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Positive solution to a higher order fractional boundary value problem with fractional integral condition*†

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Abstract

We study a boundary value problem for differential equations involving Caputo derivative of order $\alpha \in (n-1,n)$, $n \geq 2$. Existence, uniqueness and positivity results of solutions are established by virtue of fixed point theorems. Three examples are given to illustrate our results.

1 Introduction

The purpose of the present work is to investigate sufficient conditions for the existence, uniqueness and positivity of solution for the following higher order two-point fractional boundary value problem (P1):

$$^{c}D_{0^+}^{\alpha}u(t) = f(t, u(t), u'(t), ..., u^{(n-2)}(t)), 0 < t < 1 \tag{1.1}$$

$$u\,(0) = u'(0) = ... = u^{n-2}(0) = 0, u^{n-1}\,(0) = I_{0^+}^{n-1}(1), \tag{1.2}$$

where $f : [0,1] \times \mathbb{R}^{n-1} \to \mathbb{R}$ is a given function, $n = [\alpha] + 1$ ($[\alpha]$ is the entire part of α), $n \geq 2$ and $I_{0^+}^{n-1}$ denotes the Riemann-Liouville fractional integral. We mention that integral boundary conditions are encountered in various applications such as population dynamics, blood flow models, chemical engineering, cellular systems, heat transmission, plasma physics, thermoelasticity,... Non local conditions come up when values of the function on the boundary is connected to values inside the domain, they have a physical signification such total mass, moments, etc...

*Mathematics Subject Classifications: 34B10, 26A33, 34B15

†Keywords: Positive solution, Fractional Caputo derivative, Banach Contraction principle, Leray Schauder nonlinear alternative, Guo-Krasnoselskii Theorem.

It is known that the calculus of fractional derivatives and integrals does not have clear geometrical and physical interpretations. Moreover, the physical meaning is difficult to clarify and the definitions themselves are no more stringent than those of integer order.

Differential equations with fractional order derivative have recently proved to be the best tools in the modeling of many physical phenomena [4,15], consequently, many aspects of the theory need to be further investigated even the calculus of fractional derivatives and integrals does not have clear geometrical and physical interpretations.

The theory of boundary value problems for nonlinear fractional differential equations need to be explored while numerous applications and physical manifestations of fractional calculus have been found and some existence results for nonlinear fractional boundary value problems was established by the use of techniques of nonlinear analysis such Banach fixed point theorem, Leray-Schauder theory, etc, see [1,3,6,7-11,14-21].

By means of fixed point theorem for the mixed monotone operator, Zhang [19] studied the existence, multiplicity and nonexistence of positive solutions for the following higher order fractional boundary value problem

$$ {}^{c}D_{0+}^{\alpha} u\,(t) + \lambda h(t) f(u) = 0, 0 < t < 1 $$

$$ u\,(1) = u'(0) = ... = u^{n-2}\,(0) = u^{n-1}\,(0) = 0 $$

for $n \geq 2$, $n-1 < \alpha \leq n$.

Feng et all [8] considered an higher-order singular boundary value problem of fractional differential equation with integral boundary conditions:

$$ D_{0+}^{\alpha} u\,(t) + g(t) f(t, u(t)) = 0, 0 < t < 1 $$

$$ u\,(0) = u'(0) = ... = u^{n-2}(0) = 0, u(1) = \int_0^1 h(t) u(t) dt $$

where D_{0+}^{α} is the standard Rimann-Liouville fractional derivative and $n \geq 3$.

The rest of the paper is organized as follows. In the next section, we introduce some definitions, lemmas theorems and notations that will be used later. In Section 3, we investigate the uniqueness and the existence of at least one solution by using Banach contraction principle and Leray Schauder nonlinear alternative. In Section 4, we discuss some properties of Green function associated with the problem (P1), we prove the existence of positive solutions with the help of Guo-Krasnoselskii theorem, finally we give some examples to illustrate the main results.

2 Preliminaries and Lemmas

In this section, we introduce definitions and preliminary facts which are used throughout this paper.

Definition 1 *If $g \in C([a,b])$ and $\alpha > 0$, then the Riemann-Liouville fractional integral is defined by*

$$I_{a+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{g(s)}{(t-s)^{1-\alpha}} ds.$$

Definition 2 *Let $\alpha \geqq 0$, $n = [\alpha] + 1$. If $f \in AC^n[a,b]$ then the Caputo fractional derivative of order α of f defined by*

$${}^{c}D_{a+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{g^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$

exist almost everywhere on $[a,b]$ ($[\alpha]$ means the entire part of the number α).

Lemma 3 *[13] Let $\alpha, \beta > 0$ and $n = [\alpha] + 1$, then the following relations hold: ${}^{c}D_{0^+}^{\alpha} t^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} t^{\beta-\alpha-1}$, $\beta > n$ and ${}^{c}D_{0^+}^{\alpha} t^k = 0$, $k = 0, 1, 2, ..., n-1$.*

Lemma 4 *[13] For $\alpha > 0, g(t) \in C(0,1)$, the homogenous fractional differential equation*

$${}^{c}D_{a+}^{\alpha} g(t) = 0$$

has a solution

$$g(t) = c_1 + c_2 t + c_3 t^2 + ... + c_n t^{n-1},$$

where, $c_i \in R$, $i = 0, ..., n$, and $n = [\alpha] + 1$.

Denote by $L^1([0,1], \mathbb{R})$ the Banach space of Lebesgue integrable functions from $[0,1]$ into $\mathbb{R}$ with the norm $||y||_{L^1} = \int_0^1 |y(t)|\, dt$.

The following Lemmas gives some properties of Riemann-Liouville fractional integrals and Caputo fractional derivative.

Lemma 5 *[2] Let $p, q \geq 0$, $f \in L_1[a,b]$. Then $I_{0^+}^{p} I_{0^+}^{q} f(t) = I_{0^+}^{p+q} f(t) = I_{0^+}^{q} I_{0^+}^{p} f(t)$ and ${}^{c}D_{0^+}^{q} I_{0^+}^{q} f(t) = f(t)$, for all $t \in [a,b]$.*

Lemma 6 *[13] Let $\beta > \alpha > 0$. Then the formula ${}^{c}D_{0^+}^{\alpha} I_{0^+}^{\beta} f(t) = I_{0^+}^{\beta-\alpha} f(t)$, holds almost everywhere on $t \in [a,b]$, for $f \in L_1[a,b]$ and it is valid at any point $t \in [a,b]$ if $f \in C[a,b]$.*

In the following, we cite some fixed point theorems:

Lemma 7 *(Leray-Schauder nonlinear alternative) [5]. Let F be a Banach space and Ω a bounded open subset of F, $0 \in \Omega$. $T : \overline{\Omega} \to F$ be a completely continuous operator. Then, either there exists $x \in \partial\Omega$, $\lambda > 1$ such that $T(x) = \lambda x$, or there exists a fixed point $x^* \in \overline{\Omega}$.*

Theorem 8 *(Guo-Krasnosel'skii fixed point Theorem on cone) [12]. Let E be a Banach space, and let $K \subset E$, be a cone. Assume Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$ and let*

$$\mathcal{A} : K \cap \left(\overline{\Omega_2} \backslash \Omega_1\right) \to K$$

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be a completely continuous operator such that

(i) $||\mathcal{A}u|| \leq ||u||$, $u \in K \cap \partial\Omega_1$, *and* $||\mathcal{A}u|| \geq ||u||$, $u \in K \cap \partial\Omega_2$;

or

(ii) $||\mathcal{A}u|| \geq ||u||$, $u \in K \cap \partial\Omega_1$, *and* $||\mathcal{A}u|| \leq ||u||$, $u \in K \cap \partial\Omega_2$.

Then $\mathcal{A}$ *has a fixed point in* $K \cap \left(\overline{\Omega_2} \backslash \Omega_1\right)$.

3 Existence and Uniqueness results

Now we start by solving an auxiliary problem.

Lemma 9 *Let* $y \in C[0,1]$. *Then the unique solution of the fractional problem*

$$\left\{ \begin{array}{c} {}^cD_{0^+}^{\alpha} u(t) = y(t), \quad 0 < t < 1 \\ u(0) = u'(0) = ... = u^{(n-2)}(0) = 0, \quad u^{(n-1)}(0) = I_{0^+}^{n-1} u(1) \end{array} \right. \tag{3.1}$$

is given by

$$u(t) = \int_0^1 G(t,s) y(s) ds, \tag{3.2}$$

where

$$G(t,s) = \left\{ \begin{array}{c} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\Gamma(2n-1) t^{n-1} (1-s)^{\alpha+n-2}}{\Gamma(n)\Gamma(\alpha+n-1)(\Gamma(2n-1)-1)}, \quad 0 \leq s \leq t \leq 1 \\ \frac{\Gamma(2n-1) t^{n-1} (1-s)^{\alpha+n-2}}{\Gamma(n)\Gamma(\alpha+n-1)(\Gamma(2n-1)-1)}, \quad 0 \leq t \leq s \leq 1. \end{array} \right. \tag{3.3}$$

Proof. Applying the operator $I_{0^+}^{\alpha}$ to both sides of equation (3.1) then using Lemmas 4 and 5, we get

$$u(t) = I_{0^+}^{\alpha} y(t) + c_1 + c_2 t + ... + c_n t^{n-1}, \tag{3.4}$$

condition $u(0) = 0$ implies that $c_1 = 0$. Differentiating both sides of (3.4) successively and using the initial conditions $u'(0) = ... = u^{(n-2)}(0) = 0$, it yields $c_2 = c_3 = ... = c_{n-1} = 0$. The condition $u^{(n-1)}(0) = I_{0^+}^{n-1} u(1)$ implies $c_n = \frac{\Gamma(2n-1)}{\Gamma(n)(\Gamma(2n-1)-1)} I_{0^+}^{\alpha+n-1} y(1)$. Substituting c_i by their values in (3.4), we obtain

$$u(t) = I_{0^+}^{\alpha} y(t) + \frac{t^{n-1}\Gamma(2n-1)}{\Gamma(n)\left(\Gamma(2n-1)-1\right)} I_{0^+}^{\alpha+n-1} y(1) \tag{3.5}$$

that can be written as

$$\begin{aligned} u(t) \quad = \quad & \int_0^t \left[\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\Gamma(2n-1) t^{n-1} (1-s)^{\alpha+n-2}}{\Gamma(n)\Gamma(\alpha+n-1)\left(\Gamma(2n-1)-1\right)} \right] y(s) ds \\ & + \int_t^1 \frac{\Gamma(2n-1) t^{n-1} (1-s)^{\alpha+n-2}}{\Gamma(n)\Gamma(\alpha+n-1)\left(\Gamma(2n-1)-1\right)} y(s) ds \end{aligned}$$

that is equivalent to (3.2) where G is defined by (3.3). The proof is complete. ∎

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In this section we prove the existence and uniqueness of solution in the Banach space E consisting of all functions $u \in C^n[0,1]$ into $\mathbb{R}$, with the norm $||u|| = \sum_{k=0}^{n-2} \max_{t\in[0,1]} \left|u^{(k)}(t)\right|$. Throughout this section, we suppose that $f \in C\left([0,1] \times \mathbb{R}^{n-1}, \mathbb{R}\right)$ and the integral operator $T : E \to E$ is defined by

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s),u'(s),...,u^{(n-2)}(s))ds.$$

Lemma 10 *The function $u \in E$ is solution of the fractional boundary value problem (P1) if and only if $Tu(t) = u(t)$, $\forall t \in [0,1]$.*

Proof. The proof is similar to the one in Lemma 3.2 [10] and is omitted. ∎

Theorem 11 *Assume that there exist nonnegative functions $g_i \in L^1([0,1], \mathbb{R}_+)$, $0 \le i \le n-2$, such that for all $x_i, y_i \in \mathbb{R}$ and $t \in [0,1]$ we have*

$$|f(t,x_0,...,x_{n-2}) - f(t,y_0,...,y_{n-2})| \le \sum_{i=0}^{n-2} g_i(t)|x_i - y_i|, \tag{3.6}$$

and

$$\sum_{k=0}^{n-2}\sum_{i=0}^{n-2}\left(\left\|I_{0^+}^{\alpha-1-k}g_i\right\|_{L^1} + \frac{\Gamma(2n-1)}{\Gamma(n-k)\left(\Gamma(2n-1)-1\right)}I_{0^+}^{\alpha+n-1}g_i(1)\right) < 1. \tag{3.7}$$

Then (P1) has a unique solution u in E.

To prove Theorem 11, we use the following properties of Riemann-Liouville fractional integrals.

Lemma 12 *[10]Let $\alpha > 0$, $f \in L_1([a,b], \mathbb{R}_+)$. Then, for all $t \in [a,b]$ we have*

$$I_{0^+}^{\alpha+1}f(t) \le \|I_{0^+}^{\alpha}f\|_{L^1}$$

We are now in position to prove Theorem 11.

Proof. We claim that T is a contraction. Indeed, let $u, v \in E$, then for $0 \le k \le n-2$, we get

$$(Tu)^{(k)} - (Tv)^{(k)} =$$

$$\begin{aligned} = \quad & I_{0^+}^{\alpha-k}\left(f\left(t,u(t),...,u^{(n-2)}(t)\right) - f\left(t,v(t),...,v^{(n-2)}(t)\right)\right) \\ & + \frac{t^{n-k}\Gamma(2n-1)}{\Gamma(n-k)\left(\Gamma(2n-1)-1\right)} \times \\ & I_{0^+}^{\alpha+n-1}\left[f\left(1,u(1),...,u^{(n-2)}(1)\right) - f\left(1,v(1),...,v^{(n-2)}(1)\right)\right] \end{aligned}$$

By (3.6) and Lemma 12, the following estimate holds

$$\left|(Tu)^{(k)} - (Tv)^{(k)}\right| \le$$

$$\|u-v\| \sum_{i=0}^{n-2}\left(\left\|I_{0^+}^{\alpha-1-k} g_i\right\|_{L^1}+\frac{\Gamma(2n-1)}{\Gamma(n-k)\left(\Gamma(2n-1)-1\right)} I_{0^+}^{\alpha+n-1} g_i(1)\right),$$

hence

$$\|Tu-Tv\| \leq \tag{3.8}$$

$$\|u-v\| \sum_{k=0}^{n-2}\sum_{i=0}^{n-2}\left(\left\|I_{0^+}^{\alpha-1-k} g_i\right\|_{L^1}+\frac{\Gamma(2n-1)}{\Gamma(n-k)\left(\Gamma(2n-1)-1\right)} I_{0^+}^{\alpha+n-1} g_i(1)\right).$$

Thanks to (3.7) it yields

$$\|Tu-Tv\|<\|u-v\|.$$

The Banach contraction principle implies the uniqueness of solution for the fractional boundary value problem (P1). This finishes the proof. ■

We give our second main result for the fractional boundary value problem (P1).

Theorem 13 *Assume that $f(t,0,...,0)\neq 0$ and there exist nonnegative functions $k_i, g \in L^1([0,1],\mathbb{R}_+)$, $\phi_i \in C\left(\mathbb{R}_+,\mathbb{R}_+^*\right)$ nondecreasing on $\mathbb{R}_+$, $0\leq i\leq n-2$ and $r>0$, such that*

$$|f(t,x_0,...,x_{n-2})| \leq g(t)+\sum_{i=0}^{n-2} k_i(t)\,\phi_i(|x_i|), \tag{3.9}$$

$$\sum_{k=0}^{n-2}\left\{\sum_{i=0}^{n-2}\phi_i(r)\left(\left\|I_{0^+}^{\alpha-1-k}k_i\right\|+\frac{\Gamma(2n-1)I_{0^+}^{\alpha+n-1}k_i(1)}{\Gamma(n-k)\left(\Gamma(2n-1)-1\right)}\right)\right. \tag{3.10}$$
$$\left.+\left\|I_{0^+}^{\alpha-1-k}g\right\|+\frac{\Gamma(2n-1)I_{0^+}^{\alpha+n-1}g(1)}{\Gamma(n-k)\left(\Gamma(2n-1)-1\right)}\right\}<r.$$

Then the problem (P1) has at least one nontrivial solution $u^*\in E$.

To prove this theorem, we need the properties of T :

Lemma 14 *$T: E\to E$ is a completely continuous mapping.*

Proof. It is easy to see that T is continuous since f and G are continuous. Let $B_r=\{u\in E; \|u\|\leq r\}$ be a bounded subset in E, we claim that $T(B_r)$ is relatively compact. Indeed:
i) For $u\in B_r$, using (3.9) and the fact that ϕ_i are nondecreasing, we obtain

$$\left|(Tu)^{(k)}(t)\right| \leq \sum_{i=0}^{n-2}\phi_i(r)\left(\left\|I_{0^+}^{\alpha-1-k}k_i\right\|+\frac{\Gamma(2n-1)I_{0^+}^{\alpha+n-1}k_i(1)}{\Gamma(n-k)\left(\Gamma(2n-1)-1\right)}\right) \tag{3.11}$$
$$+\left\|I_{0^+}^{\alpha-1-k}g\right\|+\frac{\Gamma(2n-1)I_{0^+}^{\alpha+n-1}g(1)}{\Gamma(n-k)\left(\Gamma(2n-1)-1\right)}.$$

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Consequently we get

$$\begin{aligned}\|Tu\| &= \sum_{k=0}^{n-2}\left\{\sum_{i=0}^{n-2}\phi_i(r)\left(\left\|I_{0^+}^{\alpha-1-k}k_i\right\|+\frac{\Gamma(2n-1)I_{0^+}^{\alpha+n-1}k_i(1)}{\Gamma(n-k)\left(\Gamma(2n-1)-1\right)}\right)\right.\\ &\left.+\left\|I_{0^+}^{\alpha-1-k}g\right\|+\frac{\Gamma(2n-1)I_{0^+}^{\alpha+n-1}g(1)}{\Gamma(n-k)\left(\Gamma(2n-1)-1\right)}\right\}=C,\end{aligned}$$

then $T(B_r)$ is uniformly bounded.
ii) Let $t_1, t_2 \in [0,1]$, $t_1 < t_2$, $u \in B_r$ and $C = \max\limits_{0\le t\le 1, \|u\|<r}\left|f\left(t, u(t), ..., u^{(n-2)}(t)\right)\right|$, therefore we have

$$\begin{aligned}&\left|T^{(k)}u(t_1)-T^{(k)}u(t_2)\right|\\ \le\ & C(\int_0^{t_1}\left|G_t^{(k)}(t_1,s)-G_t^{(k)}(t_2,s)\right|ds\\ &\int_{t_1}^{t_2}\left|G_t^{(k)}(t_1,s)-G_t^{(k)}(t_2,s)\right|ds+\int_{t_2}^{1}\left|G_t^{(k)}(t_1,s)-G_t^{(k)}(t_2,s)\right|ds).\end{aligned}$$

It is easy to get $G_t^{(k)}(t,s) = \begin{cases} \frac{(t-s)^{\alpha-k-1}}{\Gamma(\alpha-k)} + \frac{\Gamma(2n-1)t^{n-k-1}(1-s)^{\alpha+n-2}}{\Gamma(n-k)\Gamma(\alpha+n-1)(\Gamma(2n-1)-1)}, & s \le t \\ \frac{\Gamma(2n-1)t^{n-k}(1-s)^{\alpha+n-2}}{\Gamma(n-k)\Gamma(\alpha+n-1)(\Gamma(2n-1)-1)}, & t \le s. \end{cases}$,
thus

$$\begin{aligned}\left|T^{(k)}u(t_1)-T^{(k)}u(t_2)\right| &\le C\left[\int_0^{t_1}\left[\frac{(t_2-s)^{\alpha-1-k}}{\Gamma(\alpha-k)}-\frac{(t_1-s)^{\alpha-1-k}}{\Gamma(\alpha-k)}\right.\right. \qquad (3.12)\\ &\left.+\left(t_2^{n-1-k}-t_1^{n-1-k}\right)\frac{\Gamma(2n-1)(1-s)^{\alpha+n-2}}{\Gamma(n-k)\Gamma(\alpha+n-1)\left(\Gamma(2n-1)-1\right)}\right]ds\\ &+\int_{t_1}^{t_2}\left(\left(t_2^{n-1-k}-t_1^{n-1-k}\right)\frac{\Gamma(2n-1)(1-s)^{\alpha+n-2}}{\Gamma(n-k)\Gamma(\alpha+n-1)\left(\Gamma(2n-1)-1\right)}\right.\\ &\left.\left.+(t_2-s)^{\alpha-1-k}\right)ds+\int_{t_2}^{1}\left(t_2^{n-1-k}-t_1^{n-1-k}\right)\frac{(1-s)^{\alpha+n-2}}{\Gamma(2n-1)-1}ds\right]\end{aligned}$$

Let us consider the function $\Phi(x) = x^{\alpha-1} - (\alpha-1)x$, $\alpha \ge 1$, we see that Φ is decreasing on $[0,1]$, consequently
$(t_2-s)^{\alpha-1-k} - (t_1-s)^{\alpha-1-k} \le (\alpha-1-k)(t_2-t_1)$ and
$t_2^{n-1-k} - t_1^{n-1-k} \le (n-1-k)(t_2-t_1)$, from which we deduce

$$\left|T^{(k)}u(t_1)-T^{(k)}u(t_2)\right| \le C(t_2-t_1)\left(c_1(k)+c_2(k)\int_0^1(1-s)^{\alpha+n-2}ds\right) \qquad (3.13)$$

where $c_1(k)$ and $c_2(k)$ are two constant independent of t_1 and t_2. Letting $t_1 \to t_2$, in (3.13) then $\left|T^{(k)}u(t_1)-T^{(k)}u(t_2)\right|$ tend to 0 for all k such $0 \le k \le n-2$.

Consequently $T(B_r)$ is equicontinuous. From Arzela-Ascoli theorem we deduce that T is completely continuous mapping on E. ■

Now we can prove Theorem 13:

Proof. We apply Leray Schauder nonlinear alternative to prove that T has at least one nontrivial solution in E. Letting $\Omega = \{u \in E : \|u\| < r\}$, for any $u \in \partial\Omega$, such that $u = \lambda Tu$, $0 < \lambda < 1$, we get with the help of (3.9)

$$\begin{aligned}\|u\| \quad &\leq \quad \|Tu\| = \sum_{k=0}^{n-2} \left\{ \sum_{i=0}^{n-2} \phi_i(r) \left(\left\| I_{0^+}^{\alpha-1-k} k_i \right\| + \frac{\Gamma(2n-1) I_{0^+}^{\alpha+n-1} k_i(1)}{\Gamma(n-k)(\Gamma(2n-1)-1)} \right) \right. \\ &\left. + \left\| I_{0^+}^{\alpha-1-k} g \right\| + \frac{\Gamma(2n-1) I_{0^+}^{\alpha+n-1} g(1)}{\Gamma(n-k)(\Gamma(2n-1)-1)} \right\}.\end{aligned}$$

Taking (3.10) into account, we deduce that $\|u\| < r$, this contradicts the fact that $u \in \partial\Omega$. Therefore, Lemma 7 implies that T has a fixed point $u^* \in \overline{\Omega}$ and then the fractional boundary value problem (P1) has a nontrivial solution $u^* \in E$. The proof is complete. ■

4 Existence of positive solutions

Our final mean result concerns with the positivity of solution for the boundary value problem (P1), for this we make the following hypothesis

H) $f(t, x_0, ...x_{n-2}) = a(t) f_1(x_0, ...x_{n-2})$ where $f_1 \in C(\mathbb{R}_+ \times \mathbb{R}^{n-2}, \mathbb{R}_+)$ and $a \in C((0,1), \mathbb{R}_+)$ with $a(t) \neq 0$.

Useful estimates of the Green function G are provided in the following lemma:

Lemma 15 *The Green function $G(t,s)$ defined by (3.3) has the following properties:*

i) $G_t^{(k)}(t,s) \in C^{n-2}\left([0,1]^2, \mathbb{R}_+\right)$, for all k such $0 \leq k \leq n-2$ and for all $t, s \in]0,1[$.

ii) For $t, s \in (\tau_1, \tau_2)$, $0 < \tau_1 < \tau_2 < 1$, $0 \leq k \leq n-2$, we have

$$\delta_1(k) \leq G_t^{(k)}(t,s) \leq \delta_2(k) \tag{4.1}$$

where

$$\begin{aligned}\delta_1(k) &= \frac{\tau_1^n (1-\tau_2)^{\alpha+n}}{\Gamma(n-k)(\Gamma(2n-1)-1)} \\ \delta_2(k) &= \frac{1}{\Gamma(\alpha-k)} + \frac{\Gamma(2n-1)}{\Gamma(n-k)\Gamma(\alpha+n-1)(\Gamma(2n-1)-1)}\end{aligned}$$

Proof. It is obvious that $G_t^{(k)}(t,s) \in C^{n-2}([0,1] \times [0,1])$, moreover $G_t^{(k)}(t,s)$ is nonnegative for all $t, s \in]0,1[$.

ii) Let $t, s \in (\tau_1, \tau_2)$, $0 < \tau_1 \leq s \leq t \leq \tau_2 < 1$, we get

$$G_t^{(k)}(t,s) \leq \frac{1}{\Gamma(\alpha-k)} + \frac{\Gamma(2n-1)}{\Gamma(n-k)\Gamma(\alpha+n-1)(\Gamma(2n-1)-1)} = \delta_2(k).$$

If $0<\tau_1\leq t\leq s\leq\tau_2<1$ it yields

$$G_t^{(k)}(t,s)\leq\frac{\Gamma(2n-1)}{\Gamma(n-k)\Gamma(\alpha+n-1)\left(\Gamma(2n-1)-1\right)}\leq\delta_2(k).$$

Now we look for lower bounds of $G_t^{(k)}(t,s)$

$$\begin{aligned}G_t^{(k)}(t,s) &\geq \frac{\Gamma(2n-1)t^{n-k-1}\left(1-s\right)^{\alpha+n-2}}{\Gamma(n-k)\Gamma(\alpha+n-1)\left(\Gamma(2n-1)-1\right)}\\ &\geq \frac{\tau_1^n\left(1-\tau_2\right)^{\alpha+n}}{\Gamma(n-k)\left(\Gamma(2n-1)-1\right)}=\delta_1(k),0<\tau_1\leq s\leq t\leq\tau_2<1.\end{aligned}$$

and

$$G_t^{(k)}(t,s)\geq\frac{\tau_1^n\left(1-\tau_2\right)^{\alpha+n}}{\Gamma(n-k)\left(\Gamma(2n-1)-1\right)}=\delta_1(k),0<\tau_1\leq t\leq s\leq\tau_2<1.$$

This completes the proof. ■

We recall the definition of positive solution

Definition 16 *A function u is called positive solution of problem (P1) if $u(t)\geq 0$, $\forall t\in[0,1]$ and it satisfies (1.1)-(1.2).*

Lemma 17 *If $u\in E$ is a positive solution of the problem (P1) then it satisfies*

$$\min_{t\in(\tau_1,\tau_2)}\sum_{k=0}^{n-2}u^{(k)}(t)\geq\xi\left\|u\right\|,$$

where $\xi=\frac{\sum_{k=0}^{n-2}\delta_1(k)}{\sum_{k=0}^{n-2}\delta_2(k)}$.

Proof. From Lemma 10, we know that

$$u(t)=\int_0^1 G(t,s)a(s)f_1(u(s),...,u^{(n-2)}(s))ds,$$

moreover we have

$$u^{(k)}(t)=\int_0^1 G_t^{(k)}(t,s)a(s)f_1(u(s),...,u^{(n-2)}(s))ds.$$

By virtue of the right hand side of inequality (4.1) and hypothesis H, we get

$$u^{(k)}(t)\leq\delta_2(k)\int_0^1 a(s)f_1(u(s),...,u^{(n-2)}(s))ds, \tag{4.2}$$

Consequently

$$\|u\|\leq\left(\sum_{k=0}^{n-2}\delta_2(k)\right)\int_0^1 a(s)f_1(u(s),...,u^{(n-2)}(s))ds,$$

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hence

$$\int_0^1 a(s)f_1(u(s),...,u^{(n-2)}(s))ds \geq \frac{1}{\sum_{k=0}^{n-2}\delta_2(k)}\|u\|. \tag{4.3}$$

In addition, the left hand side of (4.1) implies

$$u^{(k)}(t) \geq \delta_1(k)\int_0^1 a(s)f_1(u(s),...,u^{(n-2)}(s))ds, \tag{4.4}$$

for any $t \in (\tau_1, \tau_2)$, summing up, we get

$$\sum_{k=0}^{n-2} u^{(k)}(t) \geq \left(\sum_{k=0}^{n-2}\delta_1(k)\right)\int_0^1 a(s)f_1(u(s),...,u^{(n-2)}(s))ds$$

to this estimate, we apply (4.3) then we take the minimum over (τ_1, τ_2) to arrive at

$$\min_{t\in(\tau_1,\tau_2)}\left(\sum_{k=0}^{n-2} u^{(k)}(t)\right) \geq \frac{\sum_{k=0}^{n-2}\delta_1(k)}{\sum_{k=0}^{n-2}\delta_2(k)}\|u\|.$$

This finishes the proof. ■

The following notations will be used throughout:

$$A_0 = \lim_{\left(\sum_{k=0}^{n-2}|x_k|\right)\to 0}\frac{f_1(x_0,...,x_{n-2})}{\sum_{k=0}^{n-2}|x_k|}, \quad A_\infty = \lim_{\sum_{k=0}^{n-2}|x_k|\to\infty}\frac{f_1(x_0,...,x_{n-2})}{\sum_{k=0}^{n-2}|x_k|}.$$

The case $A_0 = 0$ and $A_\infty = \infty$ is called superlinear case and the case $A_0 = \infty$ and $A_\infty = 0$ is called sublinear case.

Theorem 18 *The problem (P1) has at least one positive solution in the both cases superlinear as well as sublinear.*

Proof. We apply Guo-Krasnosel'skii fixed point Theorem on cone. Define the set K by

$$K = \left\{u \in E,\ u(t) \geq 0,\ \min_{t\in(\tau_1,\tau_2)}\sum_{k=0}^{n-2} u^{(k)}(t) \geq \xi\|u\|\right\}.$$

It is easy to check that K is a nonempty closed and convex subset of E, so it is a cone. Since the functions a, f, G are nonnegative then

$$Tu(t) = \int_0^1 G(t,s)a(s)f_1(u(s),...,u^{(n-2)}(s))ds \geq 0.$$

From Lemma 14 we know that $T : K \to E$ is completely continuous for $u \in K$. By virtue of Lemma 15, we can write the following estimate

$$\|Tu\| \leq \left(\sum_{k=0}^{n-2}\delta_2(k)\right)\int_0^1 a(s)f_1(u(s),...,u^{(n-2)}(s))ds,$$

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on the other hand we have

$$\begin{aligned} & \min_{t\in(\tau_1,\tau_2)} \left(\sum_{k=0}^{n-2} (Tu)^{(k)}(t) \right) \\ = \quad & \int_0^1 \left(\min_{t\in(\tau_1,\tau_2)} G_t^{(k)}(t,s) \right) a(s) f_1(u(s),...,u^{(n-2)}(s))ds \geq \xi \|Tu\|, \end{aligned}$$

hence $T(K) \subset K$. Now, we prove the superlinear case. Since $A_0 = 0$, then for any $\varepsilon > 0$, there exists $R_1 > 0$, such that if $0 < \sum_{k=0}^{n-2} |x_k| \leq R_1$ then

$$f_1(x_0,...,x_{n-2}) \leq \varepsilon \sum_{k=0}^{n-2} |x_k|.$$

Consider the open set $\Omega_1 = \{u \in E, \|u\| < R_1\}$, then for any $u \in K \cap \partial\Omega_1$, it yields

$$(Tu)^{(k)}(t) = \int_0^1 G_t^{(k)}(t,s) a(s) f_1(u(s),...,u^{(n-2)}(s))ds \leq \varepsilon \|u\| \delta_2(k) \int_0^1 a(s)ds, \tag{4.5}$$

consequently

$$\|Tu\| \leq \varepsilon \|u\| \|a\|_{L_1[0,1]} \sum_{k=0}^{n-2} \delta_2(k). \tag{4.6}$$

In view of hypothesis H, one can choose ε such

$$\varepsilon \leq \frac{1}{\|a\|_{L_1} \sum_{k=0}^{n-2} \delta_2(k)} \tag{4.7}$$

that implies $||Tu|| \leq ||u||$. From the hypothesis $A_\infty = \infty$, we conclude that for any $M > 0$, there exists $R_2 > 0$, such that if $\sum_{k=0}^{n-2} |x_k| \geq R_2$ then $f_1(x_0,...,x_{n-2}) \geq M \sum_{k=0}^{n-2} |x_k|$. Let $R = \max\left\{2R_1, \frac{R_2}{\xi}\right\}$ and consider the open set $\Omega_2 = \{u \in E : ||u|| < R\}$. Let $u \in K \cap \partial\Omega_2$ then

$$\min_{t\in(\tau_1,\tau_2)} \sum_{k=0}^{n-2} u^{(k)}(t) \geq \xi \|u\| = \xi R \geq R_2 \tag{4.8}$$

From the left hand side of (4.1) it yields

$$(Tu)^{(k)}(t) \geq M\delta_1(k) \|u\| \|a\|_{L_1[0,1]} \tag{4.9}$$

thus

$$\|Tu\| \geq M \|u\| \|a\|_{L_1[0,1]} \sum_{k=0}^{n-2} \delta_1(k).$$

Let us choose M such that

$$M \geq \frac{1}{\|a\|_{L_1[0,1]} \sum_{k=0}^{n-2} \delta_1(k)},$$

then we get

$$||Tu|| \geq ||u|| \,,\ \forall u \in K \cap \partial\Omega_2.$$

The first part of Theorem 8 implies that T has a fixed point in $K \cap \left(\bar{\Omega}_2 \diagdown \Omega_1\right)$ such that $R_2 \leq ||u|| \leq R$. We apply similar techniques to prove the sublinear case, then we omit its proof. The proof of Theorem 18 is complete. ■

In order to illustrate our results, we give two examples.

Example 19 *The boundary value problem*

$$\begin{cases} {}^cD_{0^+}^{\frac{17}{2}}u(t) = tu + \sum_{k=1}^{7} t^k u^{(k)} - e^t, \quad 0 < t < 1, \\ u\left(0\right) = u'(0) = ... = u^{(7)}(0) = 0, u^{(8)}\left(0\right) = I_{0^+}^{8}(1), \end{cases} \tag{4.10}$$

has a unique solution in E.

Proof. We have $f\left(t, x_0..., x_8\right) = tx_0 + \sum_{k=1}^{7} t^k x_k - e^t$ and

$$\left|f\left(t, x_0..., x_8\right) - f\left(t, y_0..., y_8\right)\right| \leq t\left|x_0 - y_0\right| + \sum_{k=1}^{7} t^k \left|x_k - y_k\right|,$$

then $g_0(t) = \frac{t}{10}, g_k(t) = \frac{t^k}{10}$, $k = 1, ..., 7$. Some computations lead to

$$\sum_{k=0}^{7}\sum_{i=0}^{7}\left(\left\|I_{0^+}^{\frac{15}{2}-k} g_i\right\|_{L^1} + \frac{\Gamma(17)}{\Gamma(9-k)\left(\Gamma(17)-1\right)} I_{0^+}^{\frac{33}{2}} g_i(1)\right) = 0.14547 < 1$$

Thus, Theorem 11 implies that problem (4.10) has a unique in E.

Example 20 *The boundary value problem*

$$\begin{cases} u^{(6)}(t) = \sum_{k=0}^{4} \frac{t^2}{10}\left(u^{(k)}\right)^2 - (1+t)\,, \quad 0 < t < 1, \\ u\left(0\right) = u'(0) = ... = u^{(4)}(0) = 0, u^{(5)}\left(0\right) = I_{0^+}^{5}(1), \end{cases} \tag{4.11}$$

has at least one nontrivial solution in E.

Proof. We apply Theorem 13 to prove that problem (4.11) has at least one nontrivial solution. We have

$$\left|f\left(t, x_0, ..., x_4\right)\right| \leq \sum_{i=0}^{4} \frac{t^2}{10}\left|x_i\right|^2 + (1+t) = \sum_{i=0}^{4} k_i\left(t\right)\psi_i\left(\left|x_i\right|\right) + g\left(t\right).$$

where $k_i\left(t\right) = \frac{t^2}{10}$, $g\left(t\right) = 1+t$, $\psi_i\left(x_i\right) = x_i^2$ is a nondecreasing function on $\mathbb{R}_+$ and $f\left(t, 0, ..., 0\right) \neq 0$. Let us evaluate (3.10), we have

$$\begin{aligned} &\sum_{k=0}^{4}\left\{\sum_{i=0}^{4}\phi_i\left(r\right)\left(\left\|I_{0^+}^{5-k}k_i\right\| + \frac{\Gamma(11)I_{0^+}^{11}k_i\left(1\right)}{\Gamma(6-k)\left(\Gamma(11)-1\right)}\right)\right. \\ &\left.+\left\|I_{0^+}^{5-k}g\right\| + \frac{\Gamma(11)I_{0^+}^{11}g\left(1\right)}{\Gamma(6-k)\left(\Gamma(11)-1\right)}\right\} - r \\ =\ & 5\left(r^2\left(0.103\,22\times 10^{-1}\right)\right) + 0.936\,31 - r, \end{aligned}$$

which is negative for $r \geq 2$.

Example 21 *Let us consider the following problem*

$$\left\{\begin{array}{c} {}^{c}D_{0^+}^{5.8}u(t) = t^2 + \sum_{k=0}^{4} t^2 e^{-x_k^2}, \quad 0 < t < 1, \\ u\,(0) = u'(0) = ... = u^{(4)}(0) = 0, u^{(5)}\,(0) = I_{0^+}^{5}(1), \end{array}\right. \tag{4.12}$$

we have $f\,(t, x_0, ..., x_4) = a(t) f_1\,(x_0, ..., x_4)$ $a(t) = t^2$, $f_1\,(x_0, ..., x_4) = 1 + \sum_{k=0}^{4} e^{-x_k^2}$, *we have*

$$\frac{f_1\,(x_0, ..., x_4)}{\sum_{k=0}^{4} |x_k|} = \frac{1 + \left(\sum_{k=0}^{4} e^{-x_k^2}\right)}{\sum_{k=0}^{4} |x_k|} \to \left\{\begin{array}{lll} \infty & if & \sum_{k=0}^{4} |x_k| \to 0 \\ 0 & if & \sum_{k=0}^{4} |x_k| \to \infty \end{array}\right. ,$$

so we have the sublinear case $A_0 = \infty$ and $A_\infty = 0$. Theorem 18 implies that there exists at least one positive solution.

Acknowledgements. The authors would like to express their thanks to the referee for his helpful comments and suggestions.

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