DARBOUX VECTORS AND THE SPHERICAL INDICATRIX CURVES SATISFYING THE TZITZÉICA CONDITION IN MINKOWSKI SPACE

F.HATHOUT AND Y.YAYLI

Abstract. In this present paper, we characterize the Darboux vectors $\mathbb{D}$ with any orthogonal frame apparatus in Minkowski 3-space $\mathbb{E}_1^3$. We give a necessary and sufficient conditions for $\mathbb{D}$ to satisfies Tzitzéica condition and we show the relationships between the Darboux vectors and the spherical indicatrix curves when its satisfying the Tzitzéica condition in $\mathbb{E}_1^3$.

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1. Introduction

The notion of the Darboux vector is introduced by Gaston Darboux in the theory of space curves ([6]) as an areal velocity vector of the Frenet frame of a space curve. In terms of the Frenet-Serret frame apparatus, the Darboux vector D in Euclidian space is given by

$$D = \tau T(s) + \kappa B(s)$$

A new vectors are introduced in ([3],[4],[9] and [10]) called modified Darboux and the new modified Darboux vectors are defined, respectively, by

$$\overline{\mathcal{D}} = \frac{\tau}{\kappa}(s)T(s) + B(s) \text{ and } \overline{\overline{\mathcal{D}}} = T(s) + \frac{\kappa}{\tau}(s)B(s)$$

We denote the set of these Darboux vectors by D (i.e. $D = \{D, D, \overline{\overline{\mathcal{D}}}\}$).

In Minkowski 3-space E_1^3, the Darboux vectors D are defined in Frenet frame apparatus as

$$\mathbb{D} : \begin{cases} \mathcal{D} = \varepsilon_1 \tau T(s) + \varepsilon_2 \varepsilon_3 \kappa B(s) \\ \overline{\mathcal{D}} = \varepsilon_1 \frac{\tau}{\kappa}(s)T(s) + \varepsilon_2 \varepsilon_3 B(s) \\ \overline{\overline{\mathcal{D}}} = \varepsilon_1 T(s) + \varepsilon_2 \varepsilon_3 \frac{\kappa}{\tau}(s)B(s) \end{cases}$$

where $\varepsilon_1, \varepsilon_2$ and ε_3 are the causal characters of the vectors T, N and B, respectively.

On the other hand, Gheorghe Tzitzéica has introduced a class of curves (See [11]), called Tzitzéica curves, for which the ratio of its torsion and the square of the distance d

from the origin to the osculating plane at arbitrary point of the curves is constant, i.e.,

$$\frac{\tau}{d^2} \text{ is nonzero constant} \tag{1.1}$$

in this paper, we call condition given in equation (1.1) by the Tzitzéica condition for curves.

Our study are devoted to any orthogonal frame apparatus in E_1^3, we characterize the Darboux vectors D satisfying the Tzitzéica condition and we give relationship between D and the spherical indicatrix curves when its satisfies the Tzitzéica condition. A particular case, when this frame coincide with a Frenet-Serret frame, and $\{N, C, W, f, g\}$ frame (For more detail see [12]) are given in E_1^3.

The paper is organized as follows:
A basic definitions in Minkowski 3-space E_1^3 are given in section 2.
In section 3, we give a Frenet frame of the Darboux vectors ($\overline{\mathcal{D}}$, $\overline{\overline{\mathcal{D}}}$) in E_1^3.
The section 4 is devoted to the characterization of the Darboux vectors D satisfying the Tzitzéica condition in E_1^3 given in the Theorem 4.1 with any frame apparatus defined in the Proposition 3.1.
Finally in section 5, some relationships are given in the Theorems 5.4, 5.5 and 5.8 between the Darboux vectors D and the spherical indicatrix curves to satisfying the Tzitzéica condition with any orthogonal frame apparatus and a results in particular case of frames (Frenet and $\{N, C, W, f, g\}$) are given.

2. Preliminaries

Let E_1^3 be a 3-dimensional Minkowski space endowed with the standard flat metric given by

$$g = dx_1^2 + dx_2^2 - dx_3^2$$

where (x_1, x_2, x_3) is a standard rectangular coordinate system of E_1^3. An arbitrary vector $x = (x_1, x_2, x_3)$ in E_1^3 can be one of three Lorentzian causal characters; it can be space-like if $g(x, x) > 0$ or time-like if $g(x, x) < 0$ or null (light-like) if $g(x, x) = 0$. A curve $\alpha : I \subset R \to E_1^3$ with arc-length parameter s can be a locally space-like, timelike or null (light-like), if all of its velocity vectors $\alpha'(s)$ are respectively space-like, time-like or null (light-like). We say that a time-like vector is future pointing or past pointing if the first compound of the vector is positive or negative, respectively. A space-like or time-like curve $\alpha(s)$ has a unit speed, if $g(\alpha'(s), \alpha'(s)) = \pm 1$.

The Minkowski wedge product of two vectors in E_1^3 is defined by

$$x \wedge_1 y = (x_3y_2 - x_2y_3, x_1y_3 - x_3y_1, x_1y_2 - x_2y_1).$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ are any vectors in E_1^3. The norm of x is denoted by $\|x\| = \sqrt{|g(x, x)|}$ and two vectors x and y in E_1^3 are said to be orthogonal, if $g(x, y) = 0$. The Lorentzian (or de Sitter) unit sphere and hyperbolic unit sphere in E_1^3, are given, respectively, by

$$
\begin{aligned}
S_1^2 &= \{x = (x_1, x_2, x_3) \in \mathbb{E}_1^3 / \ g(x, x) = 1\} \text{ and} \\
H^2 &= \{x = (x_1, x_2, x_3) \in \mathbb{E}_1^3 / \ g(x, x) = -1\}.
\end{aligned}
$$

Let $\alpha : I \subset R \to E_1^3$ be a unit speed curve with arc-length parameter s. In general, a moving frame of a curve α with arclength parameter s in Minkowski 3-space E_1^3 is defined by 4-tuple $(\alpha(s); N_1, N_2, N_3)$; where $\alpha(s)$ is a choice on the curve α and (N_1, N_2, N_3) is an any orthonormal basis of the vector space E_1^3 based at $\alpha(t)$. The vector $N_1(s)$ is a unit normal vector, $N_2(s)$ is perpendicular to the vector $N_1(s)$ and $N_3(s) = N_1(s) \times N_2(s)$ for every parameter s.

Since, the frame (N_1, N_2, N_3) is orthonormal under the inner product of Minkowski 3-space E_1^3, the change of the frame and its derivative is given by the following matrix

$$
\begin{bmatrix} N_1' \\ N_2' \\ N_3' \end{bmatrix} = \begin{bmatrix} 0 & \varkappa_1 & \varepsilon_3 \varkappa_2 \\ -\varepsilon_1 \varepsilon_2 \varkappa_1 & 0 & \varkappa_3 \\ \varkappa_2 & -\varepsilon_2 \varepsilon_3 \varkappa_3 & 0 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \tag{2.1}
$$

and

$$
\begin{aligned}
&< N_1, N_1 >_1 = \varepsilon_1; \ < N_2, N_2 >_1 = \varepsilon_2; \ < N_3, N_3 >_1 = \varepsilon_3 \text{ and} \\
&< N_1, N_2 >_1 = < N_2, N_3 >_1 = < N_3, N_1 >_1 = 0
\end{aligned}
$$

where $\varkappa_i(s)_{i=\overline{1,3}}$ are a curvature functions and $(\varepsilon_i)_{i=1,2} = \pm 1$.

A curve $\alpha : I \subset R \to E_1^3$ with a unit tangent vector and any orthogonal frame apparatus $\{N_1, N_2, N_3, \varkappa_1, \varkappa_3\}$ is called a helix if for all parallel vector field X along α make a constant angle with N_1, it is characterized by the ratio of its curvature and torsion is constant i.e

$$
g(N_1, X) \text{ is constant if and only if } \frac{\varkappa_1}{\varkappa_3} \text{ is constant}
$$

We call α a N_i-helix if $g(V_i, X) = c$ non-zero constant for $i = 1, 2, 3$. Moreover, α is a Tzitzéica curve or satisfies the Tzitzéica condition if

$$
\frac{\varkappa_2}{d} \text{ is nonzero constant}
$$

The general Darboux vector field D in the terms of the moving frame apparatus $\{N_1, N_2, N_3, \varkappa_1, \varkappa_3\}$ in E_1^3, can be expressed as

$$
D = \varepsilon_1 \varkappa_3(s) N_1(s) + \varepsilon_1 \varepsilon_2 \varkappa_2(s) N_2(s) + \varepsilon_2 \varepsilon_3 \varkappa_1(s) N_3(s)
$$

with following symmetrical properties

$$
\begin{cases} \mathcal{D} \times N_1 = N_1' \\ \mathcal{D} \times N_2 = N_2' \\ \mathcal{D} \times N_3 = N_3' \end{cases} \tag{2.2}
$$

When $\varkappa_2(s) = 0$, the modified Darboux vector field is defined in ([4]) by

$$
\overline{\mathcal{D}} = \varepsilon_1 \frac{\varkappa_3}{\varkappa_1}(s) N_1(s) + \varepsilon_2 \varepsilon_3 N_3(s) \tag{2.3}
$$

with the condition $\varkappa_1(s) \neq 0$ and the new modified Darboux vector field by

$$\overline{\overline{\mathcal{D}}} = \varepsilon_1 N_1(s) + \varepsilon_2\varepsilon_3 \frac{\varkappa_1}{\varkappa_3}(s) N_3(s)$$

such that $\varkappa_3(s) \neq 0$.

3. Frame of D in E_1^3

In this section, we give a Frenet frame apparatus of the Darboux vectors $\overline{\mathcal{D}}$ and $\overline{\overline{\mathcal{D}}}$. Let α be a spacelike or timelike unit speed curve with arclength parameter s in Minkowski 3-space E_1^3 with any orthogonal frame apparatus $\{N_1, N_2, N_3, \varkappa_1, \varkappa_3\}$.

Using the formula (2.3), the differentiating of $\overline{\mathcal{D}}$ with respect to s give

$$\frac{d\overline{\mathcal{D}}}{d\widetilde{s}} \frac{d\widetilde{s}}{ds} = \varepsilon_1 (\frac{\varkappa_3}{\varkappa_1})'(s) N_1(s)$$

we can easily see that

$$d\widetilde{s} = (\frac{\varkappa_3}{\varkappa_1})'(s) ds$$

The unit tangent vector field $T_{\overline{\mathcal{D}}}$ of $\overline{\mathcal{D}}$ is parallel to the vector field $N_1(s)$; that is,

$$T_{\overline{\mathcal{D}}} = N_1(s) \tag{3.1}$$

Differentiating the equation (3.1) with respect to s,

$$\varkappa_{\overline{\mathcal{D}}} N_{\overline{\mathcal{D}}} (\frac{\varkappa_3}{\varkappa_1})'(s) = \varkappa_1(s) N_2(s)$$

then

$$\varkappa_{\overline{\mathcal{D}}} = \frac{\varkappa_1}{(\frac{\varkappa_3}{\varkappa_1})'} \tag{3.2}$$

So, $N_{\overline{\mathcal{D}}}$ is parallel to the vector field $N_2(s)$ and we have

$$N_{\overline{\mathcal{D}}} = N_2(s) \tag{3.3}$$

If we differentiate the last equation with respect to s, then we obtain

$$B_{\overline{\mathcal{D}}} = N_3(s) \tag{3.4}$$

the torsion is given by

$$\tau_{\overline{\mathcal{D}}} = \frac{\varkappa_3}{(\frac{\varkappa_3}{\varkappa_1})'}(s) \tag{3.5}$$

From the above formulas, we can give the following proposition.

Proposition 3.1. *Let α be a not helix unit speed curve in Minkowski 3-space E_1^3 with any orthogonal frame apparatus $\{N_1, N_2, N_3, \varkappa_1, \varkappa_3\}$. Then Frenet frame apparatus $\{T_{\overline{\mathcal{D}}}, N_{\overline{\mathcal{D}}}, B_{\overline{\mathcal{D}}}, \varkappa_{\overline{\mathcal{D}}}, \tau_{\overline{\mathcal{D}}}\}$ of the modified Darboux vector $\overline{\mathcal{D}}$ is given by*

$$\begin{cases} T_{\overline{\mathcal{D}}} = N_1(s) \\ N_{\overline{\mathcal{D}}} = N_2(s) \\ B_{\overline{\mathcal{D}}} = N_3(s) \end{cases}, \quad \varkappa_{\overline{\mathcal{D}}} = \frac{\varkappa_1}{(\frac{\varkappa_3}{\varkappa_1})'}(s) \text{ and } \tau_{\overline{\mathcal{D}}} = \frac{\varkappa_3}{(\frac{\varkappa_3}{\varkappa_1})'}(s). \tag{3.6}$$

Similarly, the Frenet frame $\{T_{\overline{\overline{\mathcal{D}}}}, N_{\overline{\overline{\mathcal{D}}}}, B_{\overline{\overline{\mathcal{D}}}}, \varkappa_{\overline{\overline{\mathcal{D}}}}, \tau_{\overline{\overline{\mathcal{D}}}}\}$ apparatus of the new modified Darboux vector $\overline{\overline{\mathcal{D}}}$ is

$$\left\{\begin{array}{l} T_{\overline{\overline{\mathcal{D}}}} = N_3(s) \\ N_{\overline{\overline{\mathcal{D}}}} = -N_2(s) \\ B_{\overline{\overline{\mathcal{D}}}} = N_1(s) \end{array}\right., \quad \varkappa_{\overline{\overline{\mathcal{D}}}} = \frac{\varkappa_3}{(\frac{\varkappa_1}{\varkappa_3})'}(s) \text{ and } \tau_{\overline{\overline{\mathcal{D}}}} = \frac{\varkappa_1}{(\frac{\varkappa_1}{\varkappa_3})'}(s). \tag{3.7}$$

4. The vectors D satisfying the Tzitzéica condition in E_1^3

We suppose in this section that α is a spacelike or timelike unit speed curve with arclength parameter s in Minkowski 3-space E_1^3 with any orthogonal frame apparatus $\{N_1, N_2, N_3, \varkappa_1, \varkappa_3\}$ defined in the equation (2.1) with $\varkappa_2 = 0$.

The characterizations of the Darboux vectors D to satisfying the Tzitzéica condition are given in the following propositions.

Theorem 4.1. *Let α be a not N_1 or N_3-helix curve in E_1^3. The Darboux vectors $\overline{\mathcal{D}}$ and $\overline{\overline{\mathcal{D}}}$ are a Tzitzéica curves if and only if the ratio $\frac{\varkappa_3}{(\frac{\varkappa_3}{\varkappa_1})'}$ and $\frac{\varkappa_1}{(\frac{\varkappa_1}{\varkappa_3})'}$ are constants respectively.*

Proof. Let α be a not N_1-helix curve in E_1^3 (i.e $(\frac{\varkappa_3}{\varkappa_1})'(s) \neq 0$) and considering the modified Darboux vector

$$\overline{\mathcal{D}}=\varepsilon_1 \frac{\varkappa_3}{\varkappa_1}(s) N_1(s) + \varepsilon_2\varepsilon_3 N_3(s).$$

From the equation (1.1), $\overline{\mathcal{D}}$ is a Tzitzéica curve if and only if

$$\frac{\tau_{\overline{\mathcal{D}}}}{< \overline{\mathcal{D}}, B_{\overline{\mathcal{D}}} >^2} \text{ is a constant}$$

where $\tau_{\overline{\mathcal{D}}}$ is the torsion of the $\overline{\mathcal{D}}$. Using the equation (2.3) and the Proposition 3.1, we have

$$< \overline{\mathcal{D}}, B_{\overline{\mathcal{D}}} >= 1$$

The Tzitzéica condition for $\overline{\mathcal{D}}$ give

$$\frac{\tau_{\overline{\mathcal{D}}}}{< \overline{\mathcal{D}}, B_{\overline{\mathcal{D}}} >^2} = c \Longrightarrow \frac{\varkappa_3}{(\frac{\varkappa_3}{\varkappa_1})'}(s) = c$$

Similar, when α is not N_3-helix curve in E_1^3, we can prove that $\overline{\overline{\mathcal{D}}}$ is Tzitzéica curve if and only if $\frac{\varkappa_1}{(\frac{\varkappa_1}{\varkappa_3})'}(s) = c$ (nonzero constant). $\square$

For some particular case of a curvature functions $\varkappa_1(s)$ and $\varkappa_3(s)$, the Darboux vector D satisfies the Tzitzéica condition in the following proposition.

Proposition 4.2. *Let $\varkappa_1 = \lambda e^s$ and $\varkappa_3 = c$ (constant) be a curvature functions with frame apparatus $\{N_1, N_2, N_3, \varkappa_1, \varkappa_3\}$, then the Darboux vector*

$$\mathcal{D} = c\varepsilon_1 N_1 + \varepsilon_2\varepsilon_3\varkappa_1 N_3$$

is a Tzitzéica curve.

Proof. Let α be a curve in E_1^3 with the curvatures functions $\varkappa_1 = \lambda e^s$ and $\varkappa_3 = c$. The differentiating of D with respect to s give

$$\frac{d\mathcal{D}}{d\overline{s}}\frac{d\overline{s}}{ds} = \varepsilon_2\varepsilon_3\varkappa_1{}'(s)N_3(s)$$

and

$$d\overline{s} = \varepsilon_2\varepsilon_3\varkappa_1(s)ds$$

The unit tangent vector field $T_\mathcal{D}$ is

$$T_\mathcal{D} = N_3(s) \tag{4.1}$$

and we have

$$\varkappa_\mathcal{D} N_\mathcal{D} \varkappa_1(s) = -N_2(s)$$

The curvature $\varkappa_\mathcal{D}$ of D is

$$\varkappa_\mathcal{D} = \frac{-1}{\varkappa_1(s)'} \text{ and } N_\mathcal{D} = -N_2(s) \tag{4.2}$$

we differentiate $N_\mathcal{D}$ with respect to s, then

$$B_\mathcal{D} = N_1(s) \tag{4.3}$$

and the torsion of D is

$$\tau_\mathcal{D} = \frac{\varkappa_1(s)}{(\varkappa_1)'(s)} \tag{4.4}$$

so, the orthogonal frame apparatus of D is

$$\begin{cases} T_\mathcal{D} = N_3(s) \\ N_\mathcal{D} = -N_2(s) \\ B_\mathcal{D} = N_1(s) \end{cases} \tag{4.5}$$

We have

$$\frac{\tau_\mathcal{D}}{<\mathcal{D}, B_\mathcal{D}>^2} = \frac{\tau_\mathcal{D}}{c^2} = \frac{\varkappa_1(s)}{c^2(\varkappa_1)'(s)} = \frac{\lambda e^s}{c^2\lambda e^s} = \frac{1}{c^2} \text{ constant.}$$

The ration $\frac{\tau_\mathcal{D}}{<\mathcal{D}, B_\mathcal{D}>^2}$ is constant then the Darboux vector D satisfies the Tzitzéica condition. □

Corollary 4.3. *Let α be a curve in E_1^3 with Frenet frame apparatus $\{T, N, B, \kappa, \tau\}$. If the curve α has a curvature $\kappa(s) = \lambda e^s$ and constant torsion then D is Tzitzéica curve.*

5. The vectors D and the spherical indicatrix curves satisfying the Tzitzéica condition in E_1^3

At the first, we recall the definition of the spherical indicatrix curves and these properties in different frame apparatus (See [1], [5] and [7]).

Definition 5.1. *Let α be a spacelike or timelike unit speed curve in E_1^3 with any frame apparatus $\{N_1, N_2, N_3, \varkappa_1, \varkappa_3\}$. The unit vectors N_1, N_2 and N_3 along the curve $\alpha(s)$ generate a curve $N_1(s)$, $N_2(s)$ and $N_3(s)$ on the sphere of radius 1 about the origin. The curve $N_1(s)$, $N_2(s)$ and $N_3(s)$ is called the spherical indicatrix of the vectors N_1, N_2 and N_3 respectively.*

Proposition 5.2 ([3])**.** *Let $\alpha : I \to E_1^3$ be a not N_1 or N_3-helix spacelike or timelike unit speed curve in Minkowski space E_1^3 with any orthonormal frame apparatus $\{N_1, N_2, N_3, \varkappa_1, \varkappa_3\}$. Then the following statements hold.*
i: $\frac{\varkappa_3}{\varkappa_1}(s)$ is the geodesic curvature of the curve $N_1(s)$ with $\varkappa_1(s) \neq 0$.
ii: $\frac{\varkappa_1}{\varkappa_3}(s)$ is the geodesic curvature of the curve $N_3(s)$ where $\varkappa_3(s) \neq 0$.

Proposition 5.3 ([3])**.** *Let $\alpha : I \to E_1^3$ be a not slant helix spacelike or timelike unit speed curve in E_1^3 with the alternative moving frame apparatus $\{N, C = \frac{N'}{\|N'\|}, W = N \times C, f, g\}$ defined in (*[12]*), along the curve α . Then the following statements hold.*
i: $\frac{g}{f}(s)$ is the geodesic curvature of the curve $N(s)$ with $f(s) \neq 0$.
ii: $\frac{f}{g}(s)$ is the geodesic curvature of the curve $W(s)$ with $g(s) \neq 0$.

Now, we give the relationships between D and the spherical indicatrices to satisfying the Tzitzéica condition in E_1^3.

Theorem 5.4. *Let α be a not N_1-helix spacelike or timelike unit speed curve in E_1^3. $\overline{\mathcal{D}}$ is Tzitzéica curve if $\kappa_g = c \int \varkappa_3 ds$, where κ_g is the geodesic curvature of $N_1(s)$.*

Proof. Let α be a not helix curve in E_1^3 (i.e $(\frac{\varkappa_3}{\varkappa_1})'(s) \neq 0$) and considering the modified Darboux vector

$$\overline{\mathcal{D}} = \varepsilon_1 \frac{\varkappa_3}{\varkappa_1}(s) N_1(s) + \varepsilon_2 \varepsilon_3 N_3(s).$$

From the Theorem 4.1, $\overline{\mathcal{D}}$ is a Tzitzéica curve if

$$\frac{\varkappa_3(s)}{(\frac{\varkappa_3}{\varkappa_1}(s))'} = c \text{ is a constant}$$

and from the Proposition 5.2, the geodesic curvature κ_g of $T_{\overline{\mathcal{D}}}$ is $\frac{\varkappa_3}{\varkappa_1}(s)$ then

$$\kappa_g' = c\varkappa_3(s) \text{ and } \kappa_g = c \int \varkappa_3(s) ds$$

□

Similarly, we give the Tzitzéica condition for the modified Darboux vector $\overline{\overline{\mathcal{D}}}$ in the following theorem.

Theorem 5.5. *Let α be a not N_3-helix spacelike or timelike unit speed curve in E_1^3. If $\kappa_g = c \int \varkappa_1 ds$ then $\overline{\overline{\mathcal{D}}}$ is a Tzitzéica curve where κ_g is the geodesic curvature of $N_3(s)$.*

Corollary 5.6. *Let α be a not helix spacelike or timelike unit speed and curve in E_1^3 with Frenet frame apparatus $\{T, N, B, \kappa, \tau\}$ Then*
1/ If the geodesic curvature of T is $\kappa_g = c\int \tau ds$ then the modified Darboux vector $\overline{\mathcal{D}}$ is a Tzitzéica curve.
2/ If the geodesic curvature of B is $\kappa_g = c\int \kappa ds$ then the modified Darboux vector $\overline{\overline{\mathcal{D}}}$ is a Tzitzéica curve.

Lemma 5.7 ([2])**.** *Let $\alpha(s)$ be a not N_1 or N_3-helix spacelike or timelike unit speed curve in E_1^3 with moving frame apparatus $\{N_1, N_2, N_3, \varkappa_1, \varkappa_3\}$. Then we have*
1/ If the ratio $\dfrac{\varkappa_1(\varkappa_3/\varkappa_1)'}{\varkappa_3^2}(s)$ is constant then $N_1(s)$ is a Tzitzéica curve.
2/ If the ratio $\dfrac{\varkappa_3(\varkappa_1/\varkappa_3)'}{\varkappa_1^2}(s)$ is constant then $N_3(s)$ is a Tzitzéica curve.

Theorem 5.8. *Let α be a not N_1 or N_3-helix spacelike or timelike unit speed curve in E_1^3 with moving frame apparatus $\{N_1, N_2, N_3, \varkappa_1, \varkappa_3\}$. Then $N_3(s)$ (resp $N_1(s)$) is a Tzitzéica curve if and only if $\overline{\mathcal{D}}$ (resp. $\overline{\overline{\mathcal{D}}}$) is a Tzitzéica curve.*

Proof. Let α be a not N_3-helix curve in E_1^3 (i.e. $\varkappa_1/\varkappa_3(s)$ is constant) and let $N_3(s)$ be a Tzitzéica curve. From the Lemma 5.7, we have

$$\frac{\varkappa_3(\varkappa_1/\varkappa_3)'}{\varkappa_1^2}(s) = \overline{c} \text{ (constant)} \tag{5.1}$$

using the Proposition 5.2, the geodesic curvature of $N_1(s)$ is

$$\kappa_g = \frac{\varkappa_3}{\varkappa_1}(s) \tag{5.2}$$

from the equations (5.1) and (5.2), we get the ordinary differential equation

$$\frac{\varkappa_3^2(\varkappa_1/\varkappa_3)'}{\varkappa_1^2}(s) = \kappa_g^2(1/\kappa_g)' = \overline{c}\varkappa_3(s) \tag{5.3}$$

and

$$-\kappa_g{}' = \overline{c}\varkappa_3(s) \tag{5.4}$$

the solution of the ODE given in theequation (5.4) is

$$\kappa_g = c\int \varkappa_3 ds$$

then from the Theorem 5.5, $\overline{\mathcal{D}}$ is a Tzitzéica curve.
It is similar to prove that if α is not N_1-helix curve in E_1^3, then $N_1(s)$ is a Tzitzéica curve if and only if $\overline{\overline{\mathcal{D}}}$ is a Tzitzéica curve. □

Corollary 5.9. *Let α be a not helix spacelike or timelike unit speed curve in E_1^3 with Frenet frame apparatus $\{T, N, B, \kappa, \tau\}$, then*
1/ The curve $T(s)$ is a Tzitzéica curve if and only if the modified Darboux vector

$$\overline{\overline{\mathcal{D}}} = \varepsilon_1 T(s) + \varepsilon_2\varepsilon_3 \frac{\kappa}{\tau}(s)B(s).$$

is a Tzitzéica curve.
2/ The curve $B(s)$ is a Tzitzéica curve if and only if the modified Darboux vector

$$\overline{\mathcal{D}}= \varepsilon_1 \frac{\tau}{\kappa}(s)T(s) + \varepsilon_2\varepsilon_3 B(s)$$

is a Tzitzéica curve.

Proof. It's direct consequence of the Theorem 5.8. □

From the Theorem 5.8 and the proposition 5.3, we can give the following corollary

Corollary 5.10. *Let α be a not slant helix spacelike or timelike unit speed curve in E_1^3 with frame apparatus $\{N, C = \frac{N'}{\|N'\|}, W = N \times C, f, g\}$, then*
1/ The curve $N(s)$ is a Tzitzéica curve if and only if the modified Darboux vector

$$\overline{\overline{\mathcal{D}}} = \varepsilon_1 T(s) + \varepsilon_2\varepsilon_3 \frac{f}{g}(s)B(s).$$

is a Tzitzéica curve.
2/ The curve $W(s)$ is a Tzitzéica curve if and only if the modified Darboux vector

$$\overline{\mathcal{D}}= \varepsilon_1 \frac{g}{f}(s)T(s) + \varepsilon_2\varepsilon_3 B(s)$$

is a Tzitzéica curve.

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Department of mathematics, Faculty of Science, University of Saïda, Algeria.
Email address: `f.hathout@gmail.com`

Department of Mathematics, Faculty of Science, University of Ankara, Tandoğan, Ankara, Turkey.
Email address: `yayli@science.ankara.edu.tr`