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ADDITIVE INTEGRAL FUNCTIONS IN VALUED FIELDS

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Abstract

The additive integral functions with the coefficients in a complete non-archimedean algebraically closed field of characteristic $p \neq 0$ are studied.

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1. Introduction

Let $(K, |\;|)$ be a valued field of characteristic $p \neq 0$, where $|\;|$ is a non-trivial *non-archimedean absolute value* defined on *K*, that is a mapping $|\;|: K \to [0, \infty)$ such that, for every $x, y \in K$,

(i) $|x| = 0$ if and only if $x = 0$;
(ii) $|xy| = |x||y|$;
(iii) $|x + y| \leq \max\{|x|, |y|\}$;
(iv) there exists a non-zero $x \in K$ such that $|x| \neq 1$.

For $x, y \in K$, define $d(x, y) = |x - y|$ and thus (K, d) is an ultrametric space. A formal power series

$$f(X) = \sum_{k=0}^{\infty} a_k X^k \in K[[X]] \tag{1}$$

is called an *integral function* with coefficients in *K* if, for every $x \in K$, the sequence

$$S_n(x) = \sum_{k=0}^{n} a_k x^k \tag{2}$$

is a Cauchy sequence. It follows easily that $H(K)$, the set of all integral functions with coefficients in *K*, is a $K-$algebra with respect to the ordinary addition and multiplication of integral functions. An integral function *f* with coefficients in *K* is called *additive* if, for every $x, y \in K$, in a fixed completion of *K*,

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$$f(x+y)=f(x)+f(y). \qquad (3)$$

Suppose now that *K* is an algebraically closed field of characteristic $p \neq 0$. Then, for every positive integer *r*, we may consider the Galois field $GF(p^r)$ as a subfield of *K*. In this case an additive integral function *f* is called $GF(p^r)-$*linear*, if for every $x \in K$ and $\alpha \in GF(p^r)$,

$$f(\alpha x)=\alpha f(x). \qquad (4)$$

Since, for every $\alpha \in GF(p^r)$, it follows that $\alpha^{p^r}=\alpha$ (see, for example, [3], p. 83), by (3) it follows that an additive integral function is $GF(p)-$linear.

This paper follows the ideas of Nicolae Popescu who conjectured that the additive integral functions have similar properties as the additive polynomials (see[1]).

2. Representation and zeros of additive integral functions

The following result gives a representation of the $GF(p^r)-$linear integral functions.

Theorem 1. *Let K be an algebraically closed field of characteristic* $p \neq 0$ *which is complete with respect to a non-archimedean absolute value and let f be an integral function with coefficients in K. Then f is* $GF(p^r)$ *- linear, where r is fixed, if and only if*

$$f(X)=\sum_{i=0}^{\infty} a_i X^{p^{ir}}, \text{ with } a_i \in K. \qquad (5)$$

Proof. Since $(x+y)^p=\sum_{i=0}^{p}\binom{p}{i}x^{p-i}y^i$ and $\binom{p}{i}\equiv 0(\text{mod } p)$ it follows that $(x+y)^p=x^p+y^p$. Hence $(x+y)^{p^i}=x^{p^i}+y^{p^i}$ which implies that the integral function *f* given by (5) is additive. Because, for every $\alpha \in GF(p^r)$, $\alpha^{p^r}=\alpha$, by (5) we obtain that *f* is a $GF(p^r)-$linear function.

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Conversely, we suppose that the integral function $f(X)=\sum_{j=0}^{\infty} b_j X^j$ is a $GF(p^r)-$linear function. We use the formal derivative $f'(X)=\sum_{j=1}^{\infty} jb_j X^{j-1}$. It is easy to see that this operation satisfies the standard rules of differentiation. Since $f(x+y)-f(x)-f(y)=0$, for every $x,y\in K$, because the zeros of an integral function are isolated, by taking two arbitrary sequences $\{x_n\}_{n\in\mathbf{N}}$, $\{y_n\}_{n\in\mathbf{N}}$ of elements of *K* which converge to zero, it follows that $f(X+Y)=f(X)+f(Y)$. Hence, for every $y\in K$,

$f'(y)=\frac{d}{dX}f(X+y)\Big|_{X=0}=\frac{d}{dX}\big(f(X)+f(y)\big)\Big|_{X=0}=f'(0)=b_1$. Because *f*(0) = 0 we obtain that

$$f(X)=c_0X+\sum_{j=1}^{\infty} c_j X^{n_j},\ with\ c_0=b_1, \qquad (6)$$

where $n_j>1$ and $n_j\equiv 0(\mathrm{mod}\,p)$. We write

$$f(X)=f_1(X)+f_2(X), \qquad (7)$$

where

$$f_1(X)=c_0X+\sum_{j\in I_1} c_j X^{n_j},\ f_2(X)=\sum_{j\in I_2} c_j X^{n_j}, \qquad (8)$$

$I_1=\{j: n_j \text{ is a power of } p^r\}$ and $I_2=\{j: n_j \text{ is not a power of } p^r\}$. We shall prove that $f_2=0$. Since f and f_1 are $GF(p^r)-$linear integral functions it follows that f_2 is a $GF(p^r)-$linear integral function. Because *K* is an algebraically closed field it follows that the mapping $\tau_p: K\to K$ given by $\tau_p(x)=x^p$ is an automorphism of *K*. Hence $\tau_{p^e}: K\to K$ defined by $\tau_{p^e}(x)=x^{p^e}$ is also an automorphism of *K* and we obtain that

$$f_2(X) = f_3^{p^e}(X), \tag{9}$$

where p^e is the largest power of *p* dividing all n_j, $j \in I_2$ and *e* is not divisible by *r*. Then, because τ_{p^e} is an automorphism of *K* it follows that f_3 is an additive integral function. Moreover, if there exists $\alpha \in GF(p^e)$ and $x \in K$ such that $f_3(\alpha x) \neq \alpha f_3(x)$ it follows that $\alpha^{p^e} \neq \alpha$, a contradiction which implies that f_3 is a $GF(p^e)-$linear integral function. Thus by using the form of f_2, because $1 = p^{0r}$ we obtain as above that $f_3'(y) = 0,$ for every $y \in K.$ This implies that $f_3(X) = \sum_{j=1}^{\infty} d_j X^{pm_j}$, with $d_j \in K$ and m_j a positive integer. Hence, because p^e is the largest power of *p* dividing all n_j, we obtain that $f_2 = 0$ which implies the theorem, □

Since every additive integral function is a $GF(p)-$linear integral function, by Theorem 1 we obtain the following result.

Corollary 1. *Under the hypotheses of Theorem 1 f is an additive function if and only if*

$$f(X) = \sum_{i=0}^{\infty} a_i X^{p^i}, \text{ with } a_i \in K. \tag{10}$$

Theorem 2. *Let K be an algebraically closed field of characteristic* $p \neq 0$ *which is complete with respect to a non-archimedean absolute value and let f be an integral function with coefficients in K having infinitely many distinct roots. If* $G = \{\alpha_i\}_{i \geq 0}$, *where* $\alpha_0 = 0$, *is the set of all the roots of f, then f is* $GF(p^r)-$*linear if and only if G is a* $GF(p^r)-$*linear subspace of K and there exists a chain of* $GF(p^r)-$*linear subspaces*

$$G_{n_1} \subset G_{n_2} \subset \ldots \subset G_{n_s} \subset \ldots \tag{11}$$

of G such that the order of G_{n_j} *is equal to* n_j *and* p^r *divides* n_j, *for every j.*

Proof. Let f be an additive integral function. If $\alpha_i, \alpha_j \in G$, then, because *f* is an additive integral function, it follows that $f(\alpha_i - \alpha_j) = f(\alpha_i) - f(\alpha_j) = 0$ and for every $\alpha \in GF(p^r)$, $f(\alpha\alpha_i) = \alpha f(a_i) = 0$. Hence we obtain that *G* is a $GF(p^r)-$ linear subspace of *K*.

Now we consider the critical radius of *f* (see [2], p. 291) $r_1 < r_2 < \ldots < r_k < \ldots$. Then inside the ball $B_j = \{x \in K : |x| \le r_j\}$, f has n_j roots (the proof of Theorem 1 of [2], p. 307 is the same in this case). Since $|\alpha| = 1$, for every non-zero $\alpha \in GF(p^r)$, it follows that B_j is a $GF(p^r)-$linear subspace of *G*. Hence $G_{n_j} = \{\alpha \in B_j : f(\alpha) = 0\}$, *j*=1,2,…, is a finite $GF(p^r)-$linear subspace of *K*, p^r divides n_j and (11) holds.

Conversely, if *f* is an integral function, by Theorem of [2], p. 314, it follows that $f = CX \prod_{j=1}^{\infty} \left(1 - \frac{X}{\alpha_j}\right)$. Suppose that *G* is a $GF(p^r)-$linear subspace of *K* and there a chain of $GF(p^r)-$linear subspaces G_{n_j} of *G*, of orders n_j, such that (11) holds. We consider the polynomials $P_j = CX \prod_{\substack{\alpha_j \in G_{n_j} \\ \alpha_j \ne 0}} (X - \alpha_j) = C_j Q_j$, where

$Q_j = CX \prod_{\substack{\alpha_j \in G_{n_j} \\ \alpha_j \ne 0}} \left(1 - \frac{X}{\alpha_j}\right)$ and $C_j = \prod_{\substack{\alpha_j \in G_{n_j} \\ \alpha_j \ne 0}} \alpha_j$. By Corollary 1.2.2 from [1] it follows that P_j is a $GF(p^r)-$linear polynomial. Hence Q_j is a $GF(p^r)-$linear polynomial and similarly as in the proof of Theorem of [2], p. 314 we obtain that $\lim_{j \to \infty} Q_j = f$. This implies the theorem. □

Finally we extend to integral functions a result of Ore on polynomials (see Theorem 1.4.1 of [1]).

Theorem 3. *Let K be an algebraically closed field of characteristic* $p \ne 0$ *which is complete with respect to a non-archimedean absolute value and let f be an integral function with coefficients in K having infinitely many roots. If r is a positive* *integer, then there exists a* $GF(p^r)-$ *linear integral function* $g \in H(K)$ *such that f divides g in* $H(K)$.

Proof. Suppose $f_1 = \prod_{j=1}^{\infty}\left(1-\frac{X}{\alpha_j}\right)$, where α_j, *j*=1,2,… , are all the non-zero distinct roots of *f*. Let m_j be the multiplicity of the root α_j of *f*, where $\alpha_0 = 0$. For every $m_j \geq 1$, we take k_j the smallest non-negative integer such that $m_j \leq p^{k_j r}$.

We consider the function $g = X^{p^{k_0 r}} \prod_{j=1}^{\infty}\left(1-\frac{X}{\alpha_j}\right)^{p^{k_j r}}$, where, if $m_0 = 0$, we take $k_0 = -\infty$. Then *g* is an integral function (see [2], p. 315), *f* divides *g* and by Theorem 1 it follows that *g* is $GF(p^r)-$ linear. Hence it follows the theorem. □

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