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Generalization of Titchmarsh's Theorem for the Bessel transform

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Abstract: Using a generalized translation operator, we obtain a generalization of Titchmarsh's Theorem for the Bessel transform for functions satisfying the ψ-Bessel Lipschitz condition in $\mathrm{L}_{2,p}(\mathbb{R})$.
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1 Introduction and preliminaries

Titchmarsh [6, Theorem 85] characterized the set of functions in $\mathrm{L}^2(\mathbb{R})$ satisfying the Cauchy Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transforms. We have

Theorem 1.1 *[6] Let $\alpha \in (0,1)$ and assume that $f \in \mathrm{L}^2(\mathbb{R})$. Then the following are equivalent:*

(1) $\|f(t+h) - f(t)\|_{\mathrm{L}^2(\mathbb{R})} = O(h^\alpha)$ *as* $h \longrightarrow 0$

(2) $\int_{|\lambda|\geq r} |g(\lambda)|^2 d\lambda = O(r^{-2\alpha})$ *as* $r \longrightarrow \infty$
where g stands for the Fourier transform of f.

In this paper, we obtain a generalization of Theorem 1.1 for the Bessel operator.

Bessel transform and its inverse are widely used to solve various in calculs, mechanics, mathemtical, physics, and computational mathematics (see, e.g.,[6, 7]).

Let

$$\mathrm{B} = \frac{d^2}{dx^2} + \frac{(2p+1)}{x}\frac{d}{dx}.$$

be the Bessel differential operator.

For $p \geq -\frac{1}{2}$, we introduce the Bessel normalized function of the first kind j_p defined by

$$j_p(x) = \Gamma(p+1)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+p+1)}(\frac{x}{2})^{2n}, \tag{1}$$

where $\Gamma(x)$ is the gamma-function(see[4]).

Moreover, from (1) we see that

$$\lim_{x\longrightarrow 0}\frac{j_p(x)-1}{x^2} \neq 0$$

by consequence, there exist $c > 0$ and $\eta > 0$ satisfying

$$|x| \leq \eta \Longrightarrow |j_p(x)-1| \geq c|x|^2. \tag{2}$$

The function $y = j_p(x)$ satisfies the differential equation

$$\mathrm{B}y + y = 0$$

with the initial conditions $y(0) = 1$ and $y'(0) = 0$. $j_p(x)$ is function infinitely differentiable, even, and, moreover entire analytic.

Lemma 1.2 *The following inequalities are valid for Bessel fonction* j_p

1. $|j_p(x)| \leq C, \quad \forall x \in \mathbb{R}^+$, *where* C *is a positive constant*
2. $1 - j_p(x) = O(x^2), \quad 0 \leq x \leq 1$

Proof. (See [1]). ■

$\mathrm{L}_{2,p}(\mathbb{R}^+), \quad p \geq -\frac{1}{2}$ is the Hilbert space of measurable functions $f(x)$ on $\mathbb{R}^+$ with the finite norm

$$\|f\|_{2,p} = \left(\int_0^{\infty}|f(x)|^2 x^{2p+1}dx\right)^{1/2}$$

The generalized Bessel translation T_h defined by

$$\mathrm{T}_h f(t) = c_p \int_0^\pi f(\sqrt{t^2 + h^2 - 2th\cos\varphi}) \sin^{2p} \varphi d\varphi,$$

where

$$c_p = \left(\int_0^\pi \sin^{2p} \varphi d\varphi\right)^{-1} = \frac{\Gamma(p+1)}{\Gamma(\frac{1}{2})\Gamma(p+\frac{1}{2})}.$$

The Bessel transform we call the integral transform from [4, 3, 5]

$$\widehat{f}(\lambda) = \int_0^\infty f(t) j_p(\lambda t) t^{2p+1} dt, \quad \lambda \in \mathbb{R}^+$$

The inverse Bessel transform is given by the formula

$$f(t) = (2^p\Gamma(p+1))^{-2} \int_0^\infty \widehat{f}(\lambda) j_p(\lambda t) \lambda^{2p+1} d\lambda$$

i.e the direct and inverse Bessel transforms differ by the factor $(2^p\Gamma(p+1))^{-2}$.

The following relations connect the Bessel generalized translation and the Bessel transform, in [2] we have

$$(\widehat{\mathrm{T}_h f})(\lambda) = j_p(\lambda h)\widehat{f}(\lambda) \tag{3}$$

2 Main Result

In this section we give the main result of this paper, We need first to define ψ-Bessel Lipschitz class.

Definition 2.1 *A function $f \in \mathrm{L}_{2,p}(\mathbb{R}^+)$ is said to be in the ψ-Bessel Lipschitz class, denote by $Lip(\psi, p, 2)$, if*

$$\|\mathrm{T}_h f(t) - f(t)\|_{2,p} = O(\psi(h)), \quad \textit{as } h \longrightarrow 0,$$

where $\psi(t)$ is a continuous increasing function on $[0, \infty)$, $\psi(0) = 0$ and $\psi(ts) = \psi(t)\psi(s)$ for all $t, s \in [0, \infty)$ and this function verify $\int_0^{1/h} s\psi(s^{-2})ds = O(\frac{1}{h^2}\psi(h^2))$ as $h \longrightarrow 0$

Theorem 2.2 *Let $f \in \mathrm{L}_{2,p}(\mathbb{R}^+)$. Then the following are equivalents*

1. *$f \in Lip(\psi, p, 2)$.*

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2. $\int_r^\infty |\widehat{f}(\lambda)|^2\lambda^{2p+1}d\lambda = O(\psi(r^{-2}))$ *as* $r \longrightarrow +\infty$.

Proof. $1 \Longrightarrow 2$: Suppose that $f \in Lip(\psi, p, 2)$. Then we obtain

$$\|\mathrm{T}_h f(t) - f(t)\|_{2,p} = O(\psi(h)), \quad as\ h \longrightarrow 0$$

Parseval's identity and formula (3) give

$$\|\mathrm{T}_h f(t) - f(t)\|_{2,p}^2 = \frac{1}{(2^p\Gamma(p+1))^2}\int_0^\infty |1 - j_p(\lambda h)|^2 |\widehat{f}(\lambda)|^2 \lambda^{2p+1} d\lambda.$$

From (2), we have

$$\int_{\frac{\eta}{2h}}^{\frac{\eta}{h}} |1 - j_p(\lambda h)|^2 |\widehat{f}(\lambda)|^2 \lambda^{2p+1} d\lambda \geq \frac{c^2\eta^4}{16}\int_{\frac{\eta}{2h}}^{\frac{\eta}{h}} |\widehat{f}(\lambda)|^2 \lambda^{2p+1} d\lambda$$

We see that

$$\int_{\frac{\eta}{2h}}^{\frac{\eta}{h}} |1 - j_p(\lambda h)|^2 |\widehat{f}(\lambda)|^2 \lambda^{2p+1} d\lambda \leq \int_0^\infty |1 - j_p(\lambda h)|^2 |\widehat{f}(\lambda)|^2 \lambda^{2p+1} d\lambda$$

There exists then a positive constant C_2 such that

$$\int_{\frac{\eta}{2h}}^{\frac{\eta}{h}} |\widehat{f}(\lambda)|^2 \lambda^{2p+1} d\lambda \leq C_2\psi(h^2).$$

We obtain

$$\int_r^{2r} |\widehat{f}(\lambda)|^2 \lambda^{2p+1} d\lambda \leq C_2\psi(2^{-2}\eta^2 r^{-2}).$$

Thus there exists then a positive constant K such that

$$\int_r^{2r} |\widehat{f}(\lambda)|^2 \lambda^{2p+1} d\lambda \leq K\psi(r^{-2}), \quad for\ all\ r > 0.$$

So that

$$\begin{aligned}
\int_r^\infty |\widehat{f}(\lambda)|^2 \lambda^{2p+1} d\lambda &= [\int_r^{2r} + \int_{2r}^{4r} + \int_{4r}^{8r} ....] |\widehat{f}(\lambda)|^2 \lambda^{2p+1} d\lambda \\
&= O(\psi(r^{-2}) + \psi(2^{-2}r^{-2})......) \\
&= O(\psi(r^{-2}) + \psi(r^{-2}) + ...) \\
&= O(\psi(r^{-2})).
\end{aligned}$$

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This prove that

$$\int_r^{\infty} |\widehat{f}(\lambda)|^2 \lambda^{2p+1} d\lambda = O(\psi(r^{-2})) \quad as \ r \longrightarrow +\infty$$

$2 \Longrightarrow 1$: Suppose now that

$$\int_r^{\infty} |\widehat{f}(\lambda)|^2 \lambda^{2p+1} d\lambda = O(\psi(r^{-2})) \quad as \ r \longrightarrow +\infty$$

We write

$$\int_0^{\infty} |1 - j_p(\lambda h)|^2 |\widehat{f}(\lambda)|^2 \lambda^{2p+1} d\lambda = \mathrm{I}_1 + \mathrm{I}_2,$$

where

$$\mathrm{I}_1 = \int_0^{\frac{1}{h}} |1 - j_p(\lambda h)|^2 |\widehat{f}(\lambda)|^2 \lambda^{2p+1} d\lambda$$

and

$$\mathrm{I}_2 = \int_{\frac{1}{h}}^{\infty} |1 - j_p(\lambda h)|^2 |\widehat{f}(\lambda)|^2 \lambda^{2p+1} d\lambda$$

Estimate the summands I_1 and I_2.

Firstly, we have from (1) in Lemma 1.2

$$\mathrm{I}_2 \leq (1 + C)^2 \int_{\frac{1}{h}}^{\infty} |\widehat{f}(\lambda)|^2 \lambda^{2p+1} d\lambda = O(\psi(h^2)$$

Set

$$\phi(x) = \int_x^{\infty} |\widehat{f}(\lambda)|^2 \lambda^{2p+1} d\lambda$$

We know from Lemma 1.2 that $|1 - j_p(\lambda h)| \leq C_1 \lambda^2 h^2$ for $\lambda h \leq 1$. Then $\mathrm{I}_1 \leq -C_1 h^2 \int_0^{\frac{1}{h}} x^2 \phi'(x) dx$.

We use integration by parts, we obtain

$$\begin{aligned} \mathrm{I}_1 &\leq -C_1 h^2 \int_0^{\frac{1}{h}} x^2 \phi'(x) dx \\ &\leq -C_1 \phi(\frac{1}{h}) + 2C_1 h^2 \int_0^{\frac{1}{h}} x \phi(x) dx \end{aligned}$$

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$$
\begin{aligned}
&\leq C_3 h^2 \int_0^{\frac{1}{h}} x\psi(x^{-2})dx \\
&\leq C_3 h^2 \frac{1}{h^2}\psi(h^2) \\
&\leq C_3\psi(h^2),
\end{aligned}
$$

where C_3 is a positive constant and this ends the proof. ■

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