ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2019, VOLUME 9, ISSUE 1, p.39

THE FERMI-WALKER DERIVATIVE IN THREE DIMENSIONAL LIE GROUPS WITH LEFT-INVARIANT METRIC

Fatma KARAKUŞ Yusuf YAYLI

February 05, 2019

Abstract

We introduce Fermi-Walker derivative in three-dimensional Lie group with the left-invariant metric and give its geometrical description. Fermi-Walker parallelism and non-rotating frame concepts are given for three-dimensional Lie group with left-invariant metric. We generalize the results known for the case of three-dimensional Lie group with the bi-invariant metric. We define Bishop frame (relatively parallel adapted frame accordance with dot-Frenet frame) and investigate Fermi-Walker derivative along any curve with Bishop frame. We show that the Bishop frame and Frenet frame whether they are a non-rotating frame or not.

1 Introduction

The universe is perceptible through observation. A relativistic observer γ needs reference frames, for measurements of durations and of ("geo")metric quantities: the proper time ("proper clock") is given by its own canonical parameter running on an interval of the real numbers axis; the restspaces are refered to "fixed" directions, maintained by gyroscopes focused toward "fixed" celestial bodies.

The choice of an appropriate reference frame is a fundamental and controversial problem in astronomy [8]: one needs a "center" and several "fixed" directions. In a general relativistic setting, if γ is freely falling, its restspaces are transported through Levi-Civita parallelism, so a fix spacelike direction has, by definition, a null covariant derivative [9, 10]. If γ is not freely falling i.e. for accelerated observes, the restspace are not transported by the Levi-Civita parallelism, anymore. In this case, in order to define "constant" directions, another parallelism is used: the Fermi-Walker transport which is an isometry between the tangent space along γ [4, 5, 11, 12, 13]. Fermi-Walker transport is a process used to define a coordinate system or reference frame in general relativity. All the curvatures in the reference frame in due to the presence of mass-energy density. These curvatures are not arbitrary spin or rotation of the frame.

There are different transport laws such as parallel and Fermi-Walker transport for a tensor along a given curve. The parallel transport for the tensor along the given curve is defined as the law which makes that its covariant derivative be zero [3]. If the curve is a geodesic, then the tangent vector will coincide at another point of the curve with its parallel transported vector. Otherwise, the tangent vector will not coincide with its parallel transported vector. In this case, there is Fermi-Walker's law that another transport law. The Fermi-Walker transport of the tensor along the given curve is defined as the law which makes that its the Fermi-derivative along the curve be zero [3]. If the curve is a geodesic, then Fermi-Walker's transport coincides with parallel transport. Otherwise, this is not the case. In general, the Fermi-Walker transport is not the parallel transport.

A Fermi-Walker transported set of tetrad fields is the best approximation to a non-rotating reference frame in the sense of Newtonian mechanics. It is physically realized by a system of gyroscopes. Fermi-Walker transported frames are important in lots of investigations. A frame that undergoes linear and rotational acceleration can be described by the Frenet-Serret frame. The relative rotational acceleration of a Frenet-Serret frame with respect to a Fermi-Walker transported frame is taken to characterize important phenomena, like the gyroscopic precession [16]. Non-inertial reference frames in Minkowski spacetime that undergo Fermi-Walker transport are useful, for example, in the analysis of the inertial effects on a Dirac particle [17].

Parallel vector fields have important applications in differential geometry, physics and especially in robotic kinematics. The tangent vector of the curve is parallel along the curve if and only if $\nabla_T T = 0$ in Euclidean space. In this case, the curve is a geodesic in Euclidean space $\mathbb{E}^n$. Similarly, the curve which is on the surface is a geodesic if and only if $\overline{\nabla}_T T = 0$. Namely, the tangent vector is parallel along the curve on the surface. All the straight lines are geodesic curves in Euclidean space. I wonder if all the curves will be geodesic in Euclidean space? The answer to this is hidden in the connection which is obtained by using Fermi-Walker derivative. Indeed, the solution of $\widetilde{\nabla}_T T = 0$, is provided for all curves in $\mathbb{E}^n$. Accordingly, the curves and the lines are the same. That is, the curves behave like the lines with respect to Fermi-Walker connection which is a affine connection.

The notion of Fermi-Walker derivative, it shows us one method, which is used for defining "constant" direction, that may contain lots of condition to have Fermi-Walker parallel or non-rotating frame. The condition of Fermi-Walker parallel depends on a solution that contains differential equation system which is not always easy to find the answer. Therefore, it is important to analyze this concept. Therefore, we have investigated Fermi-Walker derivative and geometric applications in various spaces like Euclidean, Lorentz and Dual space up to now [7, 14, 15].

In [1, 2], Fermi–Walker derivative along any space curve was identified and was given physical properties in $\mathbb{E}^3$.

In [7], Fermi-Walker derivative, Fermi-Walker parallelism and non-rotating frame are analyzed for Bishop, Darboux and Frenet frames along the curve in Euclidean space.

In [14], we have shown Fermi-Walker derivative, and non-rotating frame are being conditions are analyzed in Minkowski space $\mathbb{E}_1^3$.

In [15], Fermi-Walker derivative is redefined in dual space $\mathbb{D}^3$. Fermi-Walker parallelism and non-rotating frame being conditions are analyzed along the dual curves in dual space $\mathbb{D}^3$.

In this paper, we introduce Fermi-Walker derivative in three-dimensional Lie group with the left-invariant metric and give its geometrical description. Fermi-Walker parallelism and non-rotating frame concepts are given for three-dimensional Lie group with left-invariant metric. Being non-rotating frame conditions are analyzed for dot-Frenet frame in Lie groups. In case of three dimensional Lie groups with bi-invariant metric Fermi-Walker derivative was considered in [6]. We generalize the results known for the case of three-dimensional Lie group with the bi-invariant metric. We define Bishop frame (relatively parallel adapted frame accordance with dot-Frenet frame) and investigate Fermi-Walker derivative along any curve with Bishop frame. We show that the Bishop frame and Frenet frame whether they are a non-rotating frame or not.

2 Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of three dimensional Lie groups with left-invariant metric are briefly presented.

Let G^3 be a three dimensional Lie group with left-invariant metric $\langle \cdot, \cdot \rangle$, g denotes the Lie algebra for G^3 which consists of the all smooth vector fields of G^3 invariant under left translation. There are two classes of three dimensional Lie groups: unimodular and non-unimodular.

In the case of unimodular group, there is an (positively oriented) orthonormal frame of left-invariant vector fields $\{e_1, e_2, e_3\}$ such that brackets satisfy [19]:

$$[e_1, e_2] = \lambda_3 e_3, \qquad [e_3, e_1] = \lambda_2 e_2, \qquad [e_2, e_3] = \lambda_1 e_1.$$

The constant λ_i are called by the *structure constants*. The constants

$$\mu_i = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3) - \lambda_i,$$

are called by the *connection coefficients*.

In the case of non-unimodular group, there is an orthonormal frame $\{e_1, e_2, e_3\}$ such that [19]

$$[e_1, e_2] = \alpha e_2 + \beta e_3, \qquad [e_1, e_3] = -\beta e_2 + \delta e_3, \qquad [e_2, e_3] = 0;$$

with $\alpha + \delta \neq 0$ and $-\alpha\beta + \beta\delta = 0$.

Denote by $\mu(X)$ an affine transformation given by

$$\mu(X) = \begin{cases} \mu_1 X^1 e_1 + \mu_2 X^2 e_2 + \mu_3 X^3 e_3 & \text{for unimodular group,} \\ \beta X^1 e_1 + \delta X^3 e_2 - \alpha X^2 e_3 & \text{for non-unimodular group.} \end{cases}$$

In three-dimensional case one can naturally define the *cross product* by

$$e_1 \wedge e_2 = e_3, \qquad e_2 \wedge e_3 = e_1, \qquad e_3 \wedge e_1 = e_2.$$

Then for both types of groups in terms of the cross product we have

$$\nabla_{e_i} e_k = \mu(e_i) \wedge e_k$$

and hence

$$\nabla_X e_k = \mu(X) \wedge e_k$$

Let $\gamma(s)$ be a parametrized curve on the group and $T = \dot{\gamma}$ be tangent vector field. For the arbitrary vector field ξ we have

$$\nabla_T \xi = \dot{\xi}^k e_k + \mu(T) \wedge \xi, \tag{1}$$

In what follows, we call the vector field

$$\dot{\xi} = \frac{d\xi^1}{ds} e_1 + \frac{d\xi^2}{ds} e_2 + \frac{d\xi^3}{ds} e_3$$

by *dot-derivative* of the vector field ξ along the curve γ.

Let T, N, and B be the vectors of standart Frenet frame of γ, k and $\varkappa$ are called curvature and torsion function of γ, respectively. Using (2.1), we get

$$\nabla_T T = \dot{T} + \mu(T) \wedge T, \quad \nabla_T N = \dot{N} + \mu(T) \wedge N, \quad \nabla_T B = \dot{B} + \mu(T) \wedge B.$$

Assuming $k_0 = \left|\dot{T}\right| \neq 0$, we can define a new frame $\{\tau, \nu, \beta\}$ along the curve γ by

$$\tau = T, \qquad \nu = \frac{1}{k_0}\dot{\tau}, \qquad \beta = \tau \wedge \nu. \tag{2}$$

We call (2.2) by *dot-Frenet frame [18].*

Proposition 1 *The dot-Frenet frame $\{\tau, \nu, \beta\}$ satisfies the dot-Frenet formulas, namely,*

$$\dot{\tau} = k_0 \nu, \qquad \dot{\nu} = -k_0 \tau + \varkappa_0 \beta, \qquad \dot{\beta} = -\varkappa_0 \nu \tag{3}$$

where $k_0 = \left|\dot{T}\right| \neq 0$ and $\varkappa_0 = \left|\dot{\beta}\right|$ are called dot-curvature and dot-torsion, respectively [18].

In [18], the standard Frenet frame and the dot-Frenet frame has common vector $\tau = T$ and hence, the Frenet and the dot-Frenet frames are connected by

$$\tau = T, \qquad \nu = \cos\alpha N + \sin\alpha B, \qquad \beta = -\sin\alpha N + \cos\alpha B \tag{4}$$

where $\alpha = \alpha(s)$ is the angle between N and dot-principal normal ν, $\dot{\alpha} = \frac{d\alpha}{ds}$.

The inverse transformation is of the form

$$T = \tau, \qquad N = \cos\alpha\nu - \sin\alpha\beta, \qquad B = \sin\alpha\nu + \cos\alpha\beta. \tag{5}$$

Proposition 2 *In [18], the decomposition of $\mu(T)$ with respect to the Frenet frame is of the form*

$$\mu(T) = \left(\varkappa + \dot{\alpha} - \varkappa_0\right) T + k_0 \sin \alpha N + (k - k_0 \cos \alpha) B. \tag{6}$$

Remark 1 *If metric is bi-invariant, the $\mu_1 = \mu_2 = \mu_3 := \mu$ and hence, $\mu(T) = \mu.T$. As a consequence, $\alpha = 0$, $k_G = 0$, $k = k_0,$ $\varkappa_G = \varkappa - \varkappa_0$ and the dot-Frenet frame coincides with the Frenet frame.*

Definition 1 *The group-curvature k_G and the group-torsion $\varkappa_G$ of a curve have defined by*

$$k_G = |\mu(T) \wedge T|$$

$$\varkappa_G = |\mu(T) \wedge B|,$$

respectively [18].

Definition 2 *Let G^3 be a three dimensional Lie group with left-invariant metric, $\gamma(s)$ be a parametrized curve on the group and X be any vector field along the curve. $\widetilde{\nabla}_\tau X$ Fermi-Walker derivative is defined as*

$$\widetilde{\nabla}_\tau X = \dot{X} - \langle \tau, X \rangle A + \langle A, X \rangle \tau.$$

Here $\tau = \dot{\gamma}$ is the tangent vector field of curve γ and $A = \dot{\tau}$ is the acceleration of γ.

Definition 3 *Let G^3 be a three dimensional Lie group with left-invariant metric, $\gamma(s)$ be a parametrized curve on the group and X be any vector field along the curve γ. If the Fermi-Walker derivative of the vector field X vanishes, i.e., $\widetilde{\nabla}_\tau X = 0$, then X is called Fermi-Walker parallel vector field along the curve.*

Definition 4 *Let a parametrized curve $\gamma : I \subset \mathbb{R} \longrightarrow G^3$ together with orthonormal vector field U, V, W along γ be given. If the Fermi-Walker derivative of the vector fields vanish, then $\{U, V, W\}$ is called non-rotating frame.*

3 The Fermi–Walker Derivative in Lie Groups with Left-Invariant Metric

In this section, we will analyze Fermi-Walker derivative along the curve $\gamma(s)$ which is a parametrized curve on G^3 three dimensional Lie group with left-invariant metric and generalize the results known for the case of three-dimensional Lie group with the bi-invariant metric. We will investigate that the dot-Frenet frame whether it is a non-rotating frame or not. We will give the decomposition of $\mu(\tau)$ with respect to the dot-Frenet frame. For the dot-Frenet vector fields τ, ν and β we will obtain dot-Frenet formulas with respect to the connection ∇. We will define Bishop frame (relatively parallel adapted frame accordance with dot-Frenet frame) and investigate Fermi-Walker derivative along any curve with Bishop frame. We will show that the Bishop frame and Frenet frame whether they are a non-rotating frame or not. Let $\gamma : I \subset \mathbb{R} \longrightarrow G^3$ be a parametrized curve with dot-Frenet vector fields τ, ν and β, and X be any vector field along the curve γ with the corresponding connection ∇.

Proposition 3 *The decomposition of $\mu(\tau)$ with respect to the dot-Frenet frame is of the form*

$$\mu(\tau) = \left(\varkappa + \dot{\alpha} - \varkappa_0\right)\tau + (k\sin\alpha)\,\nu - (k_0 - k\cos\alpha)\,\beta. \tag{7}$$

Proof: In (2.6), using (2.5) we find

$$\begin{aligned}\mu(\tau) &= \left(\varkappa + \dot{\alpha} - \varkappa_0\right)\tau + k_0(\sin\alpha\cos\alpha)\,\nu - k_0\left(\sin^2\alpha\right)\beta \\ &\quad + (k\sin\alpha)\,\nu + (k\cos\alpha)\,\beta - k_0(\sin\alpha\cos\alpha)\,\nu - k_0\left(\cos^2\alpha\right)\beta\end{aligned}$$

$$\mu(\tau) = \left(\varkappa + \dot{\alpha} - \varkappa_0\right)\tau + (k\sin\alpha)\,\nu + (k\cos\alpha)\,\beta - k_0\beta.$$

We get

$$\mu(\tau) = \left(\varkappa + \dot{\alpha} - \varkappa_0\right)\tau + (k\sin\alpha)\,\nu - (k_0 - k\cos\alpha)\,\beta.$$

□

Proposition 4 *The dot-Frenet frame $\{\tau, \nu, \beta\}$ satisfies the dot-Frenet formulas with respect to the connection ∇, namely,*

$$\nabla_\tau \tau = (k\cos\alpha)\,\nu - (k\sin\alpha)\,\beta \tag{8}$$

$$\nabla_\tau \nu = -\left(k\cos\alpha\right)\tau + \left(\varkappa + \dot{\alpha}\right)\beta \tag{9}$$

$$\nabla_\tau \beta = \left(k\sin\alpha\right)\tau - \left(\varkappa + \dot{\alpha}\right)\nu. \tag{10}$$

Proof: Using (2.1), we get

$$\begin{aligned} \nabla_\tau \tau &= \dot{\tau} + \mu(\tau) \wedge \tau, \\ \nabla_\tau \nu &= \dot{\nu} + \mu(\tau) \wedge \nu, \\ \nabla_\tau \beta &= \dot{\beta} + \mu(\tau) \wedge \beta. \end{aligned}$$

Using (2.3) and (3.1), thus we find

$$\nabla_\tau \tau = \left(k\cos\alpha\right)\nu - \left(k\sin\alpha\right)\beta$$

$$\nabla_\tau \nu = -\left(k\cos\alpha\right)\tau + \left(\varkappa + \dot{\alpha}\right)\beta$$

$$\nabla_\tau \beta = \left(k\sin\alpha\right)\tau - \left(\varkappa + \dot{\alpha}\right)\nu.$$

□

Remark 2 *If metric is bi-invariant, the* $\mu_1 = \mu_2 = \mu_3 := \mu$ *and hence,* $\mu(\tau) = \mu.\tau$, *i.e.* $\mu(T) = \mu.T$. *As a consequence,* $\alpha = 0$, $k_G = 0$, $k = k_0$, $\varkappa_G = \varkappa - \varkappa_0$ *and the dot-Frenet formulas coincide with the Frenet formula, namely, we get [6].*

Definition 5 *Let* $T(s)$ *be a tangent to* $\gamma(s)$ *and let* $N_1(s)$ *be arbitrary orthogonal unit vector to* $T(s)$ *and* $N_2(s) = T(s) \wedge N_1(s)$. *This means that* $\{T(s), N_1(s), N_2(s)\}$ *is an orthonormal frame. The frame is called Bishop frame (relatively parallel adapted frame accordance with dot-Frenet frame). If we rotate the Bishop frame by the angle* $\phi = \int \varkappa_0 ds$ *around the tangent vector* T, *we obtain the dot-Frenet frame* $\{\tau, \nu, \beta\}$ *as below.*

$$\begin{bmatrix} \tau \\ \nu \\ \beta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}.$$

Let $k_1(s)$ and $k_2(s)$ be Bishop parameters. The derivative formulas which correspond to Bishop frame and Bishop parameters are as follow;

$$\begin{aligned}
\dot{T}(s) &= k_1(s)\, N_1(s) + k_2(s)\, N_2(s) \\
\dot{N}_1(s) &= -k_1(s)\, T(s) \\
\dot{N}_2(s) &= -k_2(s)\, T(s) \\
k_1(s) &= k_0 \cos\phi \\
k_2(s) &= k_0 \sin\phi \\
\varkappa_0 &= \dot{\phi}.
\end{aligned} \tag{11}$$

where $k_0 = \left|\dot{T}\right| \neq 0$ and $\varkappa_0 = \left|\dot{\beta}\right|$ are called dot-curvature and dot-torsion, respectively.

Lemma 5 *Let $\gamma : I \subset \mathbb{R} \longrightarrow G^3$ be a parametrized curve on the group and X be any vector field along the curve γ with the corresponding connection ∇. The Fermi-Walker derivative along the curve can be expressed as*

$$\widetilde{\nabla}_\tau X = \nabla_\tau X - \mu(\tau) \wedge X - k_0 (\beta \wedge X) \tag{12}$$

where k_0 is the dot-curvature of the curve γ at s.

Proof: *In Definition 2, using (2.1) and the features of vector product ,we get*

$$\widetilde{\nabla}_\tau X = \nabla_\tau X - \mu(\tau) \wedge X - \langle \tau, X \rangle A + \langle A, X \rangle \tau$$

$$\widetilde{\nabla}_\tau X = \nabla_\tau X - \mu(\tau) \wedge X + (A \wedge \tau) \wedge X.$$

By the dot-Frenet equations,

$$\widetilde{\nabla}_\tau X = \nabla_\tau X - \mu(\tau) \wedge X - k_0 (\beta \wedge X)$$

is obtained. Which gives the result. □

Remark 3 *If metric is bi-invariant, then $\alpha = 0$, $k_G = 0$, $k = k_0$, $\varkappa_G = \varkappa - \varkappa_0$ and we get [6].*

Theorem 6 *Let G^3 be a three dimensional Lie group with left-invariant metric, $\gamma : I \subset \mathbb{R} \longrightarrow G^3$ be a parametrized curve and $X = \lambda_1\tau + \lambda_2\nu + \lambda_3\beta$ be any vector field along the curve γ. The Fermi-Walker derivative of the vector field X along the curve can be given by*

$$\widetilde{\nabla}_\tau X = \left(\frac{d\lambda_1}{ds}\right)\tau + \left(\frac{d\lambda_2}{ds} - \varkappa_0\lambda_3\right)\nu + \left(\frac{d\lambda_3}{ds} + \varkappa_0\lambda_2\right)\beta$$

where $\varkappa_0$ is the dot-torsion of the curve γ at s and $\lambda_1, \lambda_2, \lambda_3$ are continuously differentiable functions of real parameter s.

Proof: We have

$$\begin{aligned}\nabla_\tau X &= \left\{\frac{d\lambda_1}{ds} - (k\cos\alpha)\lambda_2 + (k\sin\alpha)\lambda_3\right\}\tau + \left\{\frac{d\lambda_2}{ds} + (k\cos\alpha)\lambda_1 - \left(\varkappa + \dot{\alpha}\right)\lambda_3\right\}\nu \\ &\quad + \left\{\frac{d\lambda_3}{ds} - (k\sin\alpha)\lambda_1 + \left(\varkappa + \dot{\alpha}\right)\lambda_2\right\}\beta,\end{aligned}$$

$$\beta \wedge X = -\lambda_2\tau + \lambda_1\nu$$

and using (3.1) we get

$$\begin{aligned}\mu(\tau) \wedge X &= \lambda_1\left\{-(k_0 - k\cos\alpha)\nu - (k\sin\alpha)\beta\right\} + \lambda_2\left\{(k_0 - k\cos\alpha)\tau + \left(\varkappa + \dot{\alpha} - \varkappa_0\right)\beta\right\} \\ &\quad + \lambda_3\left\{(k\sin\alpha)\tau - \left(\varkappa + \dot{\alpha} - \varkappa_0\right)\nu\right\}.\end{aligned}$$

Hence, using Lemma 1,

$$\widetilde{\nabla}_\tau X = \left(\frac{d\lambda_1}{ds}\right)\tau + \left(\frac{d\lambda_2}{ds} - \varkappa_0\lambda_3\right)\nu + \left(\frac{d\lambda_3}{ds} + \varkappa_0\lambda_2\right)\beta$$

is obtained. □

Theorem 7 *Let G^3 be a three dimensional Lie group with left-invariant metric, $\gamma : I \subset \mathbb{R} \longrightarrow G^3$ be a parametrized curve and $X = \lambda_1\tau + \lambda_2\nu + \lambda_3\beta$ be any vector field along the curve γ. The vector field X is Fermi-Walker parallel along the curve if and only if*

$$\begin{aligned}\lambda_1(s) &= const., \\ \lambda_2(s) &= c_1\cos\left(\int_1^s \varkappa_0(s)\,ds\right) + c_2\sin\left(\int_1^s \varkappa_0(s)\,ds\right), \\ \lambda_3(s) &= -c_1\sin\left(\int_1^s \varkappa_0(s)\,ds\right) + c_2\cos\left(\int_1^s \varkappa_0(s)\,ds\right)\end{aligned}$$

where $\varkappa_0$ is the dot-torsion of the curve γ at s and $\lambda_1, \lambda_2, \lambda_3$ are continuously differentiable functions of real parameter s.

Proof: From Theorem 1,

$$\widetilde{\nabla}_\tau X = \left(\frac{d\lambda_1}{ds}\right)\tau + \left(\frac{d\lambda_2}{ds} - \varkappa_0\lambda_3\right)\nu + \left(\frac{d\lambda_3}{ds} + \varkappa_0\lambda_2\right)\beta.$$

X is Fermi-Walker parallel along the curve so,

$$\begin{aligned} \frac{d\lambda_1}{ds} &= 0 \\ \frac{d\lambda_2}{ds} - \varkappa_0\lambda_3 &= 0 \\ \frac{d\lambda_3}{ds} + \varkappa_0\lambda_2 &= 0 \end{aligned}$$

is obtained. From the solution of the equation system, we have

$$\begin{aligned} \lambda_1(s) &= const., \\ \lambda_2(s) &= c_1\cos\left(\int_1^s \varkappa_0(s)\,ds\right) + c_2\sin\left(\int_1^s \varkappa_0(s)\,ds\right), \\ \lambda_3(s) &= -c_1\sin\left(\int_1^s \varkappa_0(s)\,ds\right) + c_2\cos\left(\int_1^s \varkappa_0(s)\,ds\right). \end{aligned}$$

□

Remark 4 *If metric is bi-invariant, the $\mu_1 = \mu_2 = \mu_3 := \mu$ and hence, $\mu(\tau) = \mu.\tau$, i.e. $\mu(T) = \mu.T$. As a consequence, $\alpha = 0$, $k_G = 0$, $k = k_0$, $\varkappa_G = \varkappa - \varkappa_0$ and we get [6].*

Theorem 8 *Let G^3 be a three dimensional Lie group with left-invariant metric, $\gamma : I \subset \mathbb{R} \longrightarrow G^3$ be a parametrized curve, λ_i be constants which are not vanish and $X = \lambda_1\tau + \lambda_2\nu + \lambda_3\beta$ be any vector field along the curve γ. The vector field X is Fermi-Walker parallel along the curve if and only if the dot-torsion of the curve γ vanishes, i.e. $\varkappa_0 = 0$.*

Proof: From Theorem 1,

$$\widetilde{\nabla}_\tau X = \left(\frac{d\lambda_1}{ds}\right)\tau + \left(\frac{d\lambda_2}{ds} - \varkappa_0\lambda_3\right)\nu + \left(\frac{d\lambda_3}{ds} + \varkappa_0\lambda_2\right)\beta.$$

and from $\lambda_i = const.$,

$$\widetilde{\nabla}_\tau X = \varkappa_0(s)(-\lambda_3\nu + \lambda_2\beta)$$

is obtained. X is Fermi-Walker parallel along the curve so, we have $\varkappa_0 = 0$. □

Remark 5 *If metric is bi-invariant then* $\alpha = 0$, $k_G = 0$, $k = k_0$, $\varkappa_G = \varkappa - \varkappa_0$ *and we get [6].*

Corollary 9 *The dot-Frenet frame* $\{\tau, \nu, \beta\}$ *is a non-rotating frame along the curve if and only if* $\varkappa_0 = 0$.

Corollary 10 *The Bishop frame* $\{T, N_1, N_2\}$ *is a non-rotating frame along the all curves.*

Proof: From (3.6) and Definition 2, we have the vector field $X = \lambda_1 T + \lambda_2 N_1 + \lambda_3 N_2$ is a Fermi-Walker parallel along the all curves, i.e. $\forall\lambda_i = const.$

Hence, we get

$$\widetilde{\nabla}_\tau T = 0, \qquad \widetilde{\nabla}_\tau N_1 = 0, \qquad \widetilde{\nabla}_\tau N_2 = 0.$$

Thus, the Bishop frame $\{T, N_1, N_2\}$ is a non-rotating frame along the all curves. □

Remark 6 *If metric is bi-invariant, the* $\mu_1 = \mu_2 = \mu_3 := \mu$ *and hence,* $\mu(\tau) = \mu.\tau$, *i.e.* $\mu(T) = \mu.T$. *As a consequence,* $\alpha = 0$, $k_G = 0$, $k = k_0$, $\varkappa_G = \varkappa - \varkappa_0$ *and the Frenet frame* $\{T, N, B\}$ *is a non-rotating frame along the curve if and only if* $\varkappa_G = \varkappa$, *namely we get [6].*

Theorem 11 *Let* G^3 *be a three dimensional Lie group with left-invariant metric,* $\gamma : I \subset \mathbb{R} \longrightarrow G^3$ *be a parametrized curve and* $X = \lambda_1 T + \lambda_2 N + \lambda_3 B$ *be any vector field along the curve* γ. *The vector field* X *is Fermi-Walker parallel along the curve if and only if*

$$\begin{aligned}
\lambda_1(s) &= const., \\
\lambda_2(s) &= c_1 \cos\left(\int_1^s (\varkappa_0 - \dot{\alpha})(s)\,ds\right) + c_2 \sin\left(\int_1^s (\varkappa_0 - \dot{\alpha})(s)\,ds\right), \\
\lambda_3(s) &= -c_1 \sin\left(\int_1^s (\varkappa_0 - \dot{\alpha})(s)\,ds\right) + c_2 \cos\left(\int_1^s (\varkappa_0 - \dot{\alpha})(s)\,ds\right)
\end{aligned}$$

where $\varkappa_0$ is the dot-torsion of the curve γ at s and $\lambda_1, \lambda_2, \lambda_3$ are continuously differentiable functions of real parameter s.

Proof: Using Lemma 1, we get

$$\begin{aligned}
\widetilde{\nabla}_\tau X &= \nabla_\tau X - \mu(\tau) \wedge X - k_0 (\beta \wedge X) \\
\widetilde{\nabla}_\tau X &= \left(\frac{d\lambda_1}{ds} - \lambda_2 k_0 \cos\alpha + \lambda_2 k_0 \cos\alpha - \lambda_3 k_0 \cos\alpha + \lambda_3 k_0 \cos\alpha \right) T \\
&\quad + \left(\frac{d\lambda_2}{ds} - \left(\varkappa_0 - \dot{\alpha}\right) \lambda_3 \right) N + \left(\frac{d\lambda_3}{ds} + \left(\varkappa_0 - \dot{\alpha}\right) \lambda_2 \right) B
\end{aligned}$$

$$\widetilde{\nabla}_\tau X = \left(\frac{d\lambda_1}{ds} \right) T + \left(\frac{d\lambda_2}{ds} - \left(\varkappa_0 - \dot{\alpha}\right) \lambda_3 \right) N + \left(\frac{d\lambda_3}{ds} + \left(\varkappa_0 - \dot{\alpha}\right) \lambda_2 \right) B \tag{13}$$

is obtained. X is Fermi-Walker parallel along the curve so,

$$\begin{aligned}
\frac{d\lambda_1}{ds} &= 0 \\
\frac{d\lambda_2}{ds} - \left(\varkappa_0 - \dot{\alpha}\right) \lambda_3 &= 0 \\
\frac{d\lambda_3}{ds} + \left(\varkappa_0 - \dot{\alpha}\right) \lambda_2 &= 0
\end{aligned}$$

is obtained. From the solution of the equation system, we have

$$\begin{aligned}
\lambda_1(s) &= const., \\
\lambda_2(s) &= c_1 \cos\left(\int_1^s \left(\varkappa_0 - \dot{\alpha}\right)(s)\, ds \right) + c_2 \sin\left(\int_1^s \left(\varkappa_0 - \dot{\alpha}\right)(s)\, ds \right), \\
\lambda_3(s) &= -c_1 \sin\left(\int_1^s \left(\varkappa_0 - \dot{\alpha}\right)(s)\, ds \right) + c_2 \cos\left(\int_1^s \left(\varkappa_0 - \dot{\alpha}\right)(s)\, ds \right).
\end{aligned}$$

□

Theorem 12 *Let G^3 be a three dimensional Lie group with left-invariant metric, $\gamma : I \subset \mathbb{R} \longrightarrow G^3$ be a parametrized curve, λ_i be constants which are not vanish and $X = \lambda_1 T + \lambda_2 N + \lambda_3 B$ be any vector field along the curve γ. The vector field X is Fermi-Walker parallel along the curve if and only if the dot-torsion $\varkappa_0(s) = \dot{\alpha}(s)$.*

Proof: From (3.7),

$$\widetilde{\nabla}_\tau X = \left(\frac{d\lambda_1}{ds}\right) T + \left(\frac{d\lambda_2}{ds} - \left(\varkappa_0 - \dot{\alpha}\right)\lambda_3\right) N + \left(\frac{d\lambda_3}{ds} + \left(\varkappa_0 - \dot{\alpha}\right)\lambda_2\right) B$$

and $\lambda_i = const.$,

$$\widetilde{\nabla}_\tau X = \left(\varkappa_0 - \dot{\alpha}\right)(s)\left(-\lambda_3 N + \lambda_2 B\right)$$

is obtained. X is Fermi-Walker parallel along the curve so, we have $\varkappa_0(s) = \dot{\alpha}(s)$. □

Corollary 13 *The Frenet frame $\{T, N, B\}$ is a non-rotating frame along the curve if and only if $\varkappa_0(s) = \dot{\alpha}(s)$.*

Definition 6 *Let $\{\tau, \nu, \beta\}$ be the dot-Frenet frame of the curve. The Fermi-Walker termed Darboux vector is defined by*

$$\widetilde{w} = \varkappa_0(s)\,\tau(s)$$

where $\varkappa_0(s)$ is the dot-torsion of the curve γ at s and $\widetilde{\nabla}_\tau \tau = \widetilde{w} \wedge \tau, \widetilde{\nabla}_\tau \nu = \widetilde{w} \wedge \nu, \widetilde{\nabla}_\tau \beta = \widetilde{w} \wedge \beta$ are provided.

Theorem 14 *The Fermi-Walker termed Darboux vector is Fermi-Walker parallel along the curve if and only if the dot-torsion of the curve γ is a constant, i.e. $\varkappa_0(s) = const.$*

Proof: Using Theorem 1,

$$\widetilde{\nabla}_\tau \widetilde{w} = \frac{d\varkappa_0}{ds}\tau$$

since $\widetilde{w}(s)$ is Fermi-Walker parallel,

$$\varkappa_0(s) = const.$$

is obtained. □

Remark 7 *If metric is bi-invariant, the $\mu_1 = \mu_2 = \mu_3 := \mu$ and hence, $\mu(\tau) = \mu.\tau$, i.e. $\mu(T) = \mu.T$. As a consequence, $\alpha = 0$, $k_G = 0$, $k = k_0$, $\varkappa_G = \varkappa - \varkappa_0$ and we get [6].*

4 Conclusions

In this paper, we introduce Fermi-Walker derivative in three-dimensional Lie group with the left-invariant metric and give its geometrical description. Fermi-Walker transport and non-rotating frame concepts are given for three-dimensional Lie group with left-invariant metric. Being non-rotating frame conditions are analyzed for dot-Frenet frame in Lie groups. In case of three dimensional Lie groups with bi-invariant metric Fermi-Walker derivative was considered in [6]. We generalize the results known for the case of three-dimensional Lie group with bi-invariant metric. We give the decomposition of $\mu(\tau)$ with respect to the dot-Frenet frame. We define Bishop frame (relatively parallel adapted frame accordance with dot-Frenet frame) and investigate Fermi-Walker derivative along any curve with Bishop frame. We show that the Bishop frame and Frenet frame whether they are a non-rotating frame or not. We prove that while the dot-torsion vanishes i.e. $\varkappa_0 = 0$, the dot-Frenet frame is a non-rotating frame. We show that the Bishop frame is a non-rotating frame along the all curves on the three-dimensional Lie group with left-invariant metric. Furthermore, we show that while $\varkappa_0(s) = \dot{\alpha}(s)$, the Frenet frame a non-rotating frame along the curve on the group.

References

[1] Balakrishnan, Radha., Space curves, anholonomy and nonlinearity, Pramana Journal of Physics, 64(4) (2005) 607-615.

[2] Benn, I. M. and Tucker R. W. I, Wave mechanics and inertial guidance, Phys. Rev. D 39(6) (1989) 1594-1601.

[3] Berman, M.S., Introduction to general relativistic and scalar-tensor cosmologies, Nova Science Publishers, Inc., New York, 2007.

[4] Fermi, E., Atti Accad. Naz. Lincei Cl. Sci. Fiz. Mat. Nat., 31 (1922) 184-306.

[5] Hawking, S.W.and Ellis, G.F.R., The large scale structure of spacetime (Cambridge Univ. Press, 1973).

[6] Karakuş, F. and Yaylı, Y., The Fermi-Walker derivative in Lie Groups, Int. Journal of Geometric Methods in Modern Physics, Vol.10. No.7(2013) 1320011 (11pages).

[7] Karakuş, F. and Yaylı, Y., On the Fermi-Walker derivative and Non-rotating frame, Int. Journal of Geometric Methods in Modern Physics, Vol.9. No.8(2012) 1250066 (11pages).

[8] Capitaine, N. and Guinot, B., Systemes de reference en astronomie, C.R. Acad. Sci. Paris 324 (1997), ser. II, 725-738.

[9] Pripoae, G.T., Generalized Fermi-Walker Transport, Libertas Math., XIX (1999) 65-69.

[10] Pripoae, G.T., Generalized Fermi-Walker Parallelism Induced by Generalized Schouten Connections, Geometry Balkan Press (2000) 117-125.

[11] Sachs, R. K. and Wu, H., General Relativity for Mathematicians, Springer Verlag, N.Y., 1977.

[12] Walker, A.G., Relative co-ordinates, Proc. Royal Soc. Edinburgh 52 (1932) 345-353.

[13] Weinberg, S., Gravitation and Cosmology, J. Wiley Publ., N.Y., 1972.

[14] Karakuş, F. and Yaylı, Y., The Fermi-Walker Derivative in Minkowski Space $\mathbb{E}_1^3$, Adv. Appl. Clifford Algebras (27) (2017), 1353-1368.

[15] Şahin, T., Karakuş, F. and Yaylı, Y., The Fermi-Walker Derivative and Non-Rotating Frame in Dual Space, Adv. Appl. Clifford Algebras (March, 2018), 28:10 doi.10.1007/s00006-018-0837-z.

[16] Iyer, B.R. and Vishveshwara, C.V., Frenet-Serret description of gyroscopic precession, Physical Review D, Vol. 48, Issue 12(1993), 5706.

[17] Hehl, F.W. and Ni, W.T., Inertial effects of a Dirac particle, Physical Review D, Vol. 42(1990), 2045.

[18] Yampolsky, A. and Opariy, A., Generalized helices in three dimensional Lie groups, Turkish Journal of Mathematics, (Published Online: 26.03.2019), doi.10.3906/mat-1806-33.

[19] Milnor, J., Curvatures of Left Invariant Metrics on Lie Groups, Advances in Mathematics (21), (1976), 293-329.