

COFINITENESS AND ARTINIANNES OF GENERALIZED LOCAL COHOMOLOGY MODULES

FATEMEH DEGHANI-ZADEH

ABSTRACT. Let R be a commutative Noetherian ring, \mathfrak{a} and \mathfrak{b} ideals of R and let M and N be two finitely generated R -modules. In this paper, we study the cofiniteness of $H_{\mathfrak{b}}^j(H_{\mathfrak{a}}^i(M, N))$ in several cases.

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1. INTRODUCTION

Throughout this paper, R will denote a commutative Noetherian (not necessarily local) ring, and M , N are two finitely generated R -modules. Also, \mathfrak{a} and \mathfrak{b} will denote two proper ideals of R .

Let $H_{\mathfrak{a}}^i(M, N) = \varinjlim_{n \in \mathbb{N}} \text{Ext}_R^i(M/\mathfrak{a}^n M, N)$ be the i -th generalized local cohomology module relative to the ideal \mathfrak{a} and R -modules M and N (see [8]). For $M = R$, let us denote $H_{\mathfrak{a}}^i(R, N)$ by $H_{\mathfrak{a}}^i(N)$, the i -th ordinary local cohomology module with respect to \mathfrak{a} . In [6] Grothendieck conjectured that for any ideal \mathfrak{a} and for any finite generated R -module N , the R -module $\text{Hom}(R/\mathfrak{a}, H_{\mathfrak{a}}^i(N))$ is finite generated. In an *Inventiones Mathematicae* paper (see [7]) Hartshorn gives a counterexample to this conjecture and makes some additional assumptions to the original proposal of Grothendieck, introducing for instance the notion of \mathfrak{a} -cofiniteness for a module. He defined an R -module T to be \mathfrak{a} -cofinite if $\text{Ext}_R^i(R/\mathfrak{a}, T)$ is finitely generated for all $i \geq 0$ and $\text{Supp} T \subseteq V(\mathfrak{a})$, where $V(\mathfrak{a})$ denotes the set of prime ideals of R containing \mathfrak{a} , and asked the following question:

Let N be a finitely generated R -module and let \mathfrak{a} be an ideal of R . Then, is $H_{\mathfrak{a}}^i(N)$ \mathfrak{a} -cofinite?

This question has been studied by several authors; see for example, Yoshida [15], Zamani [14], Cuong, Goto and Hong [12], Dehghani-Zadeh [3], Bahmanpour and Naghipour [2].

In this note the following question is of interest: Are the modules $H_{\mathfrak{b}}^j(H_{\mathfrak{a}}^i(M, N))$, \mathfrak{b} -cofinite? The main purpose of this paper is to provide an affirmative answer to this question. In this direction as the result of this paper we prove $H_{\mathfrak{b}}^j(H_{\mathfrak{a}}^i(M, N))$ is \mathfrak{b} -cofinite, in the following cases:

- (i) $\dim R/\mathfrak{a} \leq 1$ and $\dim R/\mathfrak{b} \leq 1$.
- (ii) $\dim R/\mathfrak{a} = 2$ and $\dim R/\mathfrak{b} = 1$ and $i \leq f_{\mathfrak{a}}(M, N)$, where $f_{\mathfrak{a}}(M, N)$ is the least non-negative integer i such that $H_{\mathfrak{a}}^i(M, N)$ is not finitely generated.

In addition, we assume that R is a local ring with its maximal ideal \mathfrak{m} and we study in what conditions on " i " the module $H_{\mathfrak{b}}^j(H_{\mathfrak{a}}^i(M, N))$ is \mathfrak{b} -cofinite, does not matter the number $\dim R/\mathfrak{b}$ and $\dim R/\mathfrak{a}$ are.

2. COFINITENESS OF $H_{\mathfrak{b}}^j(H_{\mathfrak{a}}^i(M, N))$ FOR IDEALS OF SMALL DIMENSION.

The concept of a cofinite module plays an important role in this paper. We say that T is a cofinite module if there is a proper ideal I of R such that T is I -cofinite. In this section, we study the cofiniteness of the modules $H_{\mathfrak{b}}^j(H_{\mathfrak{a}}^i(M, N))(i, j \in \mathbb{N}_0)$, where $\mathfrak{a}, \mathfrak{b}$ are ideals in an arbitrary Noetherian (not necessarily local) ring R with $\mathfrak{a} \subseteq \mathfrak{b}$ and M, N finitely generated modules over R .

For any unexplained notation and terminology, we refer the reader to [1] and [13].

The following remark, which is needed in the proof of the next theorems, describes some of properties of cofinite modules.

- Remark 2.1.**
- (i) Assume that T is an \mathfrak{a} -cofinite R -module. Then T is \mathfrak{a} -torsion-free if and only if \mathfrak{a} contains an element x which is T -regular. (see a proof in [1, Lemma 2.1.1] for instance).
 - (ii) The class of Artinian \mathfrak{a} -cofinite modules is closed under taking submodules, quotients and extensions. (see [10, Corollary 4.4]).
 - (iii) Let T and T' be two \mathfrak{a} -cofinite modules. If $f : T \rightarrow T'$ is a homomorphism between these two \mathfrak{a} -cofinite modules and one of the three modules $\text{Ker}f$, $\text{Im}f$ and $\text{Coker}f$ is \mathfrak{a} -cofinite, then all three of them are \mathfrak{a} -cofinite.
 - (iv) If R is a local ring with its maximal ideal \mathfrak{m} , then an R -module is \mathfrak{m} -cofinite if and only if it is an Artinian R -module (see [9]).
 - (v) For each R -module T , set $\Gamma_{\mathfrak{b}}(T) = \bigcup_{n \in \mathbb{N}} (0 :_T \mathfrak{b}^n)$, the set of elements of T which are annihilated by some power of \mathfrak{b} .

Theorem 2.2. *Let $\mathfrak{a} \subseteq \mathfrak{b}$ be two ideals of R such that $\dim R/\mathfrak{b} = 0$. Let T be an \mathfrak{a} -cofinite R -module and M be a finitely generated R -module. Then $H_{\mathfrak{b}}^i(M, T)$ is an Artinian, \mathfrak{a} and \mathfrak{b} -cofinite R -module.*

Proof. Firstly, we provide some facts which are needed in the course of the proof. As T is \mathfrak{a} -cofinite, the R -module $\text{Hom}(R/\mathfrak{a}, T)$ is finitely generated. Hence $\text{Hom}(R/\mathfrak{b}, T)$ and $\text{Hom}(R/\mathfrak{b}, \Gamma_{\mathfrak{b}}(T))$ are finitely generated R -modules. Since $\dim R/\mathfrak{b} = 0$, it follows that $\text{Hom}(R/\mathfrak{b}, \Gamma_{\mathfrak{b}}(T))$ is of finite length. Therefore, by [10, Proposition 4.1], we deduce that $\Gamma_{\mathfrak{b}}(T)$ is an \mathfrak{b} -cofinite and Artinian R -module. In addition, finiteness of $\text{Hom}(R/\mathfrak{a}, T)$ shows that, $\text{Hom}(R/\mathfrak{a}, \Gamma_{\mathfrak{b}}(T))$ is finitely generated. According to Melkersson [10, Proposition 4.1], $\Gamma_{\mathfrak{b}}(T)$ is an Artinian and \mathfrak{a} -cofinite R -module. Now we use mathematical induction on " i ". If $i = 0$, then $H_{\mathfrak{b}}^0(M, N) \cong \text{Hom}(M, \Gamma_{\mathfrak{b}}(T))$, and the assertion is trivial, by Remark (2.1, ii). Let $i > 0$ and we assume that the result is true for $i - 1$. Let us consider the exact sequence

$$H_{\mathfrak{b}}^i(M, \Gamma_{\mathfrak{b}}(T)) \longrightarrow H_{\mathfrak{b}}^i(M, T) \longrightarrow H_{\mathfrak{b}}^i(M, T/\Gamma_{\mathfrak{b}}(T)),$$

in conjunction with the fact that $H_{\mathfrak{b}}^i(M, \Gamma_{\mathfrak{b}}(T)) \cong \text{Ext}_R^i(M, \Gamma_{\mathfrak{b}}(T))$, to see that $H_{\mathfrak{b}}^i(M, T)$ is Artinian and \mathfrak{b} -cofinite if and only if $H_{\mathfrak{b}}^i(M, T/\Gamma_{\mathfrak{b}}(T))$ is Artinian and \mathfrak{b} -cofinite. We assume that $\Gamma_{\mathfrak{b}}(T) = 0$. Then, in view of Remark (2.1, i), the ideal \mathfrak{b} contains an element x which is T -regular. Now, let us look at the exact sequence $0 \rightarrow T \xrightarrow{x} T \rightarrow T/xT \rightarrow 0$ which gives rise to the exact sequence

$$H_{\mathfrak{b}}^{i-1}(M, T/xT) \longrightarrow H_{\mathfrak{b}}^i(M, T) \xrightarrow{x} H_{\mathfrak{b}}^i(M, T).$$

Now, the above exact sequence is used in conjunction with the inductive hypothesis and Remark (2.1, ii) to see that $(0 :_{H_{\mathfrak{b}}^i(M, T)} x)$ is Artinian and \mathfrak{b} -cofinite. Hence, by [10, Proposition 4.1], $H_{\mathfrak{b}}^i(M, T)$ is Artinian and \mathfrak{b} -cofinite. In the same way we can prove that $H_{\mathfrak{b}}^i(M, T)$ is also \mathfrak{a} -cofinite. \square

Corollary 2.3. *Let $\mathfrak{a} \subseteq \mathfrak{b}$ be two ideals of R such that $\dim R/\mathfrak{b} = 0$ and $\dim R/\mathfrak{a} = 1$. Then for each $j, i \geq 0$, $H_{\mathfrak{b}}^j(H_{\mathfrak{a}}^i(M, N))$ is an Artinian and \mathfrak{b} and \mathfrak{a} -cofinite R -module.*

Proof. It follows from Theorem 2.2 and [3, Theorem 3.3]. \square

Theorem 2.4. *Let $\mathfrak{a} \subseteq \mathfrak{b}$ be two ideals of R such that $\dim R/\mathfrak{b} = \dim R/\mathfrak{a} = 1$. If T is \mathfrak{a} -cofinite and i a positive integer, then $\text{Ext}_R^{i-1}(R/\mathfrak{b}, H_{\mathfrak{b}}^1(T))$ is a finitely generated R -module if and only if $\text{Ext}_R^{i+1}(R/\mathfrak{b}, \Gamma_{\mathfrak{b}}(T))$ is a finitely generated R -module.*

Proof. By [11, Theorem 11.38], there exists a Grothendieck's spectral sequence

$$E_2^{p,q} = \text{Ext}_R^p(R/\mathfrak{b}, H_{\mathfrak{b}}^q(T)) \xrightarrow{p} \text{Ext}_R^{p+q}(R/\mathfrak{b}, T). \quad (*)$$

Since $\text{Supp} T \subseteq V(\mathfrak{a})$ and $\dim R/\mathfrak{a} = 1$, it follows that $\dim(T) \leq 1$. This implies that R -module $H_{\mathfrak{b}}^q(T) = 0$ for $q > 1$ (see [1, Theorem 6.1.2]). Hence $E_2^{p,q} = 0$ unless $q = 0, 1$. Therefore, using the spectral sequence (*) with [13, Exercise 5.2.2], the long exact sequence is resulted, which is following:

$$\begin{aligned} \text{Ext}_R^{i+1}(R/\mathfrak{b}, T) \xrightarrow{\varphi} \text{Ext}_R^{i+1}(R/\mathfrak{b}, H_{\mathfrak{b}}^0(T)) \xrightarrow{d} \text{Ext}_R^{i-1}(R/\mathfrak{b}, H_{\mathfrak{b}}^1(T)) \\ \xrightarrow{\psi} \text{Ext}_R^i(R/\mathfrak{b}, T) \longrightarrow \text{Ext}_R^i(R/\mathfrak{b}, H_{\mathfrak{b}}^0(T)). \end{aligned}$$

In view of hypothesis and [4, Corollary 1], $\text{Ext}_R^i(R/\mathfrak{b}, T)$ is finitely generated for all i . Hence $\text{Im} \varphi$ and $\text{Im} \psi$ are finitely generated. This proves the claim. \square

Theorem 2.5. *Let $\mathfrak{a} \subseteq \mathfrak{b}$ be two ideals of R such that $\dim R/\mathfrak{b} = \dim R/\mathfrak{a} = 1$. Then $H_{\mathfrak{b}}^j(H_{\mathfrak{a}}^i(M, N))$ is \mathfrak{b} -cofinite for all i and j .*

Proof. Consider the Grothendieck spectral sequence

$$E_2^{p,q} = H_{\mathfrak{b}}^p(H_{\mathfrak{a}}^q(M, N)) \xrightarrow{p} H_{\mathfrak{b}}^{p+q}(M, N). \quad (**)$$

Since $\text{Supp} H_{\mathfrak{a}}^q(M, N) \subseteq V(\mathfrak{a})$ and $\dim R/\mathfrak{a} = 1$, it follows that $E_2^{p,q} = 0$ unless $p = 0, 1$. Referring [13, Exercise 5.2.1], the spectral sequence (**) results to the following short exact sequence:

$$0 \longrightarrow H_{\mathfrak{b}}^1(H_{\mathfrak{a}}^{i-1}(M, N)) \longrightarrow H_{\mathfrak{b}}^i(M, N) \longrightarrow H_{\mathfrak{b}}^0(H_{\mathfrak{a}}^i(M, N)) \longrightarrow 0.$$

Thus, there is a long exact sequence

$$\begin{aligned} \cdots \longrightarrow \text{Ext}_R^n(R/\mathfrak{b}, H_{\mathfrak{b}}^i(M, N)) \longrightarrow \text{Ext}_R^n(R/\mathfrak{b}, H_{\mathfrak{b}}^0(H_{\mathfrak{a}}^i(M, N))) \longrightarrow \\ \longrightarrow \text{Ext}_R^{n+1}(R/\mathfrak{b}, H_{\mathfrak{b}}^1(H_{\mathfrak{a}}^{i-1}(M, N))) \longrightarrow \text{Ext}_R^{n+1}(R/\mathfrak{b}, H_{\mathfrak{b}}^i(M, N)) \longrightarrow \cdots (\ddagger) \end{aligned}$$

In view of [3, Theorem 3.3], $H_{\mathfrak{b}}^i(M, N)$ is \mathfrak{b} -cofinite and $H_{\mathfrak{a}}^i(M, N)$ is \mathfrak{a} -cofinite for all i . Therefore, using the exact sequence (\ddagger) and Theorem 2.4 the result follows. \square

Lemma 2.6. *Let $H_{\mathfrak{a}}^i(N)$ be Artinian for all $i < t$. Then $H_{\mathfrak{a}}^i(M, N)$ is Artinian and \mathfrak{a} -cofinite for all $i < t$.*

Proof. Since $H_{\mathfrak{a}}^i(N)$ is Artinian for all $i < t$, it follows that $\text{Supp} H_{\mathfrak{a}}^i(N)$ is a finite set. Hence, by [2, Theorem 2.6], the R -module $H_{\mathfrak{a}}^i(N)$ is also \mathfrak{a} -cofinite. The assertion follows from [4, Theorem 2.1] and Remark (2.1,ii). \square

The following Corollary is an immediate consequence of Lemma 2.6.

Corollary 2.7. *If $\dim R/\mathfrak{a} = 0$, then $H_{\mathfrak{a}}^i(M, N)$ is Artinian and \mathfrak{a} -cofinite for all i .*

Theorem 2.8. *Let $\mathfrak{a}, \mathfrak{b}$ be two ideals of R such that $\dim R/\mathfrak{b} = \dim R/\mathfrak{a} = 0$. Then $H_{\mathfrak{b}}^j(H_{\mathfrak{a}}^i(M, N))$ is \mathfrak{b} -cofinite for all i, j .*

Proof. Since, for each $i, j \geq 0$, $\text{Supp} H_{\mathfrak{b}}^j(H_{\mathfrak{a}}^i(M, N)) \subseteq V(\mathfrak{b})$, it is enough to show that

$$\text{Ext}_R^t(R/\mathfrak{b}, H_{\mathfrak{b}}^j(H_{\mathfrak{a}}^i(M, N)))$$

is finitely generated for all $t \geq 0$. By using the previous corollary $H_{\mathfrak{a}}^i(M, N)$ is \mathfrak{a} -cofinite and Artinian, and so $H_{\mathfrak{b}}^0(H_{\mathfrak{a}}^i(M, N))$ is \mathfrak{a} -cofinite and Artinian. Since $\mathfrak{a} \subseteq \mathfrak{b}$, it follows from [5, Corollary1] that $\text{Ext}_R^t(R/\mathfrak{b}, H_{\mathfrak{b}}^0(H_{\mathfrak{a}}^i(M, N)))$ is finitely generated, for all $t \geq 0$. As $\text{Supp} H_{\mathfrak{a}}^i(M, N) \subseteq V(\mathfrak{a})$ and $\dim R/\mathfrak{a} = 0$, it follows that $H_{\mathfrak{b}}^j(H_{\mathfrak{a}}^i(M, N)) = 0$ for all $j > 0$. This completes the proof. \square

Definition 2.9. Let \mathfrak{a} be a proper ideal of R . The number

$$f_{\mathfrak{a}}(M, N) = \inf \{i \in \mathbb{N}_0 \mid H_{\mathfrak{a}}^i(M, N) \text{ is not finitely generated}\},$$

is called the finiteness dimension of M and N relative to the ideal \mathfrak{a} .

The arithmetic rank of an ideal \mathfrak{a} , denoted by $\text{ara}(\mathfrak{a})$, is the least number of generates of all ideals \mathfrak{c} which have the same radical as \mathfrak{a} .

Theorem 2.10. *Let $\mathfrak{a} \subseteq \mathfrak{b}$ be two proper ideals of R such that $\dim R/\mathfrak{a} = 2$ and $\dim R/\mathfrak{b} = 1$. Let $f_{\mathfrak{a}}(M, N) = f$. Then $H_{\mathfrak{b}}^0(H_{\mathfrak{a}}^f(M, N))$ is \mathfrak{b} -cofinite and for all $i < f$ and any j , $H_{\mathfrak{b}}^j(H_{\mathfrak{a}}^i(M, N))$ is a \mathfrak{b} -cofinite R -module too.*

Proof. As $\dim R/\mathfrak{a} = 2$ and $\dim R/\mathfrak{b} = 1$, there is $x \in \mathfrak{b}$ such that $\dim R/(\mathfrak{a} + Rx) = 1$. This is by [11, Theorem 11.38], the Grothendieck spectral sequence $E_2^{p,q} = H_{xR}^p(H_{\mathfrak{a}}^q(M, N))$ converges to $H^{p+q} = H_{Rx+\mathfrak{a}}^{p+q}(M, N)$. As $\text{ara}(Rx) = 1$, it is easy to see that $E_2^{p,q} = 0$ unless $p = 0, 1$; it follows that the sequence $0 \rightarrow E_2^{1,f-1} \rightarrow H^f \rightarrow E_2^{0,f} \rightarrow 0$ is exact, which, in turn, yields the exact sequence

$$H_{xR}^1(H_{\mathfrak{a}}^{f-1}(M, N)) \rightarrow H_{Rx+\mathfrak{a}}^f(M, N) \rightarrow H_{Rx}^0(H_{\mathfrak{a}}^f(M, N)) \rightarrow 0. \quad (\S)$$

In view of Definition 2.9, the R -module $H_{\mathfrak{a}}^{f-1}(M, N)$ is finitely generated. Therefore, by [10, Proposition 5.1] the R -module $H_{Rx}^1(H_{\mathfrak{a}}^{f-1}(M, N))$ is Rx -cofinite and Artinian. So, $\text{Ext}_R^t(R/Rx, H_{Rx}^1(H_{\mathfrak{a}}^{f-1}(M, N)))$ is a finitely generated R -module for all t . In view of [5, Corollary 1], $\text{Ext}_R^t(R/(Rx+\mathfrak{a}), H_{Rx}^1(H_{\mathfrak{a}}^{f-1}(M, N)))$ is a finitely generated R -module. Also, as $\text{Supp}H_{Rx}^1(H_{\mathfrak{a}}^{f-1}(M, N)) \subseteq V(Rx+\mathfrak{a})$ we get that $H_{Rx}^1(H_{\mathfrak{a}}^{f-1}(M, N))$ is Artinian and $(Rx+\mathfrak{a})$ -cofinite. Now, since $\dim R/(Rx+\mathfrak{a}) = 1$, $H_{Rx+\mathfrak{a}}^f(M, N)$ is $(Rx+\mathfrak{a})$ -cofinite. It follows from the exact sequence (\S) and Remark (2.1,iii) that the R -module $H_{Rx}^0(H_{\mathfrak{a}}^f(M, N))$ is $(Rx+\mathfrak{a})$ -cofinite. Therefore, the result follows from $H_{\mathfrak{b}}^0(H_{Rx}^0(H_{\mathfrak{a}}^f(M, N))) \cong H_{\mathfrak{b}}^0(H_{\mathfrak{a}}^f(M, N))$ and Theorem 2.5. The last part of the theorem is clear by [15, Theorem 1.1] and the definition of $f_{\mathfrak{a}}(M, N)$. \square

3. COFINITENESS OF $H_{\mathfrak{b}}^j(H_{\mathfrak{a}}^i(M, N))$ FOR SOME INDICES i, j .

Let \mathfrak{a} and \mathfrak{b} be two ideals of R and let \mathfrak{m} be a maximal ideal of R such that $\mathfrak{a} + \mathfrak{b}$ is \mathfrak{m} -primary. The aim of this section is to study the cofiniteness of the modules $H_{\mathfrak{b}}^j(H_{\mathfrak{a}}^i(M, N))$ for some particular value of "i's".

Definition 3.1. Let us define the following number:

$$q_{\mathfrak{a}}(M, N) = \sup \{i \mid H_{\mathfrak{a}}^i(M, N) \text{ is not Artinian}\}.$$

If $H_{\mathfrak{a}}^i(M, N)$ is Artinian for all i , we write $q_{\mathfrak{a}}(M, N) = -\infty$.

In addition, $cd_{\mathfrak{a}}(M, N)$ denotes the largest non-negative integer i such that $H_{\mathfrak{a}}^i(M, N)$ is not equal to zero.

Theorem 3.2. *Let $\mathfrak{a}, \mathfrak{b}$ be two ideals of R and let \mathfrak{m} be a maximal ideal of R such that $\mathfrak{a} + \mathfrak{b}$ is \mathfrak{m} -primary. If $\text{Supp}M \subseteq V(\mathfrak{a})$, then $H_{\mathfrak{b}}^i(M)$ is Artinian and \mathfrak{b} -cofinite for all i .*

Proof. It is argued by induction on i . It is straightforward to see that the result is true when $i = 0$. Now, inductively assume that $i > 0$ and that the assertion has been proved for $i - 1$. It follows, from [1, Corollary 2.1.7(iii)] that $H_{\mathfrak{b}}^i(M) \cong H_{\mathfrak{b}}^i(M/\Gamma_{\mathfrak{b}}(M))$ for all $i \geq 1$. Also, $M/\Gamma_{\mathfrak{b}}(M)$ is a \mathfrak{b} -torsion free R -module. Then the ideal \mathfrak{b} contains an element x , which avoids all members of $\text{Ass}M$. It is clear that $\text{Supp}(M/xM) \subseteq V(\mathfrak{a})$. In addition, the exact sequence $0 \rightarrow M \rightarrow M \rightarrow M/xM \rightarrow 0$ induces the exact sequence

$$H_{\mathfrak{b}}^{i-1}(M/xM) \rightarrow H_{\mathfrak{b}}^i(M) \xrightarrow{x} H_{\mathfrak{b}}^i(M) \rightarrow H_{\mathfrak{b}}^i(M/xM),$$

that implies that the R -module $(0 :_{H_{\mathfrak{b}}^i(M)} x)$ is Artinian and \mathfrak{b} -cofinite. Therefore, in view of [10, Proposition 4.1], $H_{\mathfrak{b}}^i(M)$ is Artinian and \mathfrak{b} -cofinite. \square

Theorem 3.3. *Let \mathfrak{a} and \mathfrak{b} be two ideals of R and \mathfrak{m} be a maximal ideal of R such that $\mathfrak{a} + \mathfrak{b}$ is \mathfrak{m} -primary. Let $\text{ara}(\mathfrak{b}) = t$ and $\text{cd}_{\mathfrak{a}}(M, N) = c$. Then $H_{\mathfrak{b}}^t(H_{\mathfrak{a}}^c(M, N))$ and $H_{\mathfrak{b}}^{t-1}(H_{\mathfrak{a}}^c(M, N))$ are Artinian R -modules.*

Proof. Consider the Grothendieck spectral sequence [11, Theorem 11.38]

$$E_2^{p,q} = H_{\mathfrak{b}}^p(H_{\mathfrak{a}}^q(M, N)) \xrightarrow{p} H_{\mathfrak{m}}^{p+q}(M, N).$$

This spectral sequence induces an exact sequence of R -modules and R -homomorphisms

$$0 \longrightarrow \ker d_2^{i,j} \longrightarrow E_2^{i,j} \xrightarrow{d_2^{i,j}} E_2^{i+2,j-1} \text{ for all } i \geq 0. \quad (\#)$$

By the hypotheses $E_2^{p,q} = 0$ for all $p > t$ or $q > c$. Then the sequence $(\#)$ yields the isomorphisms below: $\text{Ker}d_2^{t,c} \cong E_2^{t,c}$, $\text{Ker}d_2^{t-1,c} \cong E_2^{t-1,c}$ and $E_2^{t,c} \cong E_r^{t,c}$ and $E_2^{t-1,c} \cong E_r^{t-1,c}$ for all $r \geq 2$, it follows that $E_{\infty}^{t-1,c} \cong E_2^{t-1,c} \cong H_{\mathfrak{b}}^{t-1}(H_{\mathfrak{a}}^c(M, N))$ and $E_{\infty}^{t,c} \cong E_2^{t,c} \cong H_{\mathfrak{b}}^t(H_{\mathfrak{a}}^c(M, N))$. Now, since $E_{\infty}^{p,q}$ is a subquotient of the Artinian R -module $H_{\mathfrak{m}}^{p+q}(M, N)$ for each $p, q \in \mathbb{N}_0$, the assertion immediately follows. \square

Remark 3.4. Let (R, \mathfrak{m}) be a local ring and let x_1, x_2, \dots, x_n be elements of R . For each $i = 1, \dots, n$, we put $N_i = N/(x_1, x_2, \dots, x_i)N$ and $\Omega = \{\mathfrak{p} \in \text{Ass}N \mid \dim R/\mathfrak{p} > 1\}$. Then the element x_1 is a generalized regular element of N in \mathfrak{a} if $x_1 \in \mathfrak{a} - \bigcup_{\mathfrak{p} \in \Omega} \mathfrak{p}$. The sequence x_1, x_2, \dots, x_n is named to be a generalized regular sequence of N in \mathfrak{a} of length n if x_i is a generalized regular element of N_i in \mathfrak{a} for all $i = 1, \dots, n$. The length of a maximal generalized regular N -sequence in \mathfrak{a} is called the generalized depth of N in \mathfrak{a} and is denoted by $\text{gdepth}(\mathfrak{a}, N)$. It is clear that $\text{gdepth}(M/\mathfrak{a}M, N)$ is a non-negative integer and it is equal to the length of any maximal generalized regular N -sequence in $\mathfrak{a} + (0 :_R M)$.

Lemma 3.5. *(see [14, Theorem 3.2]) . Let (R, \mathfrak{m}) be local ring. Then*

$$\text{gdepth}(M/\mathfrak{a}M, N) = \min \{i \mid \text{Supp}H_{\mathfrak{a}}^i(M, N) \text{ is an infinite set}\}.$$

Lemma 3.6. *(see [12, Theorem 1.2]) . Let t be a non-negative integer such that $\dim \text{Supp}(H_{\mathfrak{a}}^i(M, N)) \leq 1$ for all $i < t$. Then $H_{\mathfrak{a}}^i(M, N)$ is \mathfrak{a} -cofinite for all $i < t$.*

Theorem 3.7. *Let \mathfrak{a} and \mathfrak{b} be two ideals of R and let \mathfrak{m} be a maximal ideal of R such that $\mathfrak{a} + \mathfrak{b}$ is \mathfrak{m} -primary and let $\text{gdepth}(M/\mathfrak{a}M, N) = t$. Then $H_{\mathfrak{b}}^j(H_{\mathfrak{a}}^i(M, N))$ is Artinian and \mathfrak{b} -cofinite for all $i < t$ and $j \geq 0$. Moreover, $H_{\mathfrak{b}}^j(H_{\mathfrak{a}}^t(M, N))$ is an Artinian and \mathfrak{b} -cofinite R -module for all $j = 0, 1$.*

Proof. By Lemma 3.6 and Lemma 3.5, we have that $H_{\mathfrak{a}}^i(M, N)$ is \mathfrak{a} -cofinite for all $i < t$. It is straightforward that $H_{\mathfrak{b}}^j(H_{\mathfrak{a}}^i(M, N)) \cong H_{\mathfrak{m}}^j(H_{\mathfrak{a}}^i(M, N))$. Hence, by Theorem 2.2, $H_{\mathfrak{b}}^j(H_{\mathfrak{a}}^i(M, N))$ is Artinian and \mathfrak{b} -cofinite for all $j \geq 0$ and $i < t$. Since, by [11, Theorem 11.38], the Grothendieck's spectral sequence $E_2^{p,q} = H_{\mathfrak{b}}^p(H_{\mathfrak{a}}^q(M, N))$ converges to $H_{\mathfrak{m}}^{p+q}(M, N)$. It follows from previous paragraph that $E_2^{p,q}$ is Artinian and \mathfrak{b} -cofinite for all $q < t$. Note that $H_{\mathfrak{m}}^i(M, N)$ is Artinian for all $i \geq 0$. By using an argument similar to the proof of [4, Theorem 2.2], we obtain that $H_{\mathfrak{b}}^j(H_{\mathfrak{a}}^t(M, N))$ is Artinian for all $j = 0, 1$. Since the radical of the annihilator of $\text{Hom}(R/\mathfrak{b}, H_{\mathfrak{b}}^j(H_{\mathfrak{a}}^t(M, N)))$ is equal to \mathfrak{m} , the R -module $H_{\mathfrak{b}}^j(H_{\mathfrak{a}}^t(M, N))$ is Artinian and \mathfrak{b} -cofinite for all $j = 0, 1$. \square

Lemma 3.8. *Let $\mathfrak{a}, \mathfrak{b}$ be two ideals of R and let \mathfrak{m} be a maximal ideal of R such that $\mathfrak{a} + \mathfrak{b}$ is \mathfrak{m} -primary. Let $\Gamma_{\mathfrak{a}}(T) = T$ and we assume that T is an Artinian R -module. Then $H_{\mathfrak{b}}^i(M, T)$ is Artinian and \mathfrak{b} -cofinite for all i .*

Proof. The hypotheses says that $\text{Hom}(R/\mathfrak{b}, \Gamma_{\mathfrak{b}}(T))$ is of finite length. Therefore, by [10, Proposition 4.1], we deduce that $\Gamma_{\mathfrak{b}}(T)$ is \mathfrak{b} -cofinite and Artinian. Now, one can complete the proof by using a similar method which we used in the proof of Theorem 2.2. \square

Theorem 3.9. *Let us suppose that there exists an integer $t \geq 0$ such that $H_{\mathfrak{a}}^i(M, N)$ is \mathfrak{a} -cofinite for all $i \neq t$. Then $H_{\mathfrak{a}}^i(M, N)$ is \mathfrak{a} -cofinite for all i and $H_{\mathfrak{b}}^j(H_{\mathfrak{a}}^t(M, N))$ is an Artinian and \mathfrak{b} -cofinite R -module for all i, j , where $\mathfrak{a} + \mathfrak{b}$ is \mathfrak{m} -primary.*

Proof. Let us consider the convergent spectral sequence

$$E_2^{p,q} = \text{Ext}_R^p(R/\mathfrak{a}, H_{\mathfrak{a}}^q(M, N)) \xrightarrow{p} \text{Ext}_R^{p+q}(M/\mathfrak{a}M, \Gamma_{\mathfrak{a}}(N)).$$

Since $E_r^{p,q}$ is a subquotient of $E_2^{p,q}$ for all $r \geq 2$, our hypotheses give us that $E_r^{p,q}$ is finitely generated for all $r \geq 2$, $p \geq 0$, and $q \neq t$. For each $r \geq 2$ and $p, q \geq 0$, let $Z_r^{p,q} = \text{Ker}(E_r^{p,q} \rightarrow E_r^{p+r, q-r+1})$ and $B_r^{p,q} = \text{Im}(E_r^{p-r, q+r-1} \rightarrow E_r^{p,q})$. Note that $B_r^{p,q}$ is finitely generated for all p, q and $r \geq 2$, since either $E_r^{p-r, q+r-1}$ or $E_r^{p,q}$ is finitely generated. For all $r \geq 2$ and $p \geq 0$ we have the exact sequences

$$\begin{aligned} 0 \longrightarrow B_r^{p,t} \longrightarrow Z_r^{p,t} \longrightarrow E_{r+1}^{p,t} \longrightarrow 0 \quad \text{and} \\ 0 \longrightarrow Z_r^{p,t} \longrightarrow E_r^{p,t} \longrightarrow B_r^{p+r, t-r+1} \longrightarrow 0. \end{aligned}$$

On the other hand, $E_{\infty}^{p,t}$ is isomorphic to a subquotient of $\text{Ext}_R^{p+t}(M/\mathfrak{a}M, \Gamma_{\mathfrak{a}}(N))$. Thus it is finitely generated for all p . Since $E_{\infty}^{p,t} = E_r^{p,t}$ for r sufficiently large, it follows that $E_r^{p,t}$ is finitely generated for all p and all large r . Fix p and r and suppose $E_{r+1}^{p,t}$ is finitely generated. From the first exact sequence we obtain that $Z_r^{p,t}$ is finitely generated. From the second exact sequence we get that $E_r^{p,t}$ is finitely generated for all $r \geq 2$. In particular, $E_2^{p,t} = \text{Ext}_R^p(R/\mathfrak{a}, H_{\mathfrak{a}}^t(M, N))$ is finitely generated for all p and the result follows from Theorem 2.2. \square

The following corollary immediately follows from Theorem 3.9 and Definition 3.1.

Corollary 3.10. *Let $\mathfrak{a}, \mathfrak{b}$ be two ideals of R and let \mathfrak{m} be a maximal ideal of R such that $\mathfrak{a} + \mathfrak{b}$ is \mathfrak{m} -primary. Let $f_{\mathfrak{a}}(M, N) = q_{\mathfrak{a}}(M, N) = t$. Then $H_{\mathfrak{b}}^j(H_{\mathfrak{a}}^i(M, N))$ is an Artinian and \mathfrak{b} -cofinite R -module for all i, j .*

Proof. If $i < f_{\mathfrak{a}}(M, N)$ then, in view of the definition of $f_{\mathfrak{a}}(M, N)$ and Theorem 3.2, $H_{\mathfrak{b}}^j(H_{\mathfrak{a}}^i(M, N))$ is an Artinian and \mathfrak{b} -cofinite R -module for all j . If $i > q_{\mathfrak{a}}(M, N)$, then $H_{\mathfrak{a}}^i(M, N)$ is Artinian. It follows from Lemma 3.8 that $H_{\mathfrak{b}}^j(H_{\mathfrak{a}}^i(M, N))$ is an Artinian and \mathfrak{b} -cofinite R -module for all j . Thus we consider the case where $i = t$. To this end, consider the Grothendieck spectral sequence

$$E_2^{p,q} = H_{\mathfrak{b}}^p(H_{\mathfrak{a}}^q(M, N)) \xrightarrow{p} H_{\mathfrak{m}}^{p+q}(M, N).$$

By using an argument similar with that one used in the proof of Theorem 3.9 the result follows. \square

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DEPARTMENT OF MATHEMATICS, YAZD BRANCH, ISLAMIC AZAD UNIVERSITY, YAZD, IRAN.
E-mail address: dehghanizadeh@iauyazd.ac.ir and f.dehghanizadeh@yahoo.com