EXISTENCE OF POSITIVE PERIODIC SOLUTIONS OF A CLASS OF DELAY DIFFERENTIAL SYSTEMS

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Abstract. In this work we use some techniques of the Mawhin coincidence degree theory to prove the existence of positive periodic solutions of delay differential systems. As a consequence, we offer existence criteria and sufficient conditions on a, b, G, f for existence of positive periodic solutions to differential systems with feedback control. When these results are applied to some special delay bio-mathematics models, some new results are obtained, and many known results are improved.

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1. Introduction

The functional differential equations are not only an extension of ordinary differential equations but also provide good models in many fields including Biology, Mechanics and Economics bio-mathematics. For example, (see, [14]), in population dynamics, since a growing population consumes more (or less) food than a matured one, depending on individual species, this leads to neutral functional equations. Positive periodic solutions of differential equations have been studied extensively in recent times. We refer to the references [1]-[22] in this article and references therein for a wealth of information on this subject. In this paper, we study the existence of a positive periodic solution of systems of equations. The study on the functional differential equations is more intricate than ordinary delay differential equations. That is why comparing plenty of results on the existence of positive periodic solutions for various types of first-order or second-order ordinary delay differential equations or studies on positive periodic solutions for delay differential equations are relatively less, and most of them are confined to first-order delay differential equations, see [1-8, in [14]] which are studied by using some techniques of the Mawhin coincidence degree theory.

In [14], Li studied existence and global attractively of a positive periodic solution of a class of delay differential equation with several delays of the logistic type of equations having the form

$$x'(t) = x(t) F(t, x(t-\tau_1(t)), ..., x(t-\tau_n(t))).$$

The above equation was extensively investigated in literature as bio-mathematics models. It contains many bio-mathematics models of delay differential equations, such as the single species growth logistic equation with delay

$$x'(t) = r(t) x(t) \left[1 - \frac{x(t-\tau(t))}{R(t)}\right]. \quad (1.1)$$

It was proved in reference [22] that if $\tau(t) = m\omega$, then (1.1) has a positive ω-periodic solution, where m is a nonnegative integer. Also, the next so called "food-limited" population growth model (see [22]) was studied in reference [2], and it was shown that if $\tau(t) = m\omega$ holds, then the equation

$$x'(t) = r(t)\,x(t)\left[\frac{R(t) - x(t-\tau(t))}{R(t) + r(t)\,c(t)\,x(t-\tau(t))}\right],$$

has a positive ω-periodic solution. In reference [11], the following single-species population model exhibiting the so-called Allee effect was considered in ([1]),

$$x'(t) = x(t)\left[a(t) + b(t)\,x(t-\tau(t)) - c(t)\,x^2(t-\tau(t))\right]. \tag{1.2}$$

It was proved that if $\tau(t) = m\omega$ holds, then (1.2) has a positive ω-periodic solution (see [1–10, in [14]]).

Consider the following two types of differential systems with several delays of the logistic type

$$\begin{cases} \dfrac{dx}{dt} = \pm x(t)\,G(t, x(t-\tau_1(t)), ..., x(t-\tau_n(t)), u(t-\delta(t))), \\ \\ \dfrac{d^2u}{dt^2} = a(t)\,u(t) + \lambda b(t)\,f(x(t-\sigma(t))), \end{cases} \tag{1.3}$$

and

$$\begin{cases} \dfrac{dx}{dt} = \pm x(t)\,G(t, x(t-\tau_1(t)), ..., x(t-\tau_n(t)), u(t-\delta(t))), \\ \\ \dfrac{d^2u}{dt^2} = -a(t)\,u(t) + \lambda b(t)\,f(x(t-\sigma(t))). \end{cases} \tag{1.4}$$

Here, $G(t, x_1, ..., x_{n+1}) \in C(\mathbb{R}^{n+2}, \mathbb{R})$, $G(t+\omega, x_1, ..., x_{n+1}) = G(t, x_1, ..., x_{n+1})$ and $b, f \in C(\mathbb{R}, \mathbb{R})$, $\sigma, \delta \in C(\mathbb{R}, \mathbb{R})$, $a, \tau_i \in C(\mathbb{R}, \mathbb{R}^+)$ for $i = 1, ..., n$. All of the above functions are ω-periodic functions and $\omega > 0$ is a constant.

Special cases of (1.4) have been considered and investigated by many other authors. For example, very recently, in [7] Huo and Li discussed the existence of a positive periodic solutions of the delay to system

$$\begin{cases} \dfrac{dx}{dt} = \pm x(t)\,G(t, x(t-\tau_1(t)), ..., x(t-\tau_n(t)), u(t-\delta(t))), \\ \\ \dfrac{du}{dt} = -a(t)\,u(t) + b(t)\,x(t-\sigma(t)), \end{cases}$$

where all of the above functions are ω-periodic functions with $\omega > 0$ is a constant and G satisfies some conditions. The main tool employed in their study is based on some techniques of the Mawhin coincidence degree. For details on Mawhin techniques, we refer the reader to (Gaines and Mawhin [[5], p. 40]).

In this paper, we obtain various sufficient conditions for the existence of positive periodic solutions for both problems (1.3) and (1.4) by employing two available operator and by applying coincidence degree theorem.

The main steps of this exposition are the following. In first section, we introduce some notations and recall a lemma for coincidence degree theorem. Then, we give the Green's function of the second equation of (1.3), which plays an important role in this section. Lastly, we present our main results on existence of positive periodic solutions of (1.3).

In last section we get a few sufficient conditions ensuring the existence positive periodic solution for the system (1.4) by employing some techniques similar to those of second section. Finally, we give an example to illustrate our results.

2. Preliminaries

Let us give some notions and notations used in the theory of coincidence degree theorem [4, 5] which we will apply in the present paper.

To state existence of ω-periodic solution criteria for (1.3) and (1.4) some preparations and notations are needed. For $\omega > 0$, let C_ω be the set of all continuous scalar functions x, periodic in t of period ω. Clearly, $(C_\omega, \|\cdot\|)$ is a Banach space with the supremum norm

$$\|x\| = \sup_{t\in\mathbb{R}} |x(t)| = \sup_{t\in[0,\omega]} |x(t)| .$$

Define

$$C_\omega^- = \{x \in C_\omega \mid x < 0\}, \; C_\omega^+ = \{x \in C_\omega \mid x > 0\},$$

and

$$C_\omega^l := \{\varphi \in C_\omega \mid \|\varphi\| < l\}.$$

Throughout this section we let

$$M = \sup\{a(t) \mid t \in [0,\omega]\}, \; m = \inf\{a(t) \mid t \in [0,\omega]\}, \; \beta = \sqrt{M}.$$

Next, we state the coincidence degree theorem which enables us to prove the existence of periodic solutions to (1.3) and (1.4). The method we use in this paper involves the applications of the continuous theorem of coincidence degree (Gaines and Mawhin [14, p. 40]). This requires to introduce a few more notations. Let X and Z be two Banach spaces. Consider the operator equation

$$Lx = \eta Nx, \; \eta \in (0,1), \tag{2.1}$$

where $L : X \cap DomL \to Z$ is a linear operator and η is a parameter. Let P and Q denote two projectors such that

$$P : X \cap DomL \to \ker L \text{ and } Q : Z \to Z/ImL.$$

For convenience we cite the continuous theorem (Gaines and Mawhin ([5], p. 40)).

Lemma 2.1. *Let X and Z be two Banach spaces and L a Fredholm mapping of index zero. Assume that $N : \bar{\Omega} \to Z$ is L–compact on $\bar{\Omega}$ with Ω open bounded in X. Furthermore, we assume that*

(a) For each $\eta \in (0,1)$, $x \in \partial\Omega \cap DomL$,

$$Lx \neq \eta Nx.$$

(b) For each $x \in \partial\Omega \cap \ker L$,

$$QNx \neq 0,$$

and

$$\deg\{QNx, \partial\Omega \cap \ker L, 0\} \neq 0.$$

Then the equation $Lx = Nx$ has at least one solution in $\bar{\Omega} \cap DomL$.

Recall that a linear mapping $L : X \cap DomL \to Z$ with $\ker L = L^{-1}(0)$ and $ImL = L(DomL)$, will be called a Fredholm mapping if the following two conditions hold;

(i) ker L has a finite dimension;

(ii) ImL is closed and has a finite codimension.

Recall also that the codimension of ImL is the dimension of Z/ImL, i.e., the dimension of the cokernel $co\ker L$ of L.

When L is a Fredholm mapping, its index is the integer $Ind(L) = \dim\ker L - co\dim ImL$. We shall say that a mapping N is L–compact on Ω if the mapping $QN : \bar{\Omega} \to Z$ is continuous, $QN(\bar{\Omega})$ is bounded, and $K_P(I-Q)N : \bar{\Omega} \to X$ is compact, i.e., it is continuous and $K_P(I-Q)N(\bar{\Omega})$ is relatively compact, where $K_P : ImL \to DomL \cap \ker P$ is a inverse of the restriction L_P of L to $DomL \cap \ker P$, so that $LK_P = I$ and $K_PL = I - P$. For convenience, we shall introduce the notation

$$\bar{u} = \frac{1}{\omega}\int_0^\omega u(t)\,dt,$$

where u is a periodic continuous function with period ω.

3. Positive periodic solutions for (1.3)

As a first case, we consider the following delayed differential system with feedback control

$$
\left\{
\begin{aligned}
&\frac{dx}{dt}=-F\left(t,x\left(t-\tau_1\left(t\right)\right),...,x\left(t-\tau_n\left(t\right)\right),u\left(t-\delta\left(t\right)\right)\right),\\
&\frac{d^2u}{dt^2}=a\left(t\right)u\left(t\right)+\lambda b\left(t\right)f\left(x\left(t-\sigma\left(t\right)\right)\right).
\end{aligned}
\right. \tag{3.1}
$$

where $F\left(t,x_1,...,x_{n+1}\right)\in C\left(\mathbb{R}^{n+2},\mathbb{R}\right)$, $F\left(t+\omega,x_1,...,x_{n+1}\right)=F\left(t,x_1,...,x_{n+1}\right)$ and $b,\ f\in C\left(\mathbb{R},\mathbb{R}\right)$, $\sigma,\delta\in C\left(\mathbb{R},\mathbb{R}\right)$, $a,\tau_i\in C\left(\mathbb{R},\mathbb{R}^+\right)$ for $i=1,...,n$. All of the above functions are supposed to be ω-periodic functions with $\omega>0$ is a constant. We assume that for $t\in[0,\omega]$ and $x\in C_\omega$ we have

$$
\lambda b\left(t\right)f\left(x\left(t-\sigma\left(t\right)\right)\right)<0. \tag{3.2}
$$

The following theorem is essential for our results on existence of ω–periodic solution of (3.1).

Theorem 3.1. *Let the conditions (3.2) hold. Suppose the following conditions hold*

(i) There exists a constant $C>0$ such that, if $x\left(t\right)$ and $u\left(t\right)$ are continuous ω-periodic function and satisfy

$$
\int_0^\omega F\left(t,x\left(t-\tau_1\left(t\right)\right),...,x\left(t-\tau_n\left(t\right)\right),u\left(t-\delta\left(t\right)\right)\right)dt=0,
$$

then we have

$$
\int_0^\omega \left|F\left(t,x\left(t-\tau_1\left(t\right)\right),...,x\left(t-\tau_n\left(t\right)\right),u\left(t-\delta\left(t\right)\right)\right)\right|dt\leq C.
$$

(ii) There exists a constant $H>0$ such that when $v_i\geq H$, $i=1,2,...,n+1$,

$$
F\left(t,v_1,...,v_n,v_{n+1}\right)>0,\ F\left(t,-v_1,...,-v_n,-v_{n+1}\right)<0,
$$

uniformly hold for $[0,\infty)$.

Then, the system (3.1) has at least one positive ω-periodic solution.

To prove Theorem 3.1 we need some lemmas.

Lemma 3.2. *The equation*

$$
\frac{d^2y}{dt^2}-My\left(t\right)=h(t),h\in C_\omega^-, \tag{3.3}
$$

has a unique ω-periodic solution

$$
y\left(t\right)=\int_t^{t+\omega}K_1\left(t,s\right)\left(-h\left(s\right)\right)ds,
$$

where

$$
K_1\left(t,s\right)=\frac{\exp\left(-\beta\left(s-t\right)\right)+\exp\left(\beta\left(s-t-\omega\right)\right)}{2\beta\left(1-\exp\left(-\beta\omega\right)\right)},\ s\in\left[t,t+\omega\right]. \tag{3.4}
$$

Proof. First it is easy to see that the associate homogenous equation of (3.3) has solution

$$
y\left(t\right)=c_1\exp\left(\beta t\right)+c_2\exp\left(-\beta t\right).
$$

Applying the method of variation of parameters, we get

$$
c_1'\left(t\right)=\frac{h\left(t\right)\exp\left(-\beta t\right)}{2\beta},\ c_2'\left(t\right)=\frac{h\left(t\right)\exp\left(\beta t\right)}{-2\beta}.
$$

Since $y\left(t\right)$, $y'\left(t\right)$ are periodic functions, we have

$$
c_1\left(t\right)=\int_t^{t+\omega}\frac{\exp\left(-\beta\left(s-v\right)\right)}{2\beta\left[1-\exp\left(\beta v\right)\right]}h\left(s\right)ds,\ c_2\left(t\right)=\int_t^{t+\omega}\frac{\exp\left(-\beta\left(s-v\right)\right)}{2\beta\left[\exp\left(-\beta v\right)-1\right]}h\left(s\right)ds.
$$

Therefore,

$$y(t) = \int_t^{t+\omega} K_1(t,s)(-h(s))\,ds,$$

where $K_1(t,s)$ is as given by (3.4). □

Lemma 3.3. *$K_1(.,.)$ has the properties*

$$\begin{aligned}
&\textit{1)} \int_t^{t+\omega} K_1(t,s)\,ds = \frac{1}{M} \quad \textit{for all } t \in [0,\omega] \quad \textit{and} \quad s \in [t, t+\omega],\\
&\textit{2)}\ 0 < \frac{\exp(-\beta\omega/2)}{\beta(1-\exp(-\beta\omega))} \le K_1(t,s) \le \frac{1+\exp(-\beta\omega)}{2\beta(1-\exp(-\beta\omega))},\\
&\textit{3)}\ K_1(t+\omega, s+\omega) = K_1(t,s).
\end{aligned}$$

Proof. 1) By a simple calculation, it is easy to see that we can get

$$\begin{aligned}
\int_t^{t+\omega} K_1(t,s)\,ds &= \int_t^{t+\omega} \frac{\exp(-\beta(s-t)) + \exp(\beta(s-t-\omega))}{2\beta(1-\exp(-\beta\omega))}ds\\
&= \frac{1}{2\beta(1-\exp(-\beta\omega))}\left[-\frac{1}{\beta}\exp(-\beta(s-t)) + \frac{1}{\beta}\exp(\beta(s-t-\omega))\right]_t^{t+\omega}\\
&= \frac{1}{2\beta(1-\exp(-\beta\omega))}\left[-\frac{1}{\beta}[\exp(-\beta\omega)-1] + \frac{1}{\beta}[1-\exp(-\beta\omega)]\right]\\
&= \frac{1}{2\beta(1-\exp(-\beta\omega))}\frac{2}{\beta}[1-\exp(-\beta\omega)]\\
&= \frac{1}{\beta^2} = \frac{1}{M}.
\end{aligned}$$

2) It is clear that by the definition of $K_1(t,s)$ one can see that $\frac{\partial K_1(t,s)}{\partial s} = 0$ only if $s = t + \frac{\omega}{2}$. Then from the fact that

$$K_1\left(t, t+\frac{\omega}{2}\right) = \frac{\exp(-\beta\omega/2)}{\beta(1-\exp(-\beta\omega))},\ K_1(t,t) = \frac{1+\exp(-\beta\omega)}{2\beta(1-\exp(-\beta\omega))},$$

the condition 2) holds true. The condition 3) is straightforward. □

Lemma 3.4. *Let $x \in C_\omega$. Then, $u \in C_\omega$ is a solution of the second equation (3.1) if and only if*

$$u(t) = -\lambda(I - T_1B_1)^{-1}\int_t^{t+\omega} K_1(t,s)\,b(s)\,f(x(s-\sigma(s)))\,ds, \tag{3.5}$$

where

$$(T_1h)(t) = \int_t^{t+\omega} K_1(t,s)(-h(s))\,ds \ \textit{and}\ (B_1y)(t) = [-M + a(t)]\,y(t).$$

Proof. Let $u \in C_\omega$ be a solution of the second equation of (3.1). Since each ω-periodic solution of the equation:

$$\frac{d^2u}{dt^2} = a(t)\,u(t) + \lambda b(t)\,f(x(t-\sigma(t))). \tag{3.6}$$

(3.6) can be rewritten as

$$\frac{d^2u(t)}{dt^2} - Mu(t) = [-M + a(t)]\,u(t) + \lambda b(t)\,f(x(t-\sigma(t))).$$

Taking

$$(B_1u)(t) = [-M + a(t)]\,u(t).$$

Then, (3.6) can be transformed into

$$\frac{d^2u}{dt^2} - Mu(t) = (B_1u)(t) + \lambda h(t) f(x(t-\sigma(t))).$$

Since $\lambda b(t) f(x(t-\sigma(t))) < 0$, then from Lemma 3.2, we have

$$\begin{aligned} u(t) &= (T_1B_1u)(t) + T_1(\lambda b(t) f(x(t-\sigma(t)))) \\ &= (T_1B_1u)(t) - \lambda \int_t^{t+\omega} K_1(t,s) b(s) f(x(s-\sigma(s))) ds. \end{aligned}$$

This yields

$$(I - T_1B_1) u(t) = -\lambda \int_t^{t+\omega} K_1(t,s) b(s) f(x(s-\sigma(s))) ds.$$

Therefore, since $\|T_1B_1\| \leq 1 - \frac{m}{M} < 1$

$$u(t) = -\lambda (I - T_1B_1)^{-1} \int_t^{t+\omega} K_1(t,s) b(s) f(x(s-\sigma(s))) ds.$$

□

Now, define $S_1 : C_\omega \to C_\omega$ by

$$u(t) = -\lambda (I - T_1B_1)^{-1} \int_t^{t+\omega} K_1(t,s) b(s) f(x(s-\sigma(s))) ds := (S_1x)(t).$$

Clearly, one can, by a change of variables, have

$$u(t) = u(t+\omega).$$

Lemma 3.5. *S_1 is a continuous mapping satisfying the following condition*

$$0 < (S_1x)(t), \ x \in C_\omega.$$

Proof. To simplify notations take $h(t) = \lambda b(t) f(x(t-\sigma(t)))$ for $x \in C_\omega$. By Neumann expansions of $(I - T_1B_1)^{-1} T_1$, we have

$$\begin{aligned} &(I - T_1B_1)^{-1} T_1 \\ &= \left[I + T_1B_1 + (T_1B_1)^2 + \cdots + (T_1B_1)^n + \cdots\right] T_1 \\ &= T_1 + T_1B_1T_1 + (T_1B_1)^2 T_1 + \cdots + (T_1B_1)^n T_1 + \cdots. \end{aligned} \tag{3.7}$$

Moreover, by (3.7), recalling that $\|T_1B_1\| \leq 1 - \frac{m}{M}$ and the fat that $(T_1h)(t) > 0$ for $h < 0$, we get

$$(I - T_1B_1)^{-1} T_1(h)(t) \geq (T_1h)(t) \quad \text{for } h < 0.$$

Consequently, having in mind (3.2), we see that

$$\begin{aligned} &(I - T_1B_1)^{-1} T_1(h)(t) \\ &= (I - T_1B_1)^{-1} \int_t^{t+\omega} K_1(t,s)(-\lambda b(s) f(x(s-\sigma(s)))) ds = (S_1x)(t). \end{aligned}$$

Hence, $0 < (S_1x)(t)$, $x \in C_\omega$. □

Now, we prove Theorem 3.1.

Proof. It is obvious that $u(t)$ is the unique ω-periodic solution of (3.6) for $x \in C_\omega$. Therefore, the existence problem of ω-periodic solution of (3.1) is equivalent to that of ω-periodic solutions of the equation

$$\frac{dx}{dt} = -F[t, x(t-\tau_1(t)), ..., x(t-\tau_n(t)), (S_1x)(t-\delta(t))].$$

In order to apply Lemma 2.1, we take

$$X = Z = C_\omega := \{x \in C(\mathbb{R}, \mathbb{R}) \mid x(t+\omega) = x(t)\},$$

and use

$$\|x\| = \sup_{t\in\mathbb{R}} |x(t)| = \sup_{t\in[0,\omega]} |x(t)|.$$

Then, X and Z are Banach spaces when they are endowed with the norms $\|\cdot\|$. Set

$$Nx = -F[t, x(t-\tau_1(t)), ..., x(t-\tau_n(t)), (S_1x)(t-\delta(t))],$$

and

$$Lx = x', \quad Px = Qx = \frac{1}{\omega}\int_0^\omega x(t)\,dt, \; x \in X.$$

Obviously, $\ker L = \{x \mid x \in X, \; x = \xi, \; \xi \in \mathbb{R}\}$, $ImL = \left\{x \mid x \in X, \; \int_0^\omega x(t)\,dt = 0\right\}$ are closed in X and $\dim \ker L = co\dim ImL$. Hence, L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to L) $K_P : ImL \to \ker P \cap DomL$ has the form

$$K_P(x) = \int_0^t x(s)\,ds - \frac{1}{\omega}\int_0^\omega \int_0^t x(s)\,dsdt.$$

Thus,

$$(QN)(x) = -\frac{1}{\omega}\int_0^\omega F[t, x(t-\tau_1(t)), ..., x(t-\tau_n(t)), (S_1x)(t-\delta(t))]\,dt,$$

and

$$\begin{aligned}
&K_P(I-Q)N(x) \\
&= -\frac{1}{\omega}\int_0^\omega F[s, x(s-\tau_1(s)), ..., x(s-\tau_n(s)), (S_1x)(t-\delta(s))]\,dsdt \\
&+ \frac{1}{\omega}\int_0^\omega \int_0^t F[s, x(s-\tau_1(s)), ..., x(s-\tau_n(s)), (S_1x)(s-\delta(s))]\,dsdt \\
&+ \left(\frac{t}{\omega} - \frac{1}{2}\right)\int_0^\omega F[t, x(t-\tau_1(t)), ..., x(t-\tau_n(t)), (S_1x)(t-\delta(t))]\,dt.
\end{aligned}$$

Clearly, QN and $K_P(I-Q)N$ are continuous and, moreover, $QN(\bar{\Omega})$, $K_P(I-Q)N(\bar{\Omega})$ are relatively compact for any open bounded set $\Omega \subset X$. Hence, N is L-compact on $\bar{\Omega}$. Here Ω is any open bounded set in X. Now we reach the position to search for an appropriate open bounded subset X for the application of Lemma 2.1. Corresponding to equation $Lx = \eta Nx$, $\eta \in (0,1)$, we have

$$x'(t) = -\eta F[t, x(t-\tau_1(t)), ..., x(t-\tau_n(t)), (S_1x)(t-\delta(t))]. \tag{3.8}$$

Suppose that $x(t) \in X$ is a solution of system (3.8) for a certain $\eta \in (0,1)$. By integrating (3.8) over the interval $[0,\omega]$, we obtain

$$\int_0^\omega F[t, x(t-\tau_1(t)), ..., x(t-\tau_n(t)), (S_1x)(t-\delta(t))]\,dt = 0, \tag{3.9}$$

by this, (3.8) and (i) we find

$$\begin{aligned}
&\int_0^\omega |x'(t)|\,dt \\
&\le \eta \int_0^\omega |F[t, x(t-\tau_1(t)), ..., x(t-\tau_n(t)), (S_1x)(t-\delta(t))]|\,dt \le C.
\end{aligned}\tag{3.10}$$

Moreover, in view of (3.9) and (ii), it is easy to see that there exist an $i_0 \in \{1,..,n\}$, a point $t^* \in [0,\omega]$ and a constant $l_1 > 0$ such that

$$x(t^* - \tau_{i_0}(t^*)) < l_1, \; (S_1x)(t^* - \delta(t^*)) < l_1. \tag{3.11}$$

Otherwise, for any $l_1 > 0$ and any $t \in [0, \omega]$, one has

$$x\left(t^* - \tau_i\left(t^*\right)\right) \geq l_1,\ i = 1, .., n \text{ and } \left(S_1 x\right)\left(t - \delta\left(t\right)\right) \geq l_1.$$

In view of (ii), we see that this contradicts (3.9). Hence (3.11) holds. Denote $t^* - \tau_{i_0}\left(t^*\right) = \zeta_1 + k\omega$, $\zeta_1 \in [0, \omega]$ with k being an integer. Then

$$x\left(\zeta_1\right) < l_1. \tag{3.12}$$

In a similar way, from (3.9) and (ii), it easy to see that there exist $i_1 \in \{1, .., n\}$, a point $\zeta_1 \in [0, \omega]$ and a constant $l_2 > 0$ such that

$$x\left(\zeta_2\right) > -l_2. \tag{3.13}$$

Therefore, it follows from (3.10), (3.12) and (3.13) that

$$x\left(t\right) \leq x\left(\zeta_1\right) + \int_0^\omega \left|x'\left(t\right)\right| dt < l_1 + C,$$

and

$$x\left(t\right) \geq x\left(\zeta_2\right) - \int_0^\omega \left|x'\left(t\right)\right| dt > -\left(l_2 + C\right).$$

Thus

$$\left\|x\left(t\right)\right\| < \max\left\{l_1 + C, l_2 + C\right\} := l.$$

Now, take

$$\Omega = C_\omega^J = \left\{x \in X \mid \left\|x\left(t\right)\right\| < J\right\}.$$

where $J = \max\{l, H\}$. Notice first that Ω is a closed convex bounded subset of a Banach space so Ω satisfies the condition (a) of the Lemma 2.1. When $x \in \partial\Omega \cap \ker L = \partial\Omega \cap \mathbb{R}$, x is a constant vector in $\mathbb{R}$ with $\|x\| = J$. Then,

$$\begin{aligned}&\left(QN\right)\left(x\right)\\ &= -\frac{1}{\omega}\int_0^\omega F\left[t, x\left(t - \tau_1\left(t\right)\right), ..., x\left(t - \tau_n\left(t\right)\right), \left(S_1 x\right)\left(t - \delta\left(t\right)\right)\right] dt \neq 0.\end{aligned}$$

Set

$$\Psi\left(\theta, x\right) = \theta x + \left(1 - \theta\right) QNx,\ 0 \leq \theta \leq 1.$$

Since for $x \in \partial\Omega \cap \mathbb{R}$, $\theta \in [0, 1]$, $x\Psi\left(\theta, x\right) > 0$ one has $\Psi\left(\theta, x\right) \neq 0$. It follows from the property of invariance under a homotopy that

$$\deg\left\{QNx, \partial\Omega \cap \mathbb{R}, 0\right\} \neq 0.$$

We know that Ω verifies all the requirements of Lemma 2.1. Then system (3.1) has at least one ω-periodic solution. The proof is complete. $\square$

As a second case consider the system

$$\begin{cases} \dfrac{dx}{dt} = F\left(t, x\left(t - \tau_1\left(t\right)\right), ..., x\left(t - \tau_n\left(t\right)\right), u\left(t - \delta\left(t\right)\right)\right), \\ \\ \dfrac{d^2 u}{dt^2} = a\left(t\right) u\left(t\right) + \lambda b\left(t\right) f\left(x\left(t - \sigma\left(t\right)\right)\right). \end{cases} \tag{3.14}$$

Note that the systems (3.1) and (3.14) differ only by a sign in the first equation. So the treatment is the same as the first case. So, we have the following theorem which can be proved by similar method.

Theorem 3.6. *Suppose that the assumptions of Theorem 3.1 hold. Then the system (3.14) has at least one ω-periodic solution.*

Next recall that the delayed differential system with feedback control system (1.3) then from the results of the previous subsections we derive what follows.

Theorem 3.7. *Assume the conditions of Theorem 3.1 hold. Suppose that in (1.3) the following conditions hold.*

(i) There exists a constant $C > 0$ such that if $x(t)$ and $u(t)$ are continuous ω-periodic function and satisfies

$$\int_0^\omega G\left(t, e^{x(t-\tau_1(t))}, ..., e^{x(t-\tau_n(t))}, e^{u(t-\delta(t))}\right) dt = 0,$$

then we have

$$\int_0^\omega \left|G\left(t, e^{x(t-\tau_1(t))}, ..., e^{x(t-\tau_n(t))}, e^{u(t-\delta(t))}\right)\right| dt \le C.$$

(ii) There exists a constant $H > 0$ such that, when $v_i \ge H$, $i = 1, 2, ..., n+1$

$$G\left(t, e^{v_1}, ..., e^{v_n}, e^{v_{n+1}}\right) > 0, \;\; G\left(t, -e^{v_1}, ..., -e^{v_n}, -e^{v_{n+1}}\right) < 0,$$

uniformly hold for $[0, \infty)$.

Then, the system (1.3) has at least one positive ω-periodic solution.

Proof. Since the solutions of the initial value problem (1.3) remain positive for $t \ge 0$, we can introduce a change of variables by the formula

$$y(t) = \log(x(t)),$$

which implies that

$$x(t)\frac{dy}{dt} = \frac{dx}{dt}.$$

So, $y(t)$ satisfies the delay equation

$$\begin{aligned}\frac{dy}{dt} &= G(t, x(t-\tau_1(t)), ..., x(t-\tau_n(t)), u(t-\delta(t))) \\ &= G\left(t, e^{y(t-\tau_1(t))}, ..., e^{y(t-\tau_n(t))}, (S_1 e^y)(t-\delta(t))\right),\end{aligned}$$

where G, τ_i, $i = 1, ..., n$ and $(S_1 e^y)$ are the same as those in (1.3) (with $+$ sign) and (3.5). Set

$$\begin{aligned}&G(t, x(t-\tau_1(t)), ..., x(t-\tau_n(t)), u(t-\delta(t))) \\ &= F(t, y(t-\tau_1(t)), ..., y(t-\tau_n(t)), (S_1 e^y)(t-\delta(t))).\end{aligned}$$

Then, it is easy to see that F satisfies all the conditions of Theorem 3.6. Hence, according to Theorem 3.6, we see that (1.3) has at least one ω-periodic solution $y^*(t)$. Let

$$x^*(t) = e^{y^*(t)}.$$

Then, by Lemma 3.3 and Lemma 3.5 one can see that $\{x^*(t), u^*(t)\}$ is a positive ω–periodic solution of (1.3). In a similar way, by Theorem 3.1, one can prove that (1.3) has a positive ω-periodic solution when negative sign is selected. The proof is complete. □

4. Positive periodic solutions for 1.4

In this section we offer existence criteria for the periodic solutions of the (1.4). Throughout this part we assume that for all $t \in [0, \omega]$ and $x \in C_\omega$, whenever necessary, we shall consider

$$0 \le \lambda b(t) f(x(t-\delta(t))). \tag{4.1}$$

Let us consider the problem of existence ω-periodic solutions for following system with feedback control

$$\begin{cases}\dfrac{dx}{dt} = -F(t, x(t-\tau_1(t)), ..., x(t-\tau_n(t)), u(t-\delta(t))), \\[2ex] \dfrac{d^2u}{dt^2} = -a(t)u(t) + \lambda b(t) f(x(t-\sigma(t))),\end{cases} \tag{4.2}$$

where $F(t, x_1, ..., x_{n+1}) \in C(\mathbb{R}^{n+2}, \mathbb{R})$, $F(t+\omega, x_1, ..., x_{n+1}) = F(t, x_1, ..., x_{n+1})$ and $b, f \in C(\mathbb{R}, \mathbb{R})$, $\sigma, \delta \in C(\mathbb{R}, \mathbb{R})$, $a, \tau_i \in C(\mathbb{R}, \mathbb{R}^+)$ for $i = 1, ..., n$. All of the above functions are ω-periodic functions with $\omega > 0$ is a constant.

Lemma 4.1. *The equation*

$$\frac{d^2 y}{dt^2} + My(t) = h(t), \ h \in C_\omega^+, \tag{4.3}$$

has a unique ω*-periodic solution*

$$y(t) = \int_t^{t+\omega} K_2(t, s) h(s) ds,$$

where

$$K_2(t, s) = \frac{\cos \beta \left(\frac{\omega}{2} + t - s\right)}{2\beta \cos \frac{\beta\omega}{2}}, \ s \in [t, t+\omega]. \tag{4.4}$$

Proof. First, it is easy to see that the associate homogenous equation of (4.3) has solution

$$y(t) = c_1 \cos \beta t + c_2 \sin \beta t.$$

Applying the method of variation of parameters, we get

$$c_1'(t) = -\frac{\sin \beta t}{2\beta} h(t), \ c_2'(t) = \frac{\cos \beta t}{2\beta} h(t).$$

Noticing that $y(t)$, $y'(t)$ are periodic functions, we have

$$c_1(t) = \int_t^{t+\omega} \frac{\cos\left(s - \frac{\omega}{2}\right)}{2\beta \sin \frac{\beta\omega}{2}} h(s) ds, \quad c_2'(t) = \int_t^{t+\omega} \frac{\sin\left(s - \frac{\omega}{2}\right)}{2\beta \sin \frac{\beta\omega}{2}} h(s) ds.$$

Therefore,

$$\begin{aligned} y(t) &= c_1(t) \cos \beta t + c_2(t) \sin \beta t \\ &= \int_t^{t+\omega} K_2(t, s) h(s) ds. \end{aligned}$$

where $K_2(t, s)$ is as defined in (4.4). ☐

Lemma 4.2. $K_2(.,.)$ *in (4.4) has the properties*

1) $\int_t^{t+\omega} K_2(t, s) ds = \frac{1}{M}$ *for all* $t \in [0, \omega]$ *and* $s \in [t, t+\omega]$.

2) *If* $M < \left(\frac{\pi}{\omega}\right)^2$ *then,* $0 < \frac{\cos(\beta\omega/2)}{2\beta \sin(\beta\omega/2)} \leq K_2(t, s) \leq \frac{1}{2\beta \sin(\omega\beta/2)}$.

3) $K_2(t+\omega, s+\omega) = K_2(t, s)$.

Lemma 4.3. *Let* $x \in C_\omega$ *then,* $u \in C_\omega$ *is a solution of the second equation of (4.2) if and only if*

$$u(t) = \lambda (I - T_2 B_2)^{-1} \int_t^{t+\omega} K_2(t, s) b(s) f(x(s - \sigma(s))) ds, \tag{4.5}$$

where

$$(T_2 h)(t) = \int_t^{t+\omega} K_2(t, s) h(s) ds \ \textit{and} \ (B_2 y)(t) = [M - a(t)] y(t).$$

Proof. Let $u \in C_\omega$ be a solution of the second equation of (4.2). Rewrite the equation

$$\frac{d^2u}{dt^2} = -a(t)u(t) + \lambda b(t) f(x(t-\sigma(t))). \tag{4.6}$$

as

$$\frac{d^2u(t)}{dt^2} + Mu(t) = [M - a(t)]u(t) + \lambda b(t) f(x(t-\sigma(t))).$$

Define

$$(B_2u)(t) = [M - a(t)]u(t),$$

Then, (4.6) is transformed into

$$\frac{d^2u}{dt^2} + Mu(t) = (B_2u)(t) + \lambda h(t) f(x(t-\sigma(t))).$$

Since $0 \leq \lambda b(t) f(x(t-\sigma(t)))$, from Lemma 4.1, we have

$$\begin{aligned} u(t) &= (T_2B_2u)(t) + T_2(\lambda b(t) f(x(t-\sigma(t)))) \\ &= (T_2B_2u)(t) - \lambda \int_t^{t+\omega} K_2(t,s) b(s) f(x(s-\sigma(s))) ds. \end{aligned}$$

This yields

$$(I - T_2B_2) u(t) = \lambda \int_t^{t+\omega} K_2(t,s) b(s) f(x(s-\sigma(s))) ds.$$

Therefore, since $\|T_2B_2\| \leq 1 - \frac{m}{M} < 1$ then

$$u(t) = \lambda (I - T_2B_2)^{-1} \int_t^{t+\omega} K_2(t,s) b(s) f(x(s-\sigma(s))) ds.$$

□

Let $S_2 : C_\omega \to C_\omega$

$$u(t) = \lambda (I - T_2B_2)^{-1} \int_t^{t+\omega} K_2(t,s) b(s) f(x(s-\sigma(s))) ds := (S_2x)(t). \tag{4.7}$$

It is not difficult to establish, by a change of variable, that

$$u(t) = u(t+\omega).$$

Lemma 4.4. *S_2 is a continuous mapping satisfying the following condition*

$$0 < (S_2x)(t), \ x \in C_\omega.$$

Proof. The proof is similar to that in (3.5) by replacing $h(t) = \lambda b(t) f(x(t-\sigma(t))) < 0$ by $h(t) = \lambda b(t) f(x(t-\sigma(t))) \geq 0$. □

It is obvious that $u(t)$ is the unique ω-periodic solution of (3.6) for $x \in C_\omega$. Therefore, the existence problem of ω-periodic solution of (3.1) is equivalent to that of ω-periodic solution of the equation

$$\frac{dx}{dt} = -F[t, x(t-\tau_1(t)), ..., x(t-\tau_n(t)), (S_2x)(t-\delta(t))].$$

From the last section we deduce that by the same reasoning as in second section and by using Lemmas 4.1, 4.3 and Lemmas 4.2,4.4, and by employing techniques as in second section. It is possible to establish the existence of positive ω-periodic solution to the system (1.4). That is, from the second section follows the following theorem.

Theorem 4.5. *Under the hypotheses of theorem 3.7 and conditions (4.1), suppose further that $M < \left(\frac{\pi}{\omega}\right)^2$ holds. Then, the system (1.4) has at least one positive ω-periodic solution.*

Example 4.6. *Consider the following logistic model with several delays and feedback control*

$$
\left\{
\begin{aligned}
\frac{dx}{dt} &= x(t)\left[r(t) - \sum_{i=1}^{n} a_i(t)\, x(t-\tau_i(t)) - c(t)\, u(t-\delta(t))\right], \\
\frac{d^2u}{dt^2} &= -0.9e^{-\cos(t)} u(t) + \lambda b(t)\, x^2(t-\sigma(t)).
\end{aligned}
\right. \tag{4.8}
$$

where $\sigma, \delta \in C(\mathbb{R}, \mathbb{R})$, and for $i = 1, ..., n$ one has $c, r, b, a_i, \tau_i \in C(\mathbb{R}, \mathbb{R}^+)$ are all ω-periodic functions with $0 < \omega \leq \pi$ and $\lambda > 0$ is a constant. Then the system (4.8) has at least one positive ω-periodic solution.

Proof. Let $x(t)$ be a continuous ω-periodic solution and satisfies

$$\int_0^\omega \left[r(t) - \sum_{i=1}^{n} a_i(t)\, e^{x(t-\tau_i(t))} - c(t)\,(S_2 e^x)(t-\delta(t))\right] dt = 0,$$

where $(S_2 e^x)$ is the same as those in (4.7). Then

$$\int_0^\omega r(t)\, dt = \int_0^\omega \left[\sum_{i=1}^{n} a_i(t)\, e^{x(t-\tau_i(t))} + c(t)\,(S_2 e^x)(t-\delta(t))\right] dt = 0.$$

On the other hand

$$
\begin{aligned}
&\int_0^\omega \left| r(t) - \sum_{i=1}^{n} a_i(t)\, e^{x(t-\tau_i(t))} - c(t)\,(S_2 e^x)(t-\delta(t))\right| dt \\
&\leq \int_0^\omega |r(t)|\, dt + \int_0^\omega \left|\sum_{i=1}^{n} a_i(t)\, e^{x(t-\tau_i(t))} + c(t)\,(S_2 e^x)(t-\delta(t))\right| dt \\
&\leq 2\int_0^\omega |r(t)|\, dt = C > 0.
\end{aligned}
$$

Moreover,

$$\lim_{\{v_1, v_2, \cdots, v_{n+1}\}\to+\infty} \left(r(t) - \sum_{i=1}^{n} a_i(t)\, e^{v_i} - c(t)\,(S_2 e^{v_{n+1}})(t-\delta(t))\right) = -\infty,$$

and

$$\lim_{\{v_1, v_2, \cdots, v_{n+1}\}\to-\infty} \left(r(t) - \sum_{i=1}^{n} a_i(t)\, e^{x(t-\tau_i(t))} - c(t)\,(S_2 e^x)(t-\delta(t))\right) = r(t) > 0,$$

hold uniformly in $t \in [0, \omega]$. Furthermore,

$$
\begin{aligned}
0 &< m = \min\left\{0.9e^{-\cos^2(t)}\right\} = 0.9e^{-1}, \\
M &= \max\left\{0.9e^{-\cos^2(t)}\right\} = 0.9 < \left(\frac{\pi}{\pi}\right)^2 = 1 \leq \left(\frac{\pi}{\omega}\right)^2,
\end{aligned}
$$

and

$$\lambda b(t)\, x^2(t-\sigma(t)) \geq 0.$$

By Theorem 4.5, we see that system (4.8) has at least one positive ω-periodic solution. □

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