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On the signed total domatic numbers of directed graphs

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Abstract

Let $D = (V, A)$ be a finite simple directed graph (shortly digraph) in which $d_D^-(v) \geq 1$ for all $v \in V$. A function $f : V \longrightarrow \{-1, 1\}$ is called a *signed total dominating function* if $\sum_{u \in N^-(v)} f(u) \geq 1$ for each vertex $v \in V$. A set $\{f_1, f_2, \ldots, f_d\}$ of signed total dominating functions on D with the property that $\sum_{i=1}^{d} f_i(v) \leq 1$ for each $v \in V(D)$, is called a *signed total dominating family* (of functions) on D. The maximum number of functions in a signed total dominating family on D is the *signed total domatic number* of D, denoted by $d_{st}(D)$. In this paper we present some bounds on the signed total domatic number and we determine the signed total domatic number of some classes of digraphs.

Keywords: signed total domatic number, signed total dominating function, signed total domination number, directed graph
MSC 2010: 05C69, 05C20

1 Introduction

In this paper, D is a finite simple directed graph with vertex set $V(D)$ and arc set $A(D)$. Its underlying graph is denoted $G(D)$. We write $d_D^+(v)$ for the outdegree of a vertex v and $d_D^-(v)$ for its indegree. The minimum indegree is $\delta^-(D)$. For every vertex $v \in V$, let $N_D^-(v)$ be the set consisting of all vertices of D from which arcs go into v. Note that for any digraph D with m arcs,

$$\sum_{u \in V(D)} d^-(u) = \sum_{u \in V(D)} d^+(u) = m. \tag{1}$$

We often use the abbreviations V, $N^-(v)$, $d^-(v)$ for $V(D)$, $N_D^-(v)$, $d_D^-(v)$. Consult [6] for the notation and terminology which are not defined here.

For a real-valued function $f : V(D) \longrightarrow \mathbb{R}$, the weight of f is $w(f) = \sum_{v \in V} f(v)$. For $S \subseteq V$, we define $f(S) = \sum_{v \in S} f(v)$. So $w(f) = f(V)$.

If $k \geq 1$ is an integer, then the *signed total k-dominating function* is defined as a function $f : V(D) \longrightarrow \{-1, 1\}$ such that $f(N^-(v)) = \sum_{x \in N^-(v)} f(x) \geq k$ for every $v \in V(D)$. The *signed total k-domination number* for a digraph D is

$$\gamma_{kS}^t(D) = \min\{w(f) \mid f \text{ is a signed total } k\text{-dominating function of } D\}.$$

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A $\gamma^t_{kS}(D)$-function is a signed total k-dominating function on D of weight $\gamma^t_{kS}(D)$. As the assumption $\delta^-(D) \geq k$ is necessary, we always assume that when we discuss $\gamma^t_{kS}(D)$, all digraphs involved satisfy $\delta^-(D) \geq k$ and thus $n(D) \geq k+1$.

The signed total k-domination number of digraphs was introduced by Sheikholeslami and Volkmann [5]. When $k = 1$, the signed total k-domination number $\gamma^t_{kS}(D)$ is the usual *signed total domination number* $\gamma_{st}(D)$, which was introduced by Sheikholeslami in [3].

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed total k-dominating functions on D with the property that $\sum_{i=1}^d f_i(v) \leq k$ for each $v \in V(D)$, is called a *signed total* (k,k)*-dominating family* on D. The maximum number of functions in a signed total (k,k)-dominating family on D is the *signed total* (k,k)*-domatic number* of D, denoted by $d^k_{st}(D)$. The signed total (k,k)-domatic number of digraphs was introduced by Sheikholeslami and Volkmann [4].

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed total k-dominating functions on D with the property that $\sum_{i=1}^d f_i(v) \leq 1$ for each $v \in V(D)$, is called a *signed total* k*-dominating family* on D. The maximum number of functions in a signed total k-dominating family on D is the *signed total* k*-domatic number* of D, denoted by $d^t_{kS}(D)$. The signed total k-domatic number of digraphs was introduced by Atapour et al. [1].

If $k = 1$, then the signed total (k,k)-domatic number and the signed total k-domatic number of a digraph D are the same and is called signed total domatic number, denoted by $d_{st}(D)$. The signed total domatic number is well-defined and $d_{st}(D) \geq 1$ for all digraphs D in which $d^-_D(v) \geq 1$ for all $v \in V$ since the set consisting of any one STD function forms a STD family of D. A d_{st}*-family* of a digraph D is a STD family containing $d_{st}(D)$ STD functions.

The concept of signed total domatic number of an undirected graph was introduced by Henning in [2] as follows. The *signed total dominating function* of a graph G is defined as a function $f : V(G) \longrightarrow \{-1, 1\}$ such that $\sum_{x \in N(v)} f(x) \geq 1$ for every $v \in V(G)$. The sum $\sum_{x \in V(G)} f(x)$ is the weight $w(f)$ of f. The minimum of weights $w(f)$, taken over all signed total dominating functions f on G is called the *signed total domination number* of G, denoted by $\gamma_{st}(G)$. A set $\{f_1, f_2, \ldots, f_d\}$ of signed total dominating functions on G with the property that $\sum_{i=1}^d f_i(v) \leq 1$ for each $v \in V(G)$, is called a *signed total dominating family* on G. The maximum number of functions in a signed total dominating family on G is the *signed total domatic number* of G, denoted by $d_{st}(G)$.

In this paper we continue the study of the signed total domatic numbers in digraphs and we determine the signed total domatic number of some classes of digraphs.

The proof of the following results can be found in [1, 4].

Theorem A. Let D be a digraph of order n and positive minimum indegree with signed total domination number $\gamma_{st}(D)$ and signed total domatic number $d_{st}(D)$. Then

$$\gamma_{st}(D) \cdot d_{st}(D) \leq n.$$

Moreover if $\gamma_{st}(D) \cdot d_{st}(D) = n$, then for each d_{st}-family $\{f_1, \cdots, f_d\}$ of D, each function f_i is a γ_{st}-function and $\sum_{i=1}^d f_i(v) = 1$ for all $v \in V$.

Theorem B. The signed total domatic number of a digraph is an odd integer.

Theorem C. If D is a digraph with minimum indegree $\delta^-(D) \geq 1$, then

$$1 \leq d_{st}(D) \leq \delta^-(D).$$

Moreover if $d_{st}(D) = \delta^-(D)$, then for each function of any STD family $\{f_1, f_2, \cdots, f_d\}$ and for all vertices v of indegree $\delta^-(D)$, $\sum_{u \in N^-(v)} f_i(u) = 1$ and $\sum_{i=1}^d f_i(u) = 1$ for every $u \in N^-(v)$.

2 Bounds on the signed total domatic number

In this section we present some bounds on the signed total domatic number of a digraph.

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Theorem 1. Let D be a digraph, and let v be a vertex of even positive indegree $d^-(v) = 2t$. Then

$$d_{st}(D) \leq \begin{cases} t & \text{if } t \text{ is odd} \\ t-1 & \text{if } t \text{ is even.} \end{cases}$$

Moreover if t is odd and $d_{st} = t$, then for each function f_i of each STD family and for each vertex v of indegree $2t$ we have $\sum_{u \in N^-(v)} f_i(u) = 2$ and $\sum_{i=1}^{d} f_i(u) = 1$.

Proof. Let $d = d_{st}(D)$ and let $\{f_1, f_2, \ldots, f_d\}$ be a STD family on D . Since $d^-(v)$ is even and $f_i(u)$ is odd for each i and each u, we observe that

$$\sum_{u \in N^-(v)} f_j(u) \geq 2 \tag{2}$$

for each function f_i. Using inequality (2) and $\sum_{i=1}^{d} f_i(u) \leq 1$ for each vertex $u \in V(D)$, we obtain

$$2t = d^-(v) = \sum_{u \in N^-(v)} 1 \geq \sum_{u \in N^-(v)} \sum_{i=1}^{d} f_i(u) = \sum_{i=1}^{d} \sum_{u \in N^-(v)} f_i(u) \geq \sum_{i=1}^{d} 2 = 2d.$$

This yields $d \leq t$ immediately and if $d = t$, every inequality becomes an equality. When t is even, Theorem B implies $d \leq t - 1$. □

Restricting our attention to digraphs D of even minimum indegree, this theorem leads to a considerably improvement of the upper bound given in Theorem C.

Corollary 2. If D is a digraph of even minimum indegree $\delta^-(D) \geq 2$, then

$$d_{st}(D) \leq \begin{cases} \frac{\delta^-(D)}{2} & \text{when } \delta^-(D) \equiv 2 \pmod 4 \\ \frac{\delta^-(D)-2}{2} & \text{when } \delta^-(D) \equiv 0 \pmod 4. \end{cases}$$

Let D be an orientation of a $(4q+2)$-regular graph of order n. The number of arcs of D is $m = (2q+1)n$ and its average indegree and outdegree are both equal to $2q+1$. Hence $\delta^-(D) \leq 2q+1$ and D is inregular if and only if it is outregular. We study below such digraphs satisfying $d_{st}(D) = 2q+1$.

Definition 3. Let q be a positive integer. $\mathcal{F}_q$ is the family of the orientations of $(4q+2)$-regular graphs of order $n \equiv 0 \pmod{2q+1}$ such that there exist $2q+1$ subsets A_j of $\frac{qn}{2q+1}$ vertices such that each vertex of D is contained in exactly q of them and $|N^-(u) \cap A_j| = q$ for each vertex u in V and each subset A_j.

After Theorem 12, we give an example of a family of graphs in $\mathcal{F}_1$ which can be generalized to families of graphs in $\mathcal{F}_q$ for any q.

Proposition 4. *If $D \in \mathcal{F}_q$, then D is inregular and $\delta^-(D) = 2q+1$.*

Proof. We can associate to the digraph D a bipartite graph H with independent classes V and $\mathcal{A} = \{A_1, A_2, \cdots, A_{2q+1}\}$ where a vertex v of V is adjacent to the subsets A_j containing it. Then in H, the vertices of V have degree q and the vertices of $\mathcal{A}$ have degree $\frac{qn}{2q+1}$. If a subset U of V has exactly q elements in each A_j, then H contains $q(2q+1)$ edges between $\mathcal{A}$ and U, and thus $|U| = 2q+1$. Therefore the condition $|N^-(u) \cap A_j| = q$ for each j and each u implies that $d^-(u) = 2q+1$ for each vertex u of V. Hence D is $(2q+1)$-inregular. □

Theorem 5. *A digraph D whose underlying graph is $(4q+2)$-regular satisfies $d_{st}(D) = 2q+1$ if and only if it belongs to $\mathcal{F}_q$.*

Proof. Part only if: Let D be a digraph such that $G(D)$ is $(4q+2)$-regular and $d_{st}(D) = 2q+1$. The number of arcs of D is $m = (2q+1)n$ and its average indegree is $m/n = (2q+1)$. Therefore $\delta^- \le 2q+1$ and by Theorem C, $\delta^- = 2q+1$. Thus D is $(2q+1)$-inregular and $(2q+1)$-outregular.

Let f_j, $1 \le j \le 2q+1$, be a STD family of D and for each j, let $A_j = \{u \in V \mid f_j(u) = -1\}$. By Theorem C and since $d_{st}(D) = \delta^-$, $\sum_{j=1}^{2q+1} f_j(v) = 1$ for every v and $\sum_{u\in N^-(v)} f_i(u) = 1$ for every index i and every vertex v. From the first inequality, $q+1$ functions f_j take the value $+1$ in v and q functions f_j take the value -1 in v. Therefore each vertex v belongs to exactly q sets A_j. From the second inequality, exactly q vertices of $N^-(v)$ are in A_j, i. e., $|N^-(v) \cap A_j| = q$ for all v and all j. For each j, there are qn arcs with origin in A_j. Since $d^+(v) = 2q+1$ for all v, $(2q+1)|A_i| = qn$. Hence $n \equiv 0 \pmod{2q+1}$ and $A_i = \frac{qn}{2q+1}$ for every i.

Part if: Let D be a digraph in $\mathcal{F}_q$ and let $A_1, A_2, \cdots, A_{2q+1}$ be subsets of $V(D)$ as in Definition 3. By Theorem C and Proposition 4, $d_{st}(D) \le 2q+1$. To prove $d_{st}(D) = 2q+1$, we exhibit a STD family of $2q+1$ functions. For $j = 1, 2, \cdots, 2q+1$, let f_j be defined by $f_j(u) = -1$ if $u \in A_j$, $f_j(u) = +1$ otherwise. For each vertex u and each index j, the function f_j assigns the value -1 to the q vertices of $N^-(u) \cap A_j$ and the value $+1$ to the $q+1$ other vertices of $N^-(u)$. Hence for each j, $\sum_{v\in N^-(u)} f_j(v) = +1$ and f_j is a STD function. Since every vertex u is in exactly q subsets A_j, $f_j(u)$ takes the value -1 for exactly q indices j and the value $+1$ for the $q+1$ other ones. Hence, $\sum_{j=1}^{2q+1} f_j(u) = 1$ for every u in V and $\{f_1, f_2, \cdots, f_{2q+1}\}$ is a SDT-family. Therefore $d_{st}(D) = 2q+1$. □

Theorem 6. Let D be a digraph with positive minimum indegree, and let $\Delta = \Delta(G(D))$ be the maximum degree of $G(D)$. Then

$$d_S(D) \le \frac{\Delta}{2},$$

with equality if and only if $D \in \mathcal{F}_q$ for some q.

Proof. First of all, we show that $\delta^-(D) \le \Delta/2$. Suppose to the contrary that $\delta^-(D) > \Delta/2$. Then $\Delta^+(D) \le \Delta - \delta^-(D) < \Delta/2$, and (1) leads to the contradiction

$$\frac{\Delta \cdot |V(D)|}{2} < \sum_{u\in V(D)} d^-(u) = \sum_{u\in V(D)} d^+(u) < \frac{\Delta \cdot |V(D)|}{2}.$$

Applying Theorem C, we deduce that

$$d_S(D) \le \delta^-(D) \le \frac{\Delta}{2},$$

and this is the desired result.

Now let $d_S(D) = \frac{\Delta}{2}$. It follows that $\delta^-(D) = \frac{\Delta}{2}$ and so $\Delta \equiv 0, 2 \pmod 4$. If $\Delta \equiv 0 \pmod 4$, then Corollary 2 implies that $d_S(D) \le \frac{\Delta}{4}$ which is a contradiction.

Now let $\Delta \equiv 2 \pmod 4$, and therefore $\Delta = 4q+2$ for an integer $q \ge 0$. This implies that $\delta^-(D) = 2q+1$ and so $G(D)$ is a $(4q+2)$-regular graph. Since $d_S(D) = 2q+1$, by Theorem 5 we have $D \in \mathcal{F}_q$. □

The *associated digraph* $D(G)$ of a graph G is the digraph obtained when each edge e of G is replaced by two oppositely oriented arcs with the same ends as e. Since $N^-_{D(G)}(v) = N_G(v)$ for each vertex $v \in V(G) = V(D(G))$, the following useful observation is valid.

Observation 7. If $D(G)$ is the associated digraph of a graph G, then $d_{st}(D(G)) = d_{st}(G)$.

There are a lot of interesting applications of Observation 7, as for example the following three results:

Corollary 8. (Henning [2] 2006) If G is a graph with minimum degree $\delta(G) > 0$, then $d_{st}(G) \le \delta(G)$.

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Proof. Since $\delta(G) = \delta^-(D(G))$, it follows from Theorem C and Observation 7

$$d_{st}(G) = d_{st}(D(G)) \leq \delta^-(D(G)) = \delta(G).$$

□

Corollary 9. (Henning [2] 2006) The signed total domatic number of a graph G is an odd integer.

If K_n is the complete graph of order n, then Henning [2] have shown that $d_{st}(K_n) = \lfloor \frac{n+1}{3} \rfloor - \lceil \frac{n}{3} \rceil + \lfloor \frac{n}{3} \rfloor$ if n (≥ 3) is odd and $d_{st}(K_n) = \frac{n}{2} - \lceil \frac{n+2}{4} \rceil + \lfloor \frac{n+2}{4} \rfloor$ if n is even. This result and Observation 7 imply immediately the next corollary.

Corollary 10. If K_n^* is the complete digraph of order n, then $d_{st}(K_n^*) = \lfloor \frac{n+1}{3} \rfloor - \lceil \frac{n}{3} \rceil + \lfloor \frac{n}{3} \rfloor$ if n (≥ 3) is odd and $d_{st}(K_n^*) = \frac{n}{2} - \lceil \frac{n+2}{4} \rceil + \lfloor \frac{n+2}{4} \rfloor$ if n is even.

In the following section, we use the bounds obtained in terms of the indegrees or of the signed total dominating number to determine the value of $d_{st}(D)$ for some particular digraphs.

3 Determination of $d_{st}(D)$ for particular digraphs

For digraphs with few arcs, Theorems C and 1 immediately lead to the next results.

Theorem 11. *Let D be a digraph such that $1 \leq \delta^- \leq 2$ or such that $\delta^- \geq 3$ and with a vertex of indegree 4. Then $d_{st}(D) = 1$.*

Theorem 12. *Every orientation D with $\delta^-(D) \geq 1$ of a graph G satisfying one of the following properties is such that $d_{st}(D) = 1$:*

1. G has minimum degree at most 2.

2. G contains two adjacent vertices of degree 3.

3. G has maximum degree $\Delta(G)$ at most 6 except if G is 6-regular and $D \in \mathcal{F}_1$.

Proof. 1. and 2. If $\delta(G) \leq 2$, then $\delta^-(D) \leq 2$. If x and y are two adjacent vertices of degree 3 in G, suppose without loss of generality the edge xy is oriented from x to y in D. Then $d^-(x) \leq 2$ and again $\delta^-(D) \leq 2$.
3. If $\Delta(G) \leq 6$, the number m of arcs of D is at most $3n$ and its average indegree is at most 3. If G is not 6-regular, then the average indegree of D is less than 3 and $\delta^-(D) \leq 2$. If G is 6-regular, then $d_{st}(D) = 1$ except if $D \in \mathcal{F}_1$ by Theorem 5. □

We construct an example of digraphs in $\mathcal{F}_1$ as follows. D is a circulant digraph of order $n \equiv 0 \pmod 3$ with vertex set $V = \{u_0, u_1, \cdots, u_{n-1}\}$ and arcs $u_iu_{i+1}, u_iu_{i+2}, u_iu_{i+3}$ where the indices are taken modulo n. For $j = 0, 1, 2$, the three sets $A_j = \{u_j, u_{j+3}, u_{j+6}, \cdots, u_{j+n-3}\}$ of order $n/3$ form a partition of V and $|N^-(u) \cap A_j| = 1$ for each j and each vertex u. They allow to define three functions $f_j : V \longrightarrow \{-1, +1\}$ by $f_j(u) = -1$ if and only if $u \in A_j$. These three functions form a STD family of order $\delta^-(D) = 3$.

Now we use Theorem A to determine d_{st} for a particular class of tournaments. Let $n = 2k + 1$ be an odd positive integer. We define the circulant tournament $\mathrm{CT}(n)$ with n vertices as follows. The vertex set of $\mathrm{CT}(n)$ is $V(\mathrm{CT}(n)) = \{u_0, u_1, \ldots, u_{n-1}\}$. For each i, the arcs go from u_i to the vertices $u_{i+1}, \ldots, u_{i+k}$, the indices being taken modulo n. Sheikholeslami in [3] proved that:

Theorem D. Let $n = 2k + 1$ where k is a positive integer. Then

$$\gamma_{st}(\mathrm{CT}(n)) = \begin{cases} 3 & \text{if } k \text{ is odd} \\ 5 & \text{if } k \text{ is even.} \end{cases}$$

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Theorem 13. Let $n > 1$ be an odd integer. Then

$$d_{st}(\mathrm{CT}(n)) = \begin{cases} \frac{n}{3} & \text{if } n \equiv 3 \pmod{12} \\ \frac{n-4}{3} & \text{if } n \equiv 7 \pmod{12} \\ \frac{n-2}{3} & \text{if } n \equiv 11 \pmod{12} \\ \frac{n-6}{5} & \text{if } n \equiv 1 \pmod{20} \\ \frac{n}{5} & \text{if } n \equiv 5 \pmod{20} \\ \frac{n-4}{5} & \text{if } n \equiv 9 \pmod{20} \\ \frac{n-8}{5} & \text{if } n \equiv 13 \pmod{20} \\ \frac{n-2}{5} & \text{if } n \equiv 17 \pmod{20} \end{cases}$$

Proof. **Case 1.** $n = 2k+1$ with k is odd.

Then $n \equiv 3 \pmod 4$, which is equivalent to $n \equiv 3, 7$ or $11 \pmod{12}$.

Let $\ell = \frac{n}{3}$ if $n \equiv 3 \pmod{12}$, $\ell = \frac{n-4}{3}$ if $n \equiv 7 \pmod{12}$, $\ell = \frac{n-2}{3}$ if $n \equiv 11 \pmod{12}$ be the largest odd integer less or equal to $\lfloor \frac{n}{3} \rfloor$. By Proposition A and Theorems 1 and 3, $d_{st}(\mathrm{CT}(n)) \leq \ell$. If $n = 3$ or $n = 7$, then $\ell = 1$ and thus $d_{st}(\mathrm{CT}(n)) = 1$. We suppose henceforth $n = 2k+1$ with k odd ≥ 5. To prove $d_{st}(\mathrm{CT}(n)) = \ell$, we exhibit a STD family of order ℓ. For $j = 0, 1, \cdots, \ell-1$, let

$$A_j = A_j^1 \cup A_j^2 \text{ with}$$

$$A_j^1 = \{u_i \mid j(\frac{k-1}{2}) \leq i \leq j(\frac{k-1}{2}) + \frac{k-3}{2}\} \text{ and}$$

$$A_j^2 = \{u_i \mid \frac{n+1}{2} + j(\frac{k-1}{2}) \leq i \leq \frac{n+1}{2} + j(\frac{k-1}{2}) + \frac{k-3}{2}\},$$

where the indices i are taken modulo n, and define the function f_j by

$$f_j(u) = \begin{cases} -1 & \text{if } u \in A_j \\ +1 & \text{otherwise.} \end{cases}$$

For each $j \in \{0, 1, \cdots, \ell-1\}$, the function f_j assigns the value -1 to two sequences of $\frac{k-1}{2}$ consecutive vertices of $CT(n)$ separated by at least

$$\min\{\frac{n+1}{2} - \frac{k-3}{2} - 1,\ n - (\frac{n+1}{2} + \frac{k-3}{2}) - 1\} = \frac{k+1}{2}$$

vertices. For each vertex u, $N^-(u)$ consists of k consecutive vertices of $CT(n)$, among them at most $\frac{k-1}{2}$ receive the value -1 by f_j and the other ones the value $+1$. Therefore for each j and each u, $\sum_{v \in N^-(u)} f_j(v) \geq 1$. Hence f_j is a STD function for $0 \leq j \leq \ell-1$.

We have now to prove that $\{f_0, f_1, \cdots, f_{\ell-1}\}$ is a STD family of functions, in other words that $\sum_{j=0}^{\ell-1} f_j(u) \leq 1$ for every vertex u of $CT(n)$. Since the ℓ sets A_j^1 are disjoint, the indices j such that $u_i \in A_j^1$ are different. Similarly, the ℓ sets A_j^2 are disjoint and the indices j such that $u_i \in A_j^2$ are different. Assume $A_j^1 \cap A_j^2 \neq \emptyset$ and let $u_i \in A_j^1 \cap A_j^2$. Then there exists a non-negative integer r_1 and r_2 such that

$$j\frac{k-1}{2} \leq i + r_1 n \leq j\frac{k-1}{2} + \frac{k-3}{2} \text{ and}$$

$$\frac{n+1}{2} + j\frac{k-1}{2} \leq i + r_2 n \leq \frac{n+1}{2} + j\frac{k-1}{2} + \frac{k-3}{2}\ .$$

These integers satisfy $r_2 > r_1$ since

$$j\frac{k-1}{2} + \frac{k-3}{2} < \frac{n+1}{2} + j\frac{k-1}{2},$$

and $r_2 < r_1 + 1$ since
$$\frac{n+1}{2} + j\frac{k-1}{2} + \frac{k-3}{2} < j\frac{k-1}{2} + n.$$
This is not compatible and thus $A_j^1 \cap A_j^2 = \emptyset$. This shows that the set of indices j such that $u_i \in A_j^1$ and the set of indices j such that $u_i \in A_j^2$ are disjoint. Therefore, if a vertex u is covered $s_1(u)$ times by $A^1 = \cup_{j=0}^{\ell-1} A_j^1$ and $s_2(u)$ times by $A^2 = \cup_{j=0}^{\ell-1} A_j^2$, then
$$f_j(u) = \begin{cases} -1 & \text{for } s_1(u) + s_2(u) \text{ values of } j \\ +1 & \text{for } \ell - (s_1(u) + s_2(u)) \text{ values of } j. \end{cases}$$
To prove $\sum_{j=0}^{\ell-1} f_j(u) \le 1$ for every vertex u, it is sufficient to show that in any case, $s_1(u) + s_2(u) \ge \ell - (s_1(u) + s_2(u)) - 1$, i.e.,
$$s_1(u) + s_2(u) \ge \frac{\ell-1}{2}. \tag{3}$$
The ℓ disjoint sets A_j^1 are consecutive and
$$A^1 = \cup_{j=0}^{\ell-1} A_j^1 = \{u_i \mid 0 \le i \le \ell\frac{k-1}{2} - 1\}.$$
Similarly the ℓ disjoint sets A_j^2 are consecutive and
$$A^2 = \cup_{j=0}^{\ell-1} A_j^2 = \{u_i \mid \frac{n+1}{2} \le i \le \frac{n+1}{2} + \ell\frac{k-1}{2} - 1\}.$$
Therefore each set A^1 and A^2 covers $\ell\frac{k-1}{2}$ consecutive vertices.

Subcase 1.1 $n = 12q + 3$ with $q \ge 1$. Hence $k = 6q + 1$ and $\ell = \frac{n}{3} = 4q + 1$.

Each set A^1 and A^2 contains $\ell\frac{k-1}{2} = 3q\ell = qn$ vertices. Therefore each vertex is covered exactly q times by each of A^1, A^2 and $s_1(u) = s_2(u) = q$. Then $s_1(u) + s_2(u) = 2q = \frac{\ell-1}{2}$ for each u and (3) is satisfied.

Subcase 1.2 $n = 12q + 7$ with $q \ge 1$. Hence $k = 6q + 3$ and $\ell = \frac{n-4}{3} = 4q + 1$.

Each set A^1 and A^2 contains $\ell\frac{k-1}{2} = 3q\ell = qn + 1$ vertices. The interval A^1 covers $q + 1$ times the vertex u_0 and q times the other ones. The interval A^2 covers $q + 1$ times the vertex $u_{\frac{n+1}{2}}$ and q times the other ones. Therefore $s_1(u) + s_2(u) \ge 2q = \frac{\ell-1}{2}$ for each u and (3) is satisfied.

Subcase 1.3 $n = 12q + 11$ with $q \ge 0$. Hence $k = 6q + 5$ and $\ell = \frac{n-2}{3} = 4q + 3$.

Each set A^1 and A^2 contains $\ell\frac{k-1}{2} = qn + \frac{n+1}{2}$ vertices. The interval A^1 covers $q + 1$ times the vertices u_i with $0 \le i \le \frac{n-1}{2}$ and q times the other ones. The interval A^2 covers $q + 1$ times the vertices u_i with $\frac{n+1}{2} \le i \le n$ and q times the other ones. Therefore $s_1(u) + s_2(u) \ge 2q + 1 = \frac{\ell-1}{2}$ for each u and (3) is satisfied.

In all cases, $f_0, f_1, \cdots, f_{\ell-1}$ is a STD family and thus $d_{st}(\mathrm{CT}(n)) = \ell$.

Case 2. $n = 2k + 1$ with k even.

Then $n \equiv 1 \pmod 4$, which is equivalent to $n \equiv 1, 5, 9, 13$ or $17 \pmod{20}$.

Let $\ell = \frac{n-6}{5}$ if $n \equiv 1 \pmod{20}$, $\ell = \frac{n}{5}$ if $n \equiv 5 \pmod{20}$, $\ell = \frac{n-4}{5}$ if $n \equiv 9 \pmod{20}$, $\ell = \frac{n-8}{5}$ if $n \equiv 13 \pmod{20}$ and $\ell = \frac{n-2}{5}$ if $n \equiv 17 \pmod{20}$ be the largest odd integer less or equal to $\lfloor \frac{n}{5} \rfloor$. By Proposition A and Theorems 1 and 3, $d_{st}(\mathrm{CT}(n)) \le \ell$.

If $n = 5, 9$ or 13, then $\ell = 1$ and thus $d_{st}(\mathrm{CT}(n)) = 1$. We suppose henceforth $n = 2k + 1$ with k even ≥ 8. To prove $d_{st}(\mathrm{CT}(n)) = \ell$, we exhibit a STD family of order ℓ. The proof is similar to the proof of Case 1. We give the numerical values with less explanations.

For $j = 0, 1, \cdots, \ell - 1$, let
$$A_j = A_j^1 \cup A_j^2 \quad \text{with}$$

$$A_j^1 = \{u_i \mid j(\frac{k-2}{2}) \le i \le j(\frac{k-2}{2}) + \frac{k-4}{2}\} \text{ and}$$
$$A_j^2 = \{u_i \mid \frac{n+1}{2} + j(\frac{k-2}{2}) \le i \le \frac{n+1}{2} + j(\frac{k-2}{2}) + \frac{k-4}{2}\},$$
and define the function f_j by $f_j(u) = \begin{cases} -1 & \text{if } u \in A_j \\ +1 & \text{otherwise.} \end{cases}$

The intervals A_j^1 and A_j^2 contain each $\frac{k-2}{2}$ vertices and are separated by at least
$$\min\{\frac{n+1}{2} - \frac{k-4}{2} - 1,\ n - (\frac{n+1}{2} + \frac{k-4}{2}) - 1\} = \frac{k+2}{2}$$
vertices. For each vertex u, $N^-(u)$ consists of k consecutive vertices of $CT(n)$, among them at most $\frac{k-2}{2}$ receive the value -1 by f_j and the other ones the value $+1$. Therefore f_j is a STD function for $0 \le j \le \ell - 1$.

As above, the indices j such that $u_i \in A_j^1$ are different and the indices j such that $u_i \in A_j^2$ are different. If $u_i \in A_j^1 \cap A_j^2$, there exist non-negative integers r_1 and r_2 such that
$$j\frac{k-1}{2} \le i + r_1 n \le j\frac{k-1}{2} + \frac{k-4}{2}$$
and
$$\frac{n+1}{2} + j\frac{k-1}{2} \le i + r_2 n \le \frac{n+1}{2} + j\frac{k-1}{2} + \frac{k-4}{2}\ .$$
This is impossible, which shows that the set of indices j such that $u_i \in A_j^1$ and the set of indices j such that $u_i \in A_j^2$ are disjoint. If a vertex u is covered $s_1(u)$ times by $A^1 = \cup_{j=0}^{\ell-1} A_j^1$ and $s_2(u)$ times by $A^2 = \cup_{j=0}^{\ell-1} A_j^2$, it is sufficient, to prove that $f_0, f_1, \cdots, f_{\ell-1}$ is a STD-family, to show that (3) is satisfied. Since
$$A^1 = \cup_{j=0}^{\ell-1} A_j^1 = \{u_i \mid 0 \le i \le \ell\frac{k-2}{2} - 1\}$$
and
$$A^2 = \cup_{j=0}^{\ell-1} A_j^2 = \{u_i \mid \frac{n+1}{2} \le i \le \frac{n+1}{2} + \ell\frac{k-2}{2} - 1\},$$
each set A^1 and A^2 covers $\ell\frac{k-2}{2}$ consecutive vertices.

Subcase 2.1 $n = 20q + 1$ with $q \ge 1$. Hence $k = 10q$ and $\ell = \frac{n-6}{3} = 4q - 1$.

Each set A^1 and A^2 contains $\ell\frac{k-2}{2} = (q-1)n + \frac{n+3}{2}$ vertices. The interval A^1 covers q times the vertices u_i with $0 \le i \le \frac{n+1}{2}$ and $q-1$ times the other ones. The interval A^2 covers q times the vertices u_i with $\frac{n+1}{2} \le i \le n+1$ and $q-1$ times the other ones. Therefore $s_1(u)+s_2(u) \ge 2q-1 = \frac{\ell-1}{2}$ for each u and (3) is satisfied.

Subcase 2.2 $n = 20q + 5$ with $q \ge 1$. Hence $k = 10q + 2$ and $\ell = \frac{n}{5} = 4q + 1$.

Each set A^1 and A^2 contains $\ell\frac{k-2}{2} = qn$ vertices and covers exactly q times each vertex u_i. Therefore $s_1(u) + s_2(u) = 2q = \frac{\ell-1}{2}$ for each u and (3) is satisfied.

Subcase 2.3 $n = 20q + 9$ with $q \ge 1$. Hence $k = 10q + 4$ and $\ell = \frac{n-4}{5} = 4q + 1$.

Each set A^1 and A^2 contains $\ell\frac{k-2}{2} = qn + 1$ vertices and covers at least q times each vertex u_i. Therefore $s_1(u) + s_2(u) \ge 2q = \frac{\ell-1}{2}$ for each u and (3) is satisfied.

Subcase 2.4 $n = 20q + 13$ with $q \ge 1$. Hence $k = 10q + 6$ and $\ell = \frac{n-8}{5} = 4q + 1$.

Each set A^1 and A^2 contains $\ell\frac{k-2}{2} = qn + 2$ vertices and covers at least q times each vertex u_i. Therefore $s_1(u) + s_2(u) \ge 2q = \frac{\ell-1}{2}$ for each u and (3) is satisfied.

Subcase 2.5 $n = 20q + 17$ with $q \ge 0$. Hence $k = 10q + 8$ and $\ell = \frac{n-2}{5} = 4q + 3$.

Each set A^1 and A^2 contains $\ell\frac{k-2}{2} = qn + \frac{n+1}{2}$ vertices. The interval A^1 covers $q+1$ times the vertices u_i with $0 \le i \le \frac{n-1}{2}$ and q times the other ones. The interval A^2 covers $q+1$ times the vertices u_i with $\frac{n+1}{2} \le i \le n$ and q times the other ones. Therefore $s_1(u) + s_2(u) \ge 2q + 1 = \frac{\ell-1}{2}$ for each u and (3) is satisfied.

□

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