DETERMINATION THE IMPORTANCE OF VERTICES IN GRAPH

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ABSTRACT. Betweenness is a good measure of the centrality of a vertex in a graph modeling social or communication network. Betweenness centrality is a measure of centrality in a graph based on shortest paths. For every pair of vertices in a connected graph, there exists at least one shortest path between the vertices such that the number of edges that the path passes through is minimized. The betweenness centrality for each vertex is usually defined as the fraction of these shortest paths that pass through the vertex. Therefore, in many models, betweenness is a measure of the influence of a node in the dissemination of information over a network and can also be used to detect communities or cluster in networks. In this paper, we consider betweenness centrality of some Mycielski graphs.

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1. Introduction

In real-world analysis, one of the fundamental aims to find the most significant component of the network. The value of the significance of these components is generally expressed within central invariants. Until now, measurements related to many central invariants have been defined [8] and studied [13, 14, 15, 16, 17]. One of them is betweenness centrality. The concept of betweenness centrality was first introduced by Bavelas in 1948 [2].

Betweenness centrality finds wide application in network theory: it represents the degree of which nodes stand between each other. For example, in a telecommunications network, a node with higher betweenness centrality would have more control over the network, because more information will pass through that node. Betweenness centrality was devised as a general measure of centrality: it applies to a wide range of problems in network theory, including problems related to social network, biology, transport ans scientific cooperation [5, 6, 11, 18].

Betweenness is a good measure of the centrality of a vertex in a graph modeling social or communication network. It is usually defined as the fraction of these shortest paths that pass through the vertex. Betweenness centrality specifies how important a vertex is in the shortest paths between the other vertices pairs. Let take the information flows between each pair of vertices. The betweenness centrality measure is the value of the effect of a vertex over these information flows.

If there is only one geodesic connects each pair of vertices then the betweenness can be calculated so easily. In this situation, the communication in the network completely controlled by intermediate vertices. On the other hand, if there is more than one geodesic connecting a pair of vertices and the number of vertices n increases, then it becomes difficult to calculate the betweenness. In this case, the communication control in the network is divided into parts. Furthermore, a large networks can be thought of as it is made by joining smaller networks together. There are several graphs structure which results in a larger graph G and many of the properties of larger graphs can be derived from their constituent graphs. So, working on the parameter in some important graph structures gives us facilitie. We studied the betweenness centrality of some total graphs in [1] and now we study some Mycielski graphs for betweenness centrality in this paper. Mycielski graph is one of the important graph structure. In a search for triangle-free graphs with an arbitrarily large chromatic number, Mycielski developed a graph transformation that transforms a graph G into a new graph $\mu(G)$, which is called the Mycielski of G [7]. The Mycielski graph has fascinated graph teorists a great deal. This has resulted in studying several graph parameters of this graph [3, 4, 9, 10, 19, 20].

Definition 1.1. [12] Betweenness centrality $C_B(v)$ for a vertex v is defined as

$$C_B(v) = \sum_{s \neq v \neq t} \frac{\sigma_{st}(v)}{\sigma_{st}},$$

where σ_{st} is the number of shortest paths with vertices s and t as their end verices, while $\sigma_{st}(v)$ is the number of those shortest paths that include vertex v.

Definition 1.2. [12] The betweenness centrality of a graph G is defined as

$$C_B(G) = \frac{2\sum_{i=1}^{n}[C_B(v^*) - C_B(v_i)]}{(n-1)^2(n-2)},$$

where $C_B(v^*)$ is the largest value of $C_B(v_i)$ for any vertex v_i in the given graph G.

Throughout this paper, the following notation will be used. Let $G = (V, E)$ be a simple undirected graph of order n. The vertex set and edge set of a graph is denoted by $V(G)$ and $E(G)$, respectively. It is assumed that $V(G)$ will be abbreviated V. The shortest distance in G between two vertices u and v will be denoted $d(u, v)$. For any vertex $v \in V$, the open neighbourhood of v is $N(v) = \{u \in V(G) | uv \in E(G)\}$ and closed neighbourhood of v is $N[v] = N(v) \cup \{v\}$.

Example 1.3. Let calculate the betweenness centrality of all vertices in given the following graph (see Figure 1) with seven vertices. For the graph given in Figure 1, the shortest paths between each pair of vertices and consequently the contributions of the vertices of the graph are given in Table 1. The bottom row in Table 1 gives the betweenness centrality of each vertex. This value for any vertex is obtained from the sum of the values in the column of that vertex.

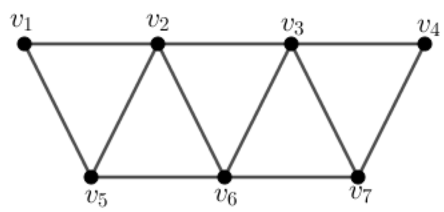

FIGURE 1. Connected graph with seven vertices

Table 1. Calculation of betweenness centrality of any vertex in Figure 1

pairs of vertices	geodesic paths	v_1	v_2	v_3	v_4	v_5	v_6	v_7
(v_1, v_3)	$v_1v_2v_3$	0	1	0	0	0	0	0
(v_1, v_4)	$v_1v_2v_3v_4$	0	1	1	0	0	0	0
(v_2, v_4)	$v_2v_3v_4$	0	0	1	0	0	0	0
(v_1, v_6)	$v_1v_2v_6$ $v_1v_5v_6$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0
(v_1, v_7)	$v_1v_2v_3v_7$ $v_1v_5v_6v_7$ $v_1v_2v_6v_7$	0	$\frac{2}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$	0
(v_2, v_7)	$v_2v_3v_7$ $v_2v_6v_7$	0	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0
(v_3, v_5)	$v_3v_2v_5$ $v_3v_6v_5$	0	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	0
(v_4, v_5)	$v_4v_3v_2v_5$ $v_4v_7v_6v_5$ $v_4v_3v_6v_5$	0	$\frac{1}{3}$	$\frac{2}{3}$	0	0	$\frac{2}{3}$	$\frac{1}{3}$
(v_4, v_6)	$v_4v_3v_6$ $v_4v_7v_6$	0	0	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$
(v_5, v_7)	$v_5v_6v_7$	0	0	0	0	0	1	0
	$C_B(v_i)$	0	4	4	0	$\frac{5}{6}$	$\frac{10}{3}$	$\frac{5}{6}$

Definition 1.4. [7] Mycielski introduced an interesting graph operation that extends a graph G to a new graph $\mu(G)$, called the Mycielski graph of G or the Mycielskian of G. Here we present the definition of the graph $\mu(G)$. For a given graph G with $V = V(G) = \{v_1, v_2, ..., v_n\}$, denote $V^{'} = \{v_1^{'}, v_2^{'}, ..., v_n^{'}\}$ to be the corresponding set of V, the Mycielski graph $\mu(G)$ of G is defined as follows: $V(\mu(G)) = V \cup V^{'} \cup \{u\}$, $E(\mu(G)) = E(G) \cup \{v_iv_j^{'} : v_iv_j \in E(G)\} \cup \{uv_i^{'} : 1 \leq i \leq n\}$. The Mycielski graph of P_6 is as shown in Figure 2.

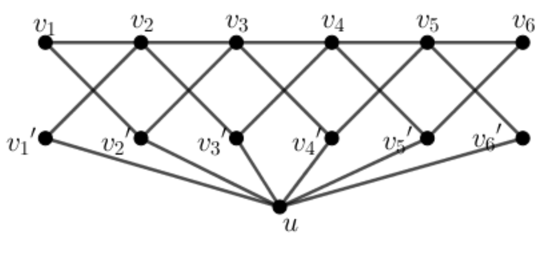

Figure 2. Mycielski graph of P_6

In this paper, we consider the betweenness centrality of some Mycielski graphs. In Section 2, we present some preliminary results about betweenness centrality. In Section 3, we calculate the exact values for betweenness centrality of Mycielski graphs underlying graphs of which are complete graphs, stars, wheel graphs, complete bipartite graphs.

2. Basic Results

In this section, we give some well-known results for the betweenness centrality in graphs.

Theorem 2.1. [12] *The betweenness centrality of a vertex v in a wheel graph W_n, $n > 5$, is given by*

$$C_B(v) = \begin{cases} \frac{(n-1)(n-5)}{2}, & \text{if } v \text{ is central vertex} \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

Theorem 2.2. [12] *Let K_n be a complete graph on n vertices and $e = (v_i, v_j)$ an edge of it. Then the betweenness centrality of vertices in $K_n - e$ is given by*

$$C_B(v) = \begin{cases} \frac{1}{n-2}, & \text{if } v \neq v_i, v_j \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 2.3. [12] *The betweenness centrality of a vertex in complete bipartite graph $K_{m,n}$ is given by*

$$C_B(v) = \begin{cases} \frac{1}{m}\binom{n}{2}, & \text{if } deg(v) = n \\ \frac{1}{n}\binom{m}{2}, & \text{if } deg(v) = m. \end{cases}$$

Theorem 2.4. [12] *The betweenness centrality of any vertex in a path graph is the product of the number of vertices on either side of that vertex in the path.*

Theorem 2.5. [12] *The betweenness centrality of a vertex v in C_n is given by*

$$C_B(v) = \begin{cases} \frac{(n-1)(n-3)}{8}, & \text{if } n \text{ is odd} \\ \frac{(n-2)^2}{8}, & \text{if } n \text{ is even.} \end{cases}$$

3. Betweenness centrality of some Mycielski graphs

In this section, we have calculated betweenness centrality of any vertex in some Mycielski graphs such as $\mu(K_n)$, $\mu(K_{1,n})$, $\mu(K_{m,n})$, $\mu(W_{1,n})$. Note that if $diam(\mu(G)) = 2$, then the number of geodesic paths between two non-adjacent vertices, which are x and y, is equal to $|N(x) \cap N(y)|$. Further, there is no need to examine the geodesic paths between two adjacent vertices, because such pairs contribute a betweenness centrality zero.

Theorem 3.1. *Let $\mu(K_n)$ be Mycielski graph of K_n. Then the betweeenness centrality of a vertex x in $\mu(K_n)$ is*

$$C_B(x) = \begin{cases} 1, & \text{if } x = v_i^{'} \in V', \\ \frac{n}{2}, & \text{otherwise} \end{cases}$$

,where $i \in \{1, 2, ..., n\}$.

Proof. We have tree cases for the betweenness centrality of any vertex x in the graph $\mu(K_n)$.
Case 1. Let $x = v_i \in V$, where $i \in \{1, 2, ..., n\}$.

Consider pairs of vertices between V and $V^{'}$. The geodesic paths between v_j and $v_j^{'}$ contain the vertex v_i, where $j \in \{1, 2, ..., n\}$ and $j \neq i$. The number of pairs of vertices $(v_j, v_j^{'})$ is $n-1$. Further, there are $n-1$ geodesic paths for such pairs and one of them contains v_i.

Consider any two vertices in $V^{'}$. The geodesic paths between $v_j^{'}$ and $v_k^{'}$ contain the vertex v_i, where $j, k \in \{1, 2, ..., n\}$ and $j \neq k \neq i$. The number of pairs of vertices $(v_j^{'}, v_k^{'})$ is $\binom{n-1}{2}$. Further, there are $n-1$ geodesic paths for such pairs and one of them contains v_i.

Consider pairs of vertices (v_j, u), where $j \in \{1, 2, ...n\}$. Since $d(v_j, u) = 2$ for any j and there is not any geodesic of length 2 contains v_i.

Consequently, these pairs of vertices contribute a betweenness centrality $\frac{1}{(n-1)}(n-1) + \frac{1}{(n-1)}\binom{n-1}{2} + 0$.

Therefore, the betweenness centrality of any vertex in V is

$$C_B(v_i) = \frac{n}{2}.$$

Case 2. Let $x = v_i^{'} \in V^{'}$, where $i \in \{1, 2, ..., n\}$.

Consider pairs of vertices between V and $V^{'}$. Since the geodesic paths of length 2 between v_j and $v_j^{'}$ do not contain $v_i^{'}$, where $j \in \{1, 2, ..., n\}$ and $j \neq i$, these pairs contribute zero to $v_i^{'}$.

Similarly, pairs of vertices in $V^{'}$ contribute zero to $v_i^{'}$.

Consider pairs of vertices (v_j, u), where $j \in \{1, 2, ..., n\}$ and $j \neq i$. The number of pairs of vertices (v_j, u) is $n - 1$. Further, there are $n - 1$ geodesic paths for such pairs and one of these paths contains $v_i^{'}$.

Consequently, these pairs of vertices contribute a betweenness centrality $0 + 0 + \frac{1}{(n-1)}(n - 1)$.

Therefore, the betweenness centrality of any vertex in $V^{'}$ is

$$C_B(v_i^{'}) = 1.$$

Case 3. Let $x = u$.

Only the geodesic paths between $v_j^{'}$ and $v_k^{'}$ contain the vertex u, where $j, k \in \{1, 2, ..., n\}$ and $j \neq k$. The number of pairs of vertices $(v_j^{'}, v_k^{'})$ is $\binom{n}{2}$. Further, there are $n - 1$ geodesic paths for such pairs and one of them contains u.

Consequently, these pairs of vertices contribute a betweenness centrality $\frac{1}{(n-1)}\binom{n}{2}$.

Therefore, the betweenness centrality of u is

$$C_B(u) = \frac{n}{2}.$$

By summing up Case 1, Case 2 and Case 3, the proof is completed. □

Theorem 3.2. *Let $\mu(K_{1,n-1})$ be Mycielski graph of $K_{1,n-1}$, v_1 be central vertex of $K_{1,n-1}$ and $v_1^{'}$ be corresponding vertex of v_1. Then the betweeenness centrality of a vertex x in $\mu(K_{1,n-1})$ is*

$$C_B(x) = \begin{cases} \frac{1}{n-1}, & \text{if } x = v_i \text{ or } x = v_i^{'},\ i \neq 1 \\ \frac{(n-1)(3n-4)}{2}, & \text{if } x = v_1 \\ \frac{(n-1)(n+2)}{4}, & \text{if } x = v_1^{'} \text{ or } x = u. \end{cases}$$

Proof. We have tree cases for the betweenness centrality of any vertex x in the graph $\mu(K_{1,n-1})$.

Case 1. Let $x = v_i \in V$, where $i \in \{1, 2, ..., n\}$.

- When $i \neq 1$, then only geodesic paths between v_1 and $v_1^{'}$ contain v_i. There are $n - 1$ geodesic paths for such pairs and one of them contains v_i.
  Therefore, the betweenness centrality of v_i is
  $$C_B(v_i) = \tfrac{1}{n-1}, \text{ where } i \neq 1.$$
- When $i = 1$, then we must examine the pairs of vertices (v_j, v_k), $(v_j, v_k^{'})$, $(v_j^{'}, v_k^{'})$, where $j, k \in \{2, 3, ..., n\}$.
  Because any geodesic between pairs of vertices (v_j, u) and $(v_j^{'}, u)$ does not contain v_1.
  Consider any two vertices in V. The number of pairs of vertices (v_j, v_k) is $\binom{n-1}{2}$, where $j \neq k$. Further, there are two geodesic paths between v_j and v_k and one of these paths contains v_1.
  Consider pairs of vertices between V and $V^{'}$. The number of pairs of vertices $(v_j, v_k^{'})$ is $(n - 1)(n - 1)$, where $j, k \in \{2, 3, ..., n\}$. Further, there is only one geodesic for such pairs and it contains v_1.
  Consider any two vertices in $V^{'}$. The number of pairs of vertices $(v_j^{'}, v_k^{'})$ is $\binom{n-1}{2}$, where $j \neq k$. There are two geodesic paths for such pairs and one of them contain v_1.

Consequently, these pairs of vertices contribute a betweenness centrality $\binom{n-1}{2}\frac{1}{2} + (n - 1)^2 + \binom{n-1}{2}\frac{1}{2}$.

Therefore, we get

$$C_B(v_1) = \frac{(n - 1)(3n - 4)}{2}.$$

Case 2. Let $x = v_i^{'} \in V^{'}$, where $i \in \{1, 2, ..., n\}$.

- When $i \neq 1$, then only geodesic paths between v_1 and u contain $v_i^{'}$. There are $n-1$ geodesic paths between such pair and one of them contains $v_i^{'}$.
  Therefore, the betweenness centrality of $v_i^{'}$ is
  $$C_B(v_i^{'}) = \frac{1}{n-1}, \text{ where } i \neq 1.$$
- When $i = 1$, then we must examine the pairs (v_j, v_k) and (v_j, u), where $j,k \in \{2,3,...,n\}$. Because any geodesic between pairs of vertices $(v_j, v_k^{'})$, $(v_j^{'}, v_k^{'})$ and $(v_j^{'}, u)$ does not contain $v_1^{'}$.
  Consider any two vertices in V. The number of pairs of vertices (v_j, v_k) is $\binom{n-1}{2}$, where $j \neq k$. Further, there are two geodesic paths for such pairs and one of these paths contains $v_1^{'}$.
  Consider pairs of vertices (v_j, u). The number of such pairs is $n-1$ since $j \neq 1$. There is only one geodesic path for such pair and it contains $v_1^{'}$.
  Consequently, these pairs of vertices contribute a betweenness centrality $\binom{n-1}{2}\frac{1}{2} + (n-1)$.
  Therefore, we get
  $$C_B(v_1^{'}) = \frac{(n-1)(n+2)}{4}.$$

Case 3. Let $x = u$.

Consider pairs of vertices $(v_j^{'}, v_k^{'})$. When $j = 1$ and $k \neq 1$, then the number of pairs of vertices $(v_1^{'}, v_k^{'})$ is $n-1$. There is only one geodesic path for such pairs and it contains the vertex u. When $j \neq k \neq 1$, then the number of pairs of vertices $(v_j^{'}, v_k^{'})$ is $\binom{n-1}{2}$. There are two geodesic paths for such pairs and one of them contains the vertex u.

Consequently, these pairs of vertices contribute a betweenness centrality $(n-1) + \binom{n-1}{2}\frac{1}{2}$.

Therefore, we get
$$C_B(u) = \frac{(n-1)(n+2)}{4}.$$

By summing up Case 1, Case 2 and Case 3, the proof is completed. □

Theorem 3.3. *Let $\mu(K_{m,n})$ be Mycielski graph of $K_{m,n}$. Then the betweeenness centrality of a vertex x in $\mu(K_{m,n})$ is*

$$C_B(x) = \begin{cases} \begin{cases} \binom{n}{2}\frac{1}{2m} + \frac{n}{m}, & \text{if } i \in \{1,2,...,m\} \\ \binom{m}{2}\frac{1}{2n} + \frac{m}{n}, & \text{if } i \in \{m+1,...,m+n\} \end{cases}, & \text{if } x = v_i^{'} \\ \begin{cases} \binom{n}{2}\frac{1}{2m} + \frac{n^2}{m} + \binom{n}{2}\frac{1}{m+1}, & \text{if } i \in \{1,2,...,m\} \\ \binom{m}{2}\frac{1}{2n} + \frac{m^2}{n} + \binom{m}{2}\frac{1}{n+1}, & \text{if } i \in \{m+1,...,m+n\} \end{cases}, & \text{if } x = v_i \\ \binom{m}{2}\frac{1}{n+1} + \binom{n}{2}\frac{1}{m+1} + mn, & \text{if } x = u \end{cases}$$

Proof. Let $V(K_{m,n}) = V_1 \cup V_2$ and $V^{'} = V_1^{'} \cup V_2^{'}$, where $V_1 = \{v_1, v_2, ..., v_m\}$, $V_2 = \{v_{m+1}, v_{m+2}, ..., v_{m+n}\}$, $V_1^{'} = \{v_1^{'}, v_2^{'}, ..., v_m^{'}\}$,$V_2^{'} = \{v_{m+1}^{'}, v_{m+2}^{'}, ..., v_{m+n}^{'}\}$. We have three cases for the betweenness centrality of any vertex x in the graph $\mu(K_{m,n})$.

Case 1. Let $x = v_i \in V$, where $i \in \{1,2,...,m+n\}$.

- When $v_i \in V_1$, then there are 2m geodesic paths between any two vertices in V_2 and one of them contains v_i. The number of such pairs of vertices is $\binom{n}{2}$. Further, there are m geodesic paths between pairs of vertices $(v_j, v_k^{'})$, where $j,k \in \{m+1, m+2, ..., m+n\}$ and one of them contains v_i. The number of such pairs is n^2. Finally there are m+1 geodesic paths between any two vertices in $V_2^{'}$ and one of them contains v_i. The number of such pairs is $\binom{n}{2}$.
  Consequently, these pairs of vertices contribute a betweenness centrality $\binom{n}{2}\frac{1}{2m} + n^2\frac{1}{m} + \binom{n}{2}\frac{1}{m+1}$.
  Therefore, the betweenness centrality of v_i is
  $$C_B(v_i) = \binom{n}{2}\frac{1}{2m} + n^2\frac{1}{m} + \binom{n}{2}\frac{1}{m+1}, \text{ where } i \in \{1,2,...,m\}.$$
- When $v_i \in V_2$, then examining is done similar to $v_i \in V_1$. Therefore, the betweenness centrality of $v_i \in V_2$ is obtained as
  $$C_B(v_i) = \binom{m}{2}\frac{1}{2n} + m^2\frac{1}{n} + \binom{m}{2}\frac{1}{n+1}, \text{ where } i \in \{m+1, m+2, ..., m+n\}.$$

Case 2. Let $x = v_i^{'} \in V^{'}$, where $i \in \{1,2,...,m+n\}$.

- When $v_i^{'} \in V_1^{'}$, then there are 2m geodesic paths between any two vertices in V_2 and one of them contains $v_i^{'}$. The number of such pairs of vertices is $\binom{n}{2}$. Further, there are m geodesic paths between v_j and u, where $j \in \{m+1, m+2, ..., m+n\}$ and one of them contains v_i. The number of such pairs is n. Consequently, these pairs of vertices contribute a betweenness centrality $\binom{n}{2}\frac{1}{2m} + n\frac{1}{m}$.
  Therefore, the betweenness centrality of $v_i^{'}$ is
  $$C_B(v_i^{'}) = \binom{n}{2}\frac{1}{2m} + \frac{n}{m}, \text{ where } i \in \{1, 2, ..., m\}.$$
- When $v_i^{'} \in V_2^{'}$, then examining is done similar to $v_i^{'} \in V_1^{'}$. Therefore, the betweenness centrality of $v_i^{'} \in V_2^{'}$ is obtained as
  $$C_B(v_i^{'}) = \binom{m}{2}\frac{1}{2n} + \frac{m}{n}, \text{ where } i \in \{m+1, m+2, ..., m+n\}.$$

Case 3. Let $x = u$. For any two vertices in $V_1^{'}$, there are $n+1$ geodesic paths and one of them contains u. Since the number of such pairs of vertices is $\binom{m}{2}$, these pairs contribute a betweenness centrality $\binom{m}{2}\frac{1}{n+1}$. Similarly, the pairs of vertices in $V_2^{'}$ contribute a betweenness centrality $\binom{n}{2}\frac{1}{m+1}$. Further there is only one geodesic between $v_j^{'} \in V_1^{'}$ and $v_k^{'} \in V_2^{'}$. Since the number of such pairs of vertices is mn, these pairs contribute a betweenness centrality mn.

Therefore, the betweenness centrality of u is
$$C_B(u) = \binom{m}{2}\frac{1}{n+1} + \binom{n}{2}\frac{1}{m+1} + mn.$$

By summing up Case 1, Case 2 and Case 3, the proof is completed. □

Theorem 3.4. *Let $\mu(W_{1,n-1})$ be Mycielski graph of $W_{1,n-1}$, c be central vertex of $W_{1,n-1}$ and $c^{'}$ be corresponding vertex of c. Then the betweeenness centrality of a vertex x in $\mu(W_{1,n-1})$ is*
$$C_B(x) = \begin{cases} \frac{35}{12} + \frac{1}{n-1}, & \text{if } x = v_i \in V - c \\ \frac{11}{12} + \frac{1}{n-1}, & \text{if } x = v_i^{'} \in V^{'} - c^{'} \\ \frac{(n-1)(18n-79)}{12}, & \text{if } x = c \\ \frac{(n-1)(3n-11)}{12}, & \text{if } x = c^{'} \\ \frac{(n-1)(3n-2)}{12}, & \text{if } x = u. \end{cases}$$

Proof. We have tree cases for the betweenness centrality of any vertex x in the graph $\mu(W_{1,n-1})$.

Case 1. Let $x = v \in V$.

- Consider the vertices $v_i \in V - c$, where $i \in \{1, 2, ..., n-1\}$. Then we must examine the pairs (v_j, v_k), $(v_j, v_k^{'})$ and $(v_j^{'}, v_k^{'})$.
  Consider any two vertices in V. There is only one pair of vertices (v_{i-1}, v_{i+1}) contains v_i. There are 4 geodesic paths for such pair and one of them contains the vertex v_i.
  Consider pairs of vertices between V and $V^{'}$. There are $n-1$ geodesic paths between pair of vertices $(c, c^{'})$ and one of them contains v_i. Further, there are 3 geodesic paths between pair of vertices $(v_{i-1}, v_{i-1}^{'})$ and pair of vertices $(v_{i+1}, v_{i+1}^{'})$. One of them contains v_i. Finally, there are 2 geodesic paths between pair of vertices $(v_{i-1}, v_{i+1}^{'})$ and pair of vertices $(v_{i+1}, v_{i-1}^{'})$. One of them contains v_i.
  Consider any two vertices in $V^{'}$. There are 3 geodesic paths between pair of vertices $(v_{i-1}^{'}, v_{i-1}^{'})$. One of them contains v_i. Further there are 3 geodesic paths between pair of vertices $(c^{'}, v_{i-1}^{'})$ and pair of vertices$(c^{'}, v_{i+1}^{'})$. One of them contains v_i.
  Consequently, these pairs of vertices contribute a betweenness centrality $\frac{1}{4} + \frac{1}{n-1} + 2\frac{1}{3} + 2\frac{1}{2} + \frac{1}{3} + 2\frac{1}{3}$.
  Therefore, the betweenness centrality of v_i is
  $$C_B(v_i) = \frac{35}{12} + \frac{1}{n-1}, \text{ where } i = \{1, 2, ..., n-1\}.$$
- Consider the vertex $c \in V$. Then we must examine the pairs of vertices (v_j, v_k), $(v_j, v_k^{'})$ and $(v_j^{'}, v_k^{'})$.
  Consider any two vertices in $V - c$. Each pair of adjacent vertices in $V - c$ contributes betweenness centrality of c is zero. Further, geodesic paths between pair of vertices (v_j, v_{j+2}) contain the vertex c. There are 4 geodesic paths for such pairs and one of them contains c. The number of such pairs of vertices is $n-1$. Further, there are 2 geodesic paths between the remaining $\binom{n-1}{2} - 2(n-1)$ pairs of

vertices and one of them contains the vertex c.
Consider pairs of vertices between V and $V^{'}$. There are 3 geodesic paths between v_j and $v_j^{'}$. One of them contains the vertex c. The number of such pairs is $n-1$. Further, there are 2 geodesic paths between pairs of vertices $(v_j, v_{j+2}^{'})$ and pairs of vertices $(v_j, v_{j-2}^{'})$. One of them contains the vertex c. The number of such pairs of vertices is $2(n-1)$. Finally, there is only geodesic between remaining $(n-1)^2 - 2(n-1) - 2(n-1) - (n-1) = (n-1)(n-6)$ pair of vertices and one of them contains the vertex c.
Consider any two vertices in $V^{'} - c'$. There are 3 geodesic paths between pairs of vertices $(v_j^{'}, v_{j+2}^{'})$ and one of them contains the vertex c. Since $j \in \{1, 2, ..., n-1\}$, the number of such pairs is $(n-1)$. Further, there are 2 geodesic paths between the remaining $\binom{n-1}{2} - (n-1)$ pairs of vertices in $V' - c^{'}$ and one of them contains the vertex c.
Consequently, these pairs of vertices contribute a betweenness centrality $(n-1)\frac{1}{4} + \frac{(n-1)(n-6)}{2}\frac{1}{2} + (n-1)\frac{1}{3} + 2(n-1)\frac{1}{2} + (n-1)(n-6) + (n-1)\frac{1}{3} + (\binom{n-1}{2} - (n-1))\frac{1}{2}$.
Therefore, the betweenness centrality of the vertex c is

$$C_B(c) = \frac{18n - 79}{12}(n-1).$$

Case 2. Let $x = v^{'} \in V^{'}$.

- Consider the vertices $v_i^{'} \in V^{'} - c^{'}$. Then we must examine the pairs (v_j, v_k) and (v_j, u).
  Consider any two vertices in $V - \{c\}$. There are 4 geodesic paths between pair of vertices (v_{i-1}, v_{i+1}) and one of them contains $v_i^{'}$. Consider pairs of vertices between any vertex in V and u. There are $n-1$ geodesic paths between c and u. One of them contains v_i. Finally, there are 3 geodesic paths between pair of vertices (v_{i-1}, u) and pair of vertices (v_{i+1}, u). One of them contains $v_i^{'}$.
  Consequently, these pairs of vertices contribute a betweenness centrality $\frac{1}{4} + \frac{1}{n-1} + 2\frac{1}{3}$.
  Therefore, the betweenness centrality of $v_i^{'}$ is
  $$C_B(v_i^{'}) = \tfrac{11}{12} + \tfrac{1}{n-1}, \text{ where } i = \{1, 2, ..., n-1\}.$$
- Consider the vertex $c^{'} \in V^{'}$. Then we must examine the pairs (v_j, v_k) and (v_j, u).
  Consider any two vertices in $V - c$. Similar to examining betweenness centrality of c, the contribution of these pairs is obtained $(n-1)\frac{1}{4} + \frac{(n-1)(n-6)}{2}\frac{1}{2}$.
  Consider pairs of vertices between V and u. There are 3 geodesic paths between v_j and u. One of them contains the vertex $c^{'}$. The number of such pairs is $n-1$.
  Consequently, these pairs of vertices contribute a betweenness centrality $(n-1)\frac{1}{4} + \frac{(n-1)(n-6)}{2}\frac{1}{2} + (n-1)\frac{1}{3}$.
  Therefore, we get
  $$C_B(c^{'}) = \frac{(n-1)(3n-11)}{12}.$$

Case 3. Let $x = u$.

We must only examine geodesic paths between the pairs of vertices in $V^{'}$. Because none of the other pairs contains the vertex u. There are 3 geodesic paths between the pairs of vertices $(c, v_j^{'})$ and one of them contains the vertex u. The number of such pairs is $(n-1)$, where $j \in \{1, 2, ..., n-1\}$. Further There are 3 geodesic paths between pairs of vertices $(v_j^{'}, v_{j+2}^{'})$ and pairs of vertices $(v_j^{'}, v_{j-2}^{'})$. One of them contains the vertex u. The number of such pairs of vertices is $2(n-1)$. Finally, there are 2 geodesic paths between remaining $\binom{n-1}{2} - 2(n-1)$ pairs of vertices and one of them contains the vertex u.

Consequently, these pairs of vertices contribute a betweenness centrality $(n-1)\frac{1}{3} + 2(n-1)\frac{1}{2} + (\binom{n-1}{2} - 2(n-1))\frac{1}{2}$.
Therefore, the betweenness centrality of u is

$$C_B(u) = \frac{(n-1)(3n-2)}{12}.$$

By summing up Case 1, Case 2 and Case 3, the proof is completed. □

4. Conclusion

In this paper, we evaluate the betweenness centrality of some mychielski graphs. It is important to calculate betweenness centrality for simple graph types. If a large and complex network can divide into smaller networks, then under some conditions the solutions of the optimization problem in smaller networks can be combined with a solution for the optimization problem in the larger network. By calculating the betweenness centrality for some real networks, very good practical results can be achieved. As a further study, exact formulas or bounds may be obtained for generalized Mycielski graphs.

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