A DIFFERENTIAL EQUATIONS FOR FRENET CURVES IN EUCLIDEAN 3-SPACE AND ITS APPLICATIONS

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ABSTRACT. First we prove that the distance function of every Frenet curve in $\mathbb{E}^3$ satisfies a 4-th order differential equation. Next, we show that several well-known characterizations of spherical and rectifying curves are consequences of this differential equation. Then we derive a new characterization of helices via this 4-th order differential equation. Finally, we prove a simple new characterization of spherical curves in terms of centrode and co-centrode.

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1. Introduction

Let $\alpha : I \to \mathbb{E}^3$ be a unit speed curve in the Euclidean 3-space $\mathbb{E}^3$ with Frenet-Serret apparatus $\{\kappa, \tau, T, N, B\}$, where κ, τ, T, N and B denote the curvature, the torsion, the unit tangent T, the unit principal normal N and the unit binormal of α, respectively (cf. [7]). Then α is called a *Frenet curve* if $\kappa > 0$ and $\tau \neq 0$. The *distance function* d of α is given by $d(s) = \|\alpha(s)\|$. It is worth noting that the distance function d plays an important role in addition to the curvature and torsion for getting the characterizations of rectifying curves (see [3, 6]).

Some important types of Frenet curves are helices (characterized by $\tau/\kappa = c$ with a nonzero constant c), spherical curves (characterized by $(\rho'\sigma)' + \rho/\sigma = 0$ with $\rho = \kappa^{-1}$, $\sigma = \tau^{-1}$ as we did in Corollary 4.1) and rectifying curves (characterized by $\tau/\kappa = as + b$ with constants $a \neq 0$ and b). One interesting question on Frenet curves is to find different characterizations of spherical curves, helices as well as of rectifying curves. Several interesting characterizations of spherical curves are obtained in [1, 8, 9]; and of rectifying curves in [3, 4, 5, 6].

In this article, first we prove that the distance function of every Frenet curve satisfies a 4-th order differential equation. Then we show that several well-known characterizations of spherical and rectifying curves are consequences of this differential equation and derive a new characterization of helices via this differential equation. Finally, we prove a simple new characterization of spherical curves in terms of centrode and co-centrode.

2. Preliminaries

Let $\alpha : I \to \mathbb{E}^3$ be a unit speed Frenet curve. We have

$$T' = \kappa N, \quad N' = -\kappa T + \tau B, \quad B' = -\tau N. \tag{2.1}$$

It follows immediately from (2.1) that

$$\begin{cases} \langle \alpha(s), T \rangle' = 1 + \kappa \langle \alpha(s), N \rangle, \\ \langle \alpha(s), N \rangle' = -\kappa \langle \alpha(s), T \rangle + \tau \langle \alpha(s), B \rangle, \\ \langle \alpha(s), B \rangle' = -\tau \langle \alpha(s), N \rangle. \end{cases} \tag{2.2}$$

The curve α is called a *helix* if there exists a constant unit vector u such that $\langle T, u \rangle$ is a constant. The Lancret's theorem states that a unit speed curve α is a helix if and only if τ/κ is a nonzero constant (cf. [7]).

A unit speed Frenet curve α is called a *rectifying curve* if the position vector of α always lies in the rectifying plane (cf. [3, 4, 5]). It was proved in [3] that the distance function $d(s) = \|\alpha(s)\|$ of a rectifying curve α satisfies

$$d(s) = \sqrt{s^2 + c_1 s + c_2}, \tag{2.3}$$

for some constants c_1, c_2 and vise versa. Also, it is known that α is a rectifying curve if and only if the ratio $\tau : \kappa$ satisfies $\tau/\kappa = as + b$, for some constants a, b with $a \neq 0$ (see, e.g., [3, 5]). A unit speed curve is called *spherical* if it lies on a sphere. It is well-known that a unit speed Frenet curve is spherical if and only if we have $\rho^2 + (\rho'\sigma)^2 = constant$ with $\rho = \kappa^{-1}$ and $\sigma = \tau^{-1}$ (cf. [7]).

3. A differential equation for Frenet curves in $\mathbb{E}^3$

First, we derive a general differential equation satisfied by the distance function for every Frenet curve in $\mathbb{E}^3$.

Proposition 3.1. *Every unit speed Frenet curve $\alpha = \alpha(s)$ in $\mathbb{E}^3$ satisfies*

$$\rho\sigma h''' + (\rho\sigma' + 2\rho'\sigma)h'' + \left\{(\sigma\rho')' + \frac{\rho}{\sigma} + \frac{\sigma}{\rho}\right\} h' + \left(\frac{\sigma}{\rho}\right)' h = (\sigma\rho')' + \frac{\rho}{\sigma}, \tag{3.1}$$

where $\rho = \kappa^{-1}$, $\sigma = \tau^{-1}$ and $h(s) = d(s)d'(s)$.

Proof. By differentiating $d(s) = \|\alpha(s)\|$ and using (2.2), we find

$$h = \langle \alpha, T \rangle, \tag{3.2}$$

which, again on differentiating and using first equation in (2.2) gives

$$\rho\,(h' - 1) = \langle \alpha, N \rangle. \tag{3.3}$$

By differentiating (3.3) we find

$$\rho h'' + \rho'(h' - 1) = -\kappa \langle \alpha, T \rangle + \tau \langle \alpha, B \rangle. \tag{3.4}$$

It follows from (3.2) and (3.4) that

$$\rho\sigma h'' + \sigma\rho' h' - \sigma\rho' + \frac{\sigma}{\rho} h = \langle \alpha(s), B \rangle. \tag{3.5}$$

After differentiating (3.5) and applying (3.3) we obtain

$$\rho\sigma h''' + (\rho\sigma' + 2\rho'\sigma)h''')'h' - (\sigma\rho')' + \left(\frac{\sigma}{\rho}\right)' h + \frac{\sigma h'}{\rho} = \frac{\rho}{\sigma}(1 - h'),$$

which implies the required differential equation (3.1). □

4. Some applications of Proposition 3.1

In this section, we show that Proposition 3.1 implies easily several well-known characterizations of spherical and rectifying curves.

Corollary 4.1. [7] *A unit speed Frenet curve α in $\mathbb{E}^3$ is a spherical if and only if it satisfies*

$$(\rho'\sigma)' + \frac{\rho}{\sigma} = 0.$$

Proof. Let α be a spherical curve lying on a sphere with radius r. Without loss of generality we may assume that the center of the sphere is at the origin.Then we have the distance function $d(s) = r$. Thus we have $h = dd' = 0$. Hence, the differential equation in Proposition 3.1 reduces to

$$(\sigma\rho')' + \frac{\rho}{\sigma} = 0.$$

Thus α is a spherical curve. The converse is trivial. □

Corollary 4.2. [3] *A unit speed Frenet curve $\alpha(s)$ is a rectifying curve if and only if it satisfies*

$$(s+c)\left(\frac{\kappa}{\tau}\right)' + \frac{\kappa}{\tau} = 0, \tag{4.1}$$

for some constant c.

Proof. Assume that α is a rectifying curve. Then the distance function is given by $d(s) = \sqrt{s^2 + c_1 s + c_2}$, where c_1 and c_2 are constants [3]. Thus we have $h = s + c$, where $c = c_1/2$. Hence (3.1) reduces to

$$(\sigma\rho')' + \frac{\sigma}{\rho} + (s+c)\left(\frac{\sigma}{\rho}\right)' = (\sigma\rho')',$$

which implies condition (4.1).

Conversely, if (4.1) holds, then by integrating (4.1), we find $(s+c)\kappa = \bar{c}\tau$ for a constant $\bar{c}$, which implies that α is a rectifying curve (cf.[3]). □

Another easy consequence of Proposition 3.1 is the following.

Corollary 4.3. [2] *Let $\alpha : I \to \mathbb{E}^3$ be the unit speed Frenet curve. Then*

$$\langle \alpha, N \rangle^2 + \langle \alpha, B \rangle^2 = c^2 \tag{4.2}$$

holds for a constant c if and only if either α lies in a sphere centered at the origin or α is a rectifying curve.

Proof. Assume α is a unit speed Frenet curve that satisfies condition (4.2). Then it follows from (4.2) that the distance function of α satisfies

$$d = \sqrt{\langle \alpha, T \rangle^2 + c^2}. \tag{4.3}$$

After differentiating (4.3) and using (3.2) and (3.3), we find $h = dd' = hh'$. Thus either $h = 0$ or $h' = 1$. So we have $h = 0$ or $h = s + b$ for a constant b. Consequently, either α lies in a sphere centered at the origin or it is a rectifying curve.

The converse is trivial. □

5. Another application of Proposition 3.1

Proposition 3.1 gives the following new characterization of helices as well.

Theorem 5.1. *A unit speed Frenet curve α in $\mathbb{E}^3$ is a helix if and only if, with respect to a suitable arc-length parameter s, the function $h(s) = d(s)d'(s)$ satisfies*

$$(\rho' h)' + \left(\frac{\rho}{\sigma} + \frac{\sigma}{\rho}\right)\tau h = \rho' + \frac{s\rho}{\sigma^2}, \tag{5.1}$$

where $d = ||\alpha||$, $\rho = \kappa^{-1}$ and $\sigma = \tau^{-1}$.

Proof. Assume that α is a helix with its axis parallel to a unit vector u. Then we have $\langle T, u\rangle = c$ for some constant c and so that $\langle N, u\rangle = 0$. Thus we get

$$u = cT + \sqrt{1-c^2}B, \tag{5.2}$$

$$ck = \sqrt{1-c^2}\tau. \tag{5.3}$$

Since $\langle \alpha, u\rangle' = c$ holds, we have

$$\langle \alpha, u\rangle = cs + \bar{c} \tag{5.4}$$

for some constant $\bar{c}$. Now, using (5.2) and (5.4), we find

$$\langle \alpha, T\rangle = \frac{1}{c}(cs + \bar{c}) - \frac{\sqrt{1-c^2}}{c}\langle \alpha, B\rangle.$$

Combining this with equation (3.2) yields

$$h = s + b - \frac{\sqrt{1-c^2}}{c}\langle \alpha, B\rangle, \quad b = \frac{\bar{c}}{c}. \tag{5.5}$$

By differentiating (5.5) and using equation (5.3), we get

$$\rho(h' - 1) = \langle \alpha, N\rangle,$$

which again by differentiation leads to

$$\begin{aligned} \rho h'' + \rho'(h' - 1) &= \tau\langle \alpha, B\rangle - \kappa h \\ &= \frac{c(s+b)\tau}{\sqrt{1-c^2}} - \left(\frac{\sqrt{1-c^2}}{c} + \frac{c}{\sqrt{1-c^2}}\right)\tau h, \end{aligned} \tag{5.6}$$

where we have applied (5.3) and (5.5). Notice that (5.3) gives

$$\frac{\sigma}{\rho} = \frac{\sqrt{1-c^2}}{c}, \quad \frac{\rho}{\sigma} = \frac{c}{\sqrt{1-c^2}},$$

Substituting these into equation (5.6) leads to (5.1) up a suitable translation in s.

Conversely, assume that α is a unit speed Frenet curve that satisfies (5.1). Then, by differentiating (5.1) we derive

$$\begin{aligned} &\rho\sigma h''' + \{\rho\sigma' + 2\sigma\rho'\}h'' + \left\{(\sigma\rho')' + \frac{\rho}{\sigma} + \frac{\sigma}{\rho}\right\}h' + \left(\frac{\sigma}{\rho} + \frac{\rho}{\sigma}\right)' h - (\sigma\rho')' \\ &= \frac{\rho}{\sigma} + (s+b)\left(\frac{\rho}{\sigma}\right)'. \end{aligned} \tag{5.7}$$

Comparing (5.7) with (3.1) in Proposition 3.1 gives

$$\left(\frac{\rho}{\sigma}\right)'(h - s - b) = 0. \tag{5.8}$$

If $h = s + b$ holds, then (5.1) reduces to

$$\left(\frac{\rho}{\sigma} + \frac{\sigma}{\rho}\right)(s+b) - \left(\frac{\rho}{\sigma}\right)(s+b) = 0,$$

which implies $\sigma = 0$ which is impossible since α is assumed to be a Frenet curve. Hence we get $(\rho/\sigma)' = 0$ from (5.8), which implies that α is a helix. ☐

6. A characterization of spherical curves in terms of centrode and co-centrode

Consider a unit speed Frenet curve $\alpha : I \to \mathbb{E}^3$. Let $\delta(s)$ and δ^* denote the centrode and co-centrode of α defined respectively by

$$\delta = \tau T + \kappa B, \quad \delta^* = -\kappa T + \tau B. \tag{6.1}$$

Obviously, $\{N, \delta_1, \delta^*\}$ forms an orthogonal moving frame along α.

It follows easily from (6.1) that the Frenet curve α *is a helix if and only if δ' is always orthogonal to the co-centrode δ^* at each $s \in I$.* This fact can be easily seen as follows: By differentiating (6.1) and by using (2.1), we have $\delta' = \tau' T + \kappa' B$. Hence δ' is orthogonal to co-centrode δ^* if and only if $\kappa'\tau = \kappa\tau'$, which implies that α is a helix if and only if δ' and δ^* are orthogonal at each point.

In this final section, we prove the following simple characterization of spherical curves in terms of the centrode δ and the co-centrode δ^* of Frenet curves.

Theorem 6.1. *A unit speed Frenet curve α in $\mathbb{E}^3$ lies in a sphere centered at the origin if and only if*

$$\frac{\tau}{\kappa} = \frac{\langle \alpha, \delta^* \rangle}{\langle \alpha, \delta \rangle} \tag{6.2}$$

holds identically.

Proof. Since $\{N, \delta_1, \delta^*\}$ is an orthogonal frame along α, it follows from (6.1) that the distance function d satisfies

$$d^2 = \frac{(\kappa^2 + \tau^2)\langle \alpha, N \rangle^2 + \langle \alpha, \delta \rangle^2 + \langle \alpha, \delta^* \rangle^2}{\kappa^2 + \tau^2}, \tag{6.3}$$

Thus, after differentiating (6.3) and using (2.1) and (6.1), we find

$$\begin{aligned} dd' &= \langle \alpha, N \rangle \langle \alpha, \delta^* \rangle - \frac{(\kappa\kappa' + \tau\tau')}{(\kappa^2 + \tau^2)^2}\{\langle \alpha, \delta \rangle^2 + \langle \alpha, \delta^* \rangle^2\} \\ &+ \frac{\langle \alpha, \delta \rangle \langle \alpha, \delta \rangle' + \langle \alpha, \delta^* \rangle \langle \alpha, \delta^* \rangle'}{\kappa^2 + \tau^2} \end{aligned} \tag{6.4}$$

On the other hand, it follows from (2.1) and (6.1) that

$$\begin{aligned} \langle \alpha, \delta \rangle' &= \tau + \langle \alpha, \tau' T + \kappa' B \rangle, \\ \langle \alpha, \delta^* \rangle' &= -\kappa - \langle \alpha, \kappa' T - \tau' B \rangle - (\kappa^2 + \tau^2)\langle \alpha, N \rangle \end{aligned} \tag{6.5}$$

After substituting (6.1) and (6.5) into (6.4), we obtain

$$\begin{aligned} dd' = &\frac{\langle \alpha, \delta \rangle \{\langle \alpha, \tau' T + \kappa' B \rangle\}}{\kappa^2 + \tau^2} - \frac{(\kappa\kappa' + \tau\tau')}{(\kappa^2 + \tau^2)^2}\{\langle \alpha, \delta \rangle^2 + \langle \alpha, \delta^* \rangle^2\} \\ &+ \frac{\langle \alpha, \delta^* \rangle \{\langle \alpha, -\kappa' T + \tau' B \rangle\}}{\kappa^2 + \tau^2} + \frac{\tau \langle \alpha, \delta \rangle - \kappa \langle \alpha, \delta^* \rangle}{\kappa^2 + \tau^2}. \end{aligned} \tag{6.6}$$

Now, by combining (6.6) with (6.1) and by a long direct computation, we arrive at

$$dd' = \frac{\tau \langle \alpha, \delta \rangle - \kappa \langle \alpha, \delta^* \rangle}{\kappa^2 + \tau^2},$$

which implies that α is spherical if and only if (6.2) holds identically. □

References

[1] S. Breuer and D. Gottliebb, *Explicit characterizations of spherical curves.* Proc. Amer. Math. Soc. 27(1971), 126-127.

[2] B.-Y. Chen, *Geometry of position functions of Riemannian submanifolds in pseudo-Euclidean space.* J. Geom. **74** (2002), 61–77.

[3] B.-Y. Chen, *When doe the position vector of a space curve always lie in its rectifying plane?.* Amer. Math. Monthly **110** (2003), 147–152.

[4] B.-Y. Chen, *Rectifying curves and geodesics on a cone in the Euclidean 3-space.* Tamkang J. Math. **48** (2017), no. 2, 209–214.

[5] B.-Y. Chen and F. Dillen, *Rectifying curves as centrodes and extremal curves.* Bull. Inst. Math. Acad. Sinica **33** (2005), no. 2, 77–90.

[6] S. Deshmukh, B.-Y. Chen and S. H. Alshammari, *On rectifying curves in Euclidean 3-space.* Turk. J. Math. **42** (2018), no. 2, DOI: 10.3906/mat-1701-52,

[7] R. S. Millman and G. D. Parker, *Elements of Differential Geometry.* Prentice-Hall, Englewood Cliffs, NJ, 1977.

[8] Y. Wong, *A global formulation of the condition for a curve to lie in a sphere.* Monatsch. Math. **67** (1963), 363–365.

[9] Y. Wong, *On an explicit characterization of spherical curves.* Proc. Amer. Math. Soc. **34** (1972), 239–242.

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