

# $q$ -ANALOGUE OF THE ALZER'S INEQUALITY

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ABSTRACT. In this article, we are interested in giving a  $q$ -analogue of the Alzer's inequality.

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## 1. INTRODUCTION

In 1964 H. Mink and L. Sathre [15] proved the following inequality

$$(1.1) \quad \frac{n}{n+1} < \frac{(n!)^{\frac{1}{n}}}{((n+1)!)^{\frac{1}{n+1}}}, n \in \mathbf{N}.$$

The inequality (1.1) was generalized and refined by H. Alzer in [2]-[4]. He proved in [4] the following inequality:

$$(1.2) \quad \frac{n}{n+1} \leq \left[ \frac{(n+1) \sum_{i=1}^n i^r}{n \sum_{i=1}^{n+1} i^r} \right]^{\frac{1}{r}} < \frac{(n!)^{\frac{1}{n}}}{((n+1)!)^{\frac{1}{n+1}}}, n \in \mathbf{N}, r \in \mathbf{R}_+.$$

The lower and upper bounds are the best possible.

Many proofs of the inequality (1.2) and some generalizations were given in ([1],[5]-[7],[9],[10],[12]-[14],[16]-[23]).

The left hand side of the Alzer's inequality (1.2) was generalized by Feng Qi [8] as follows:

$$(1.3) \quad \frac{n+k}{n+m+k} \leq \left[ \frac{\frac{1}{n} \sum_{i=k+1}^{n+k} i^r}{\frac{1}{n+m} \sum_{i=k+1}^{n+m+k} i^r} \right]^{\frac{1}{r}}, n, m \in \mathbf{N},$$

where  $k$  is a nonnegative integer and  $r \in \mathbf{R}_+$ . The lower bound is best possible.

The main purpose of this paper is to give a  $q$ -analogue of inequalities (1.2) and (1.3).

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## 2. $q$ -ANALOGUE OF THE ALZER'S INEQUALITY

Throughout this paper, we consider a positive integer  $q \neq 1$  and for  $x \in \mathbb{C}$ , we write

$$(2.1) \quad [x]_q = \frac{1 - q^x}{1 - q}.$$

Note that  $[x]_q$  tends to  $x$  when  $q$  tends to 1 (we refer to [11] for more details about  $q$ -calculus). To prove the main result of the paper, we need the following lemma.

**Lemma 2.1.** *For all  $q > 1$ , for all nonnegative integers  $n$  and  $k$  and for all nonnegative real number  $r$ , we have*

$$(2.2) \quad \sum_{i=k+1}^{n+k} [i]_q^r > \frac{[n]_q [n+k]_q^r [n+k+1]_q^r}{[n+1]_q [n+k+1]_q^r - [n]_q [n+k]_q^r}.$$

*Proof.* Let  $k$  be a nonnegative integer and  $r$  be a nonnegative real number. We prove the result by induction on  $n$ .

For  $n = 1$ , we have

$$\begin{aligned} [k+1]_q^r - \frac{[k+1]_q^r [k+2]_q^r}{[2]_q [k+2]_q^r - [k+1]_q^r} &= [k+1]_q^r \frac{[2]_q [k+2]_q^r - [k+1]_q^r - [k+2]_q^r}{[2]_q [k+2]_q^r - [k+1]_q^r} \\ &= [k+1]_q^r \frac{q[k+2]_q^r - [k+1]_q^r}{(1+q)[k+2]_q^r - [k+1]_q^r} \\ &= [k+1]_q^r \frac{q(1 - q^{k+2})^r - (1 - q^{k+1})^r}{(1+q)(1 - q^{k+2})^r - (1 - q^{k+1})^r} \\ &= [k+1]_q^r \frac{q - \left(\frac{1 - q^{k+1}}{1 - q^{k+2}}\right)^r}{1 + q - \left(\frac{1 - q^{k+1}}{1 - q^{k+2}}\right)^r}. \end{aligned}$$

Using the fact that  $\frac{1 - q^{k+1}}{1 - q^{k+2}} < 1 < q$ , we get

$$[k+1]_q^r > \frac{[k+1]_q^r [k+2]_q^r}{[2]_q [k+2]_q^r - [k+1]_q^r},$$

which achieves the proof of the result for  $n = 1$ .

Suppose, now, that it is valid for  $n > 1$  and let's prove that it's valid for  $n + 1$ . Using the fact that  $\sum_{i=k+1}^{n+k+1} [i]_q^r = \sum_{i=k+1}^{n+k} [i]_q^r + [n+k+1]_q^r$ , calculating straightforwardly, and simplifying easily, the induction step can be written as

$$\frac{[n+2]_q [n+k+2]_q^r - [n+1]_q [n+k+1]_q^r}{[n+1]_q [n+k+1]_q^r - [n]_q [n+k]_q^r} > \left( \frac{[n+k+2]_q}{[n+k+1]_q} \right)^r.$$

Consider the functions  $f$  and  $g$  defined on  $[n, n+1]$  as follows

$$f(x) = [x+1]_q [x+k+1]_q^r \quad \text{and} \quad g(x) = [x]_q [x+k]_q^r.$$

Simple derivation gives

$$f'(x) = \frac{\ln q}{q-1} [q^{x+1} [x+k+1]_q^r + r q^{x+k+1} [x+1]_q [x+k+1]_q^{r-1}]$$

and

$$g'(x) = \frac{\ln q}{q-1} [q^x [x+k]_q^r + r q^{x+k} [x]_q [x+k]_q^{r-1}].$$

So, for all  $x \in [n, n+1]$ , we have

$$\begin{aligned} \frac{f'(x)}{g'(x)} &= \frac{q^{x+1}[x+k+1]_q^r + rq^{x+k+1}[x+1]_q[x+k+1]_q^{r-1}}{q^x[x+k]_q^r + rq^{x+k}[x]_q[x+k]_q^{r-1}} \\ &= \frac{q[x+k+1]_q^r \left(1 + rq^k \frac{[x+1]_q}{[x+k+1]_q}\right)}{[x+k]_q^r \left(1 + rq^k \frac{[x]_q}{[x+k]_q}\right)} \\ &> \left(\frac{[x+k+1]_q}{[x+k]_q}\right)^r. \end{aligned}$$

From Cauchy's mean-value theorem and the previous inequality, there exists one point  $\xi \in (n, n+1)$  such that

$$(2.3) \quad \frac{[n+2]_q[n+k+2]_q^r - [n+1]_q[n+k+1]_q^r}{[n+1]_q[n+k+1]_q^r - [n]_q[n+k]_q^r} = \frac{f'(\xi)}{g'(\xi)} > \left(\frac{[\xi+k+1]_q}{[\xi+k]_q}\right)^r.$$

But,

$$\begin{aligned} \frac{[\xi+k+1]_q}{[\xi+k]_q} - \frac{[n+k+2]_q}{[n+k+1]_q} &= \frac{(1-q^{\xi+k+1})(1-q^{n+k+1}) - (1-q^{n+k+2})(1-q^{\xi+k})}{(1-q^{\xi+k})(1-q^{n+k+1})} \\ &= \frac{q^{n+k+2} + q^{\xi+k} - q^{\xi+k+1} - q^{n+k+1}}{(1-q^{\xi+k})(1-q^{n+k+1})} \\ &= \frac{q^k(q-1)(q^{n+1} - q^\xi)}{(1-q^{\xi+k})(1-q^{n+k+1})} > 0. \end{aligned}$$

Then,

$$\left(\frac{[\xi+k+1]_q}{[\xi+k]_q}\right)^r > \left(\frac{[n+k+2]_q}{[n+k+1]_q}\right)^r.$$

This inequality together with (2.3) gives

$$(2.4) \quad \frac{[n+2]_q[n+k+2]_q^r - [n+1]_q[n+k+1]_q^r}{[n+1]_q[n+k+1]_q^r - [n]_q[n+k]_q^r} > \left(\frac{[n+k+2]_q}{[n+k+1]_q}\right)^r,$$

which proves that the result is valid for  $n+1$ . □

Now, we are in a situation to prove the main result of this paper.

**Theorem 2.2.**

$$(2.5) \quad \frac{[n+k]_q}{[n+m+k]_q} \leq \begin{cases} \frac{1}{q^{m(1+\frac{1}{r})}} \left(\frac{[n+m]_q \sum_{i=k+1}^{n+k} q^{-ir} [i]_q^r}{[n]_q \sum_{i=k+1}^{n+k+m} q^{-ir} [i]_q^r}\right)^{\frac{1}{r}}, & \text{if } q \in ]0, 1[, \\ \left(\frac{[n+m]_q \sum_{i=k+1}^{n+k} [i]_q^r}{[n]_q \sum_{i=k+1}^{n+k+m} [i]_q^r}\right)^{\frac{1}{r}}, & \text{if } q \in ]1, +\infty[, \end{cases}$$

where  $n, m \in \mathbf{N}$ ,  $k$  is a nonnegative integer and  $r \in \mathbf{R}_+$ . The lower bounds are best possible.

*Proof.* It is easy to verify that for all positive real  $q \neq 1$ , we have

$$[n]_q = q^{n-1} [n]_{\frac{1}{q}}$$

and so,

$$\frac{[n+k]_q}{[n+m+k]_q} = \frac{q^{n+k-1} [n+k]_{\frac{1}{q}}}{q^{n+m+k-1} [n+m+k]_{\frac{1}{q}}} = \frac{[n+k]_{\frac{1}{q}}}{q^m [n+m+k]_{\frac{1}{q}}}.$$

Then, to prove the result, it suffices to focus on the case  $q > 1$ .

Let  $q > 1$  and  $r$  be a nonnegative real number.  
From the previous lemma and the fact that

$$(2.6) \quad \sum_{i=k+1}^{n+k+1} [i]_q^r = \sum_{i=k+1}^{n+k} [i]_q^r + [n+k+1]_q^r$$

we obtain for all  $n \in \mathbf{N}$  and  $k$  nonnegative integer

$$(2.7) \quad \frac{1}{[n]_q [n+k]_q^r} \sum_{i=k+1}^{n+k} [i]_q^r > \frac{1}{[n+1]_q [n+k+1]_q^r} \sum_{i=k+1}^{n+k+1} [i]_q^r.$$

So, by induction on  $m$ , we get for all  $n \in \mathbf{N}$  and  $k, m$  nonnegative integers

$$\frac{1}{[n]_q [n+k]_q^r} \sum_{i=k+1}^{n+k} [i]_q^r > \frac{1}{[n+m]_q [n+m+k]_q^r} \sum_{i=k+1}^{n+m+k} [i]_q^r.$$

Then,

$$\left( \frac{[n+k]_q}{[n+m+k]_q} \right)^r < \frac{[n+m]_q \sum_{i=k+1}^{n+k} [i]_q^r}{[n]_q \sum_{i=k+1}^{n+m+k} [i]_q^r},$$

which achieves the proof. □

The limit case is given by  
 $\forall q \in ]1, +\infty[$ ,

$$(2.8) \quad \lim_{r \rightarrow +\infty} \left( \frac{[n+m]_q \sum_{i=k+1}^{n+k} [i]_q^r}{[n]_q \sum_{i=k+1}^{n+m+k} [i]_q^r} \right)^{\frac{1}{r}} = \frac{[n+k]_q}{[n+m+k]_q}.$$

$\forall q \in ]0, 1[$ ,

$$(2.9) \quad \lim_{r \rightarrow +\infty} \frac{1}{q^{m(1+\frac{1}{r})}} \left( \frac{[n+m]_q \sum_{i=k+1}^{n+k} q^{-ir} [i]_q^r}{[n]_q \sum_{i=k+1}^{n+m+k} q^{-ir} [i]_q^r} \right)^{\frac{1}{r}} = \frac{[n+k]_q}{[n+m+k]_q}.$$

Thus, the lower bound is best possible.

Indeed, using the fact that  $0 \leq \frac{[i]_q}{[j]_q} < 1$ ,  $\forall 1 \leq i < j$ ,  $\forall q \in ]1, +\infty[$ ,

$$\begin{aligned} \lim_{r \rightarrow +\infty} \left( \frac{[n+m]_q \sum_{i=k+1}^{n+k} [i]_q^r}{[n]_q \sum_{i=k+1}^{n+m+k} [i]_q^r} \right)^{\frac{1}{r}} &= \lim_{r \rightarrow +\infty} \left( \frac{[n+m]_q}{[n]_q} \right)^{\frac{1}{r}} \frac{[n+k]_q}{[n+m+k]_q} \left( \frac{1 + \sum_{i=k+1}^{n+k-1} \left( \frac{[i]_q}{[n+k]_q} \right)^r}{1 + \sum_{i=k+1}^{n+m+k-1} \left( \frac{[i]_q}{[n+m+k]_q} \right)^r} \right)^{\frac{1}{r}} \\ &= \frac{[n+k]_q}{[n+m+k]_q}. \end{aligned}$$

$\forall q \in ]0, 1[$ ,

$$\begin{aligned} \lim_{r \rightarrow +\infty} \frac{1}{q^{m(1+\frac{1}{r})}} \left( \frac{[n+m]_q \sum_{i=k+1}^{n+k} q^{-ir} [i]_q^r}{[n]_q \sum_{i=k+1}^{n+k+m} q^{-ir} [i]_q^r} \right)^{\frac{1}{r}} &= \lim_{r \rightarrow +\infty} \frac{1}{q^{m(1+\frac{1}{r})}} \left( \frac{[n+m]_q}{[n]_q} \right)^{\frac{1}{r}} \frac{[n+k]_{\frac{1}{q}}}{[n+m+k]_{\frac{1}{q}}} \\ &\times \left( \frac{1 + \sum_{i=k+1}^{n+k-1} \left( \frac{[i]_{\frac{1}{q}}}{[n+k]_{\frac{1}{q}}} \right)^r}{1 + \sum_{i=k+1}^{n+k+m-1} \left( \frac{[i]_{\frac{1}{q}}}{[n+m+k]_{\frac{1}{q}}} \right)^r} \right)^{\frac{1}{r}} \\ &= \frac{1}{q^m} \frac{[n+k]_{\frac{1}{q}}}{[n+m+k]_{\frac{1}{q}}} = \frac{[n+k]_q}{[n+m+k]_q}. \end{aligned}$$

For  $k = 0$  and  $m = 1$ , we find the following special case:

**Corollary 2.3.** *If  $r, q$  are positive real numbers and  $n$  is a positive integer, then*

$$(2.10) \quad \frac{[n]_q}{[n+1]_q} \leq \begin{cases} \frac{1}{q^{1+\frac{1}{r}}} \left( \frac{[n+1]_q \sum_{i=1}^n q^{-ir} [i]_q^r}{[n]_q \sum_{i=1}^{n+1} q^{-ir} [i]_q^r} \right)^{\frac{1}{r}}, & \text{if } q \in ]0, 1[, \\ \left( \frac{[n+1]_q \sum_{i=1}^n [i]_q^r}{[n]_q \sum_{i=1}^{n+1} [i]_q^r} \right)^{\frac{1}{r}}, & \text{if } q \in ]1, +\infty[. \end{cases}$$

*The lower bounds are best possible.*

**Remark 2.4.** When  $q$  tends to 1 ( $q \rightarrow 1^+$  or  $q \rightarrow 1^-$ ),  $[n]_q$  tends to  $n$  and the inequality (2.10) tends to the Alzer's one.

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