i=1}^{2} \int_{2r}^{\infty} \|f_i\|_{L^{p_i}(B_{t_i})} \, t_i^{-\frac{n}{p_i}-1} dt_i \\
& \le Cr^{\frac{n}{p}} \|b_1\|_{BMO} \|b_2\|_{BMO} \prod_{i=1}^{2} \int_{2r}^{\infty} (1+\ln \frac{t_i}{r}) \|f_i\|_{L^{p_i}(B_{t_i})} \, t_i^{-\frac{n}{p_i}-1} dt_i.
\end{aligned}$$

By the similarity of (3.22), we have

$$\begin{aligned}
\widetilde{H}_{42} & \le C \|b_1 - b_{B_r}^1\|_{L^p(B_r)} \|b_2\|_{BMO} \prod_{i=1}^{2} \int_{2r}^{\infty} (1+\ln \frac{t_i}{r}) \|f_i\|_{L^{p_i}(B_{t_i})} \, t_i^{-\frac{n}{p_i}-1} dt_i \\
& \le Cr^{\frac{n}{p}} \|b_1\|_{BMO} \|b_2\|_{BMO} \prod_{i=1}^{2} \int_{2r}^{\infty} (1+\ln \frac{t_i}{r}) \|f_i\|_{L^{p_i}(B_{t_i})} \, t_i^{-\frac{n}{p_i}-1} dt_i.
\end{aligned}$$

By (3.22), we have

$$\widetilde{H}_{43} \le Cr^{\frac{n}{p}} \|b_1\|_{BMO} \|b_2\|_{BMO} \prod_{i=1}^{2} \int_{2r}^{\infty} (1+\ln \frac{t_i}{r}) \|f_i\|_{L^{p_i}(B_{t_i})} \, t_i^{-\frac{n}{p_i}-1} dt_i.$$

By the condition (1.2), Hölder's inequality and the similarity of (3.21), we have

$$\begin{aligned}
& \left|T\big((b_1 - b_{B_r}^1) f_1^2, (b_2 - b_{B_r}^2) f_2^2\big)(x)\right| \\
& \le C \left(\int_{(B_{2r})^c} \frac{|(b_1(y_1) - b_{B_r}^1) f_1(y_1)|}{|x_0 - y_1|^n} dy_2 \right) \left(\int_{(B_{2r})^c} \frac{|(b_2(y_2) - b_{B_r}^2) f_2(y_2)|}{|x_0 - y_2|^n} dy_2 \right) \\
& \le C \|b_1\|_{BMO} \|b_2\|_{BMO} \prod_{i=1}^{2} \int_{2r}^{\infty} (1+\ln \frac{t_i}{r}) \|f_i\|_{L^{p_i}(B_{t_i})} \, t_i^{-\frac{n}{p_i}-1} dt_i.
\end{aligned}$$

Thus,

$$\widetilde{H}_{44} \leq Cr^{\frac{n}{p}} \|b_1\|_{BMO} \|b_2\|_{BMO} \prod_{i=1}^{2} \int_{2r}^{\infty} (1+\ln \frac{t_i}{r}) \|f_i\|_{L^{p_i}(B_{t_i})}\ t_i^{-\frac{n}{p_i}-1} dt_i.$$

By combining the above inequalities for $\widetilde{H}_{41} \sim \widetilde{H}_{44}$, we have

$$\widetilde{H}_{4} \leq Cr^{\frac{n}{p}} \|b_1\|_{BMO} \|b_2\|_{BMO} \prod_{i=1}^{2} \int_{2r}^{\infty} (1+\ln \frac{t_i}{r}) \|f_i\|_{L^{p_i}(B_{t_i})}\ t_i^{-\frac{n}{p_i}-1} dt_i.$$

Combining the estimates of $\widetilde{H}_j (j=1,\cdots,4)$, we get

$$\|T_{\Pi\vec{b}}(\vec{f})\|_{L^p(B_r)} \leq Cr^{\frac{n}{p}} \|b_1\|_{BMO} \|b_2\|_{BMO} \prod_{i=1}^{2} \int_{2r}^{\infty} (1+\ln \frac{t_i}{r}) \|f_i\|_{L^{p_i}(B_{t_i})} t_i^{-\frac{n}{p_i}-1} dt_i,$$

which completes the proof of Theorem 3.5. □

Theorem 3.6. *Let $m \geq 2$, T be an m-linear singular integral operator with generalized kernel and $\sum_{k_i=1}^{\infty} k_i C_{k_i} < \infty$, $i=1,\cdots,m$. Suppose for fixed $1 \leq r_1,\cdots,r_m \leq s'$ with $1/r = 1/r_1 + \cdots + 1/r_m$, T is bounded from $L^{r_1} \times \cdots \times L^{r_m}$ into $L^{r,\infty}$. Let $x_0 \in \mathbb{R}^n$ and $\vec{b} \in BMO^m$. If the functions φ, $\varphi_j : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ $(i=1\cdots m)$ and $(\varphi_1,\varphi_2,\cdots,\varphi_m,\varphi)$ satisfy the condition*

$$\prod_{i=1}^{m} \int_{2r}^{\infty} (1+\ln \frac{t_i}{r}) \frac{\operatorname*{ess\,inf}_{t_i<\tau_i<\infty} \varphi_i(x_0,\tau_i)\, \tau_i^{\frac{n}{p_i}}}{t_i^{\frac{n}{p_i}+1}} \mathrm{d}t_i \leq C\varphi(x_0,r), \tag{3.24}$$

where C does not depend on r, then for any $s' < p_1,\cdots,p_m < \infty$ with $1/p = \sum_{i=1}^{m} 1/p_i$ and $1<p<\infty$, the operator $T_{\vec{b}}$ and $T_{\Pi\vec{b}}$ is bounded from $\mathrm{LM}_{p_1,\varphi_1}^{\{x_0\}} \times \cdots \times \mathrm{LM}_{p_m,\varphi_m}^{\{x_0\}}$ into $\mathrm{LM}_{p,\varphi}^{\{x_0\}}$.

Proof. By (3.17) and the condition (3.24), we have

$$\begin{aligned}
&\prod_{i=1}^{m} \int_{2r}^{\infty} (1+\ln \frac{t_i}{r}) \|f_i\|_{L^{p_i}(B_{t_i})} t_i^{-\frac{n}{p_i}-1} dt_i \\
&= \int_{2r}^{\infty} \cdots \int_{2r}^{\infty} \frac{\prod_{i=1}^{m} \|f_i\|_{L^{p_i}(B_{t_i})}}{\prod_{i=1}^{m} \operatorname*{ess\,inf}_{0<t_i<\tau_i<\infty} \varphi_i(x_0,\tau_i)\tau_i^{\frac{n}{p_i}}} \frac{\prod_{i=1}^{m} \operatorname*{ess\,inf}_{0<t_i<\tau_i<\infty} \varphi_i(x_0,\tau_i)\tau_i^{\frac{n}{p_i}}}{\prod_{i=1}^{m} (1+\ln \frac{t_i}{r})^{-1}\ t_i^{\frac{n}{p_i}+1}} dt_1 \cdots dt_m \\
&\leq \prod_{i=1}^{m} \|f_i\|_{LM_{p_i,\varphi_i}^{\{x_0\}}} \prod_{i=1}^{m} \int_{2r}^{\infty} (1+\ln \frac{t_i}{r}) \frac{\operatorname*{ess\,inf}_{t_i<\tau_i<\infty} \varphi_i(x_0,\tau_i)\, \tau_i^{\frac{n}{p_i}}}{t_i^{\frac{n}{p_i}+1}} dt_i \\
&\leq C\varphi(x_0,r) \prod_{i=1}^{m} \|f_i\|_{LM_{p_i,\varphi_i}^{\{x_0\}}}.
\end{aligned}$$

Then by Theorem 3.5, we have

$$\begin{aligned}
&\|T_{\vec{b}}(\vec{f})\|_{LM_{p_i,\varphi_i}^{\{x_0\}}} \\
&= \sup_{r>0} \varphi(x_0,r)^{-1} |B_r|^{-\frac{1}{p}} \|T_{\vec{b}}(\vec{f})\|_{L^p(B_r)} \\
&\leq C \sup_{r>0} \varphi(x_0,r)^{-1} |B_r|^{-\frac{1}{p}}\ r^{\frac{n}{p}} \|\vec{b}\|_{BMO^m} \prod_{i=1}^{m} \int_{2r}^{\infty} (1+\ln \frac{t_i}{r}) \|f_i\|_{L^{p_i}(B_{t_i})}\ t_i^{-\frac{n}{p_i}-1} dt_i \\
&\leq C\|\vec{b}\|_{BMO^m} \prod_{i=1}^{m} \|f_i\|_{LM_{p_i,\varphi_i}^{\{x_0\}}},
\end{aligned}$$

and

$$\begin{aligned}\|T_{\Pi_{\vec{b}}}(\vec{f})\|_{LM^{\{x_0\}}_{p_i,\varphi_i}} &= \sup_{r>0}\varphi(x_0,r)^{-1}|B_r|^{-\frac{1}{p}}\|T_{\Pi_{\vec{b}}}(\vec{f})\|_{L^p(B_r)}\\ &\le C\sup_{r>0}\varphi(x_0,r)^{-1}|B_r|^{-\frac{1}{p}}\ r^{\frac{n}{p}}\prod_{j=1}^m\|b_j\|_{BMO}\\ &\quad\times\prod_{i=1}^m\int_{2r}^{\infty}(1+\ln\frac{t_i}{r})\|f_i\|_{L^{p_i}(B_{t_i})}\ t_i^{-\frac{n}{p_i}-1}dt_i\\ &\le C\prod_{j=1}^m\|b_j\|_{BMO}\prod_{i=1}^m\|f_i\|_{LM^{\{x_0\}}_{p_i,\varphi_i}},\end{aligned}$$

which completes the proof of Theorem 3.6. □

Since the generalized Morrey space is the generalization of the classical Morrey space mentioned as in Remark 1.9, we can obtain the following two results as corollaries.

Corollary 3.7. *Let $m\ge 2$, T be an m-linear singular integral operator with generalized kernel and $\sum_{k_i=1}^{\infty}C_{k_i}<\infty$, $i=1,\cdots,m$. Suppose for fixed $1\le r_1,\cdots,r_m\le s'$ with $1/r=1/r_1+\cdots+1/r_m$, T is bounded from $L^{r_1}\times\cdots\times L^{r_m}$ into $L^{r,\infty}$. Then for any $s'<p_j\le q_j<\infty$ $(j=1,\cdots,m)$ with $1/p=\sum_{i=1}^m 1/p_i$, $1/q=\sum_{i=1}^m 1/q_i$ and $1<p\le q<\infty$, T is bounded from $M^{q_1}_{p_1}(\mathbb{R}^n)\times\cdots\times M^{q_m}_{p_m}(\mathbb{R}^n)$ into $M^q_p(\mathbb{R}^n)$.*

Corollary 3.8. *Let $m\ge 2$, T be an m-linear singular integral operator with generalized kernel and $\sum_{k_i=1}^{\infty}k_iC_{k_i}<\infty$, $i=1,\cdots,m$. Suppose for fixed $1\le r_1,\cdots,r_m\le s'$ with $1/r=1/r_1+\cdots+1/r_m$, T is bounded from $L^{r_1}\times\cdots\times L^{r_m}$ into $L^{r,\infty}$. If $\vec{b}\in BMO^m$, then for any $s'<p_j\le q_j<\infty$ $(j=1,\cdots,m)$ with $1/p=\sum_{i=1}^m 1/p_i$, $1/q=\sum_{i=1}^m 1/q_i$ and $1<p\le q<\infty$, $T_{\vec{b}}$ and $T_{\Pi\vec{b}}$ are bounded from $M^{q_1}_{p_1}(\mathbb{R}^n)\times\cdots\times M^{q_m}_{p_m}(\mathbb{R}^n)$ into $M^q_p(\mathbb{R}^n)$.*

4. Applications

Due to the relationship of the standard m-linear Calderón-Zygmund operator and the m-linear singular integral operator with generalized kernel, the conclusions we get above also can be applied to the standard m-linear Calderón-Zygmund operator.

Moreover, as we have discussed in Section 1, the condition (1.4) implies the condition (1.5) by putting $C_{k_i}=\rho(2^{-k_i})^{\frac{1}{m}}, i=1,\dots,m$, for any $1<s<\infty$. The m-linear singular integral operator with kernel of type ρ is exactly a special case of the m-linear singular integral operator with generalized kernel.

Let $\rho(t)$ be a non-negative and non-decreasing function on $\mathbb{R}^+$ with $0<\rho(1)<\infty$. For $a>0$, we say $\rho\in\text{Dini}(a)$ if

$$[\rho]_{\text{Dini}(a)}:=\int_0^1\frac{\rho^a(t)}{t}dt<\infty.$$

And ρ is said to satisfy the log-Dini(a) condition if the following inequality holds:

$$[\rho]_{\log\text{-Dini}(a)}:=\int_0^1\frac{\rho^a(t)}{t}\left(1+\log\frac{1}{t}\right)dt<\infty.$$

If $\rho\in\text{Dini}(\frac{1}{m})$ and $C_{k_i}=\rho(2^{-k_i})^{\frac{1}{m}}$, then for any $i=1,\dots,m$,

$$\sum_{k_i=1}^{\infty}C_{k_i}=\sum_{k_i=1}^{\infty}\rho(2^{-k_i})^{\frac{1}{m}}\le C\int_0^1\frac{\rho^{\frac{1}{m}}(t)}{t}dt<\infty.$$

And if $\rho \in \log\text{-Dini}(\frac{1}{m})$ and $C_{k_i} = \rho(2^{-k_i})^{\frac{1}{m}}$, then for any $i = 1, \ldots, m$,

$$\sum_{k_i=1}^{\infty} k_i C_{k_i} = \sum_{k_i=1}^{\infty} k_i \rho(2^{-k_i})^{\frac{1}{m}} \leq C \int_0^1 \frac{\rho^{\frac{1}{m}}(t)}{t} \left(1 + \log \frac{1}{t}\right) dt < \infty.$$

Thus, the conclusions of Theorem 3.1, Theorem 3.3, Theorem 3.4 and Corollary 3.7 can be applied to the m-linear singular integral operator with kernel of type ρ when $\rho \in \text{Dini}(\frac{1}{m})$. And the results of Theorem 3.2, Theorem 3.5, Theorem 3.6 and Corollary 3.8 can be applied to the m-linear singular integral operator with kernel of type ρ if $\rho \in \log\text{-Dini}(\frac{1}{m})$. We omit the proof of the applications since the similar process can be found in [19].

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