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A study on fuzzy (prime) hyperideals of Γ-hypersemirings

Bijan Davvaz[a] and **Bayram Ali Ersoy**[b]
[a]Department of Mathematics, Yazd University, Yazd, Iran
[b]Department of Mathematics, Yildiz Technical University, 81270, Istanbul, Turkey
davvaz@yazd.ac.ir
ersoya@gmail.com

Abstract

A Γ-semihyperring is a generalization of a semiring, a generalization of a semihyperring and a generalization of a Γ-semiring. In this paper, we define the notion of a fuzzy (prime) Γ-hyperideal of a Γ-semihyperring. Then we prove some results in this respect. Also, by using the notion of fuzzy Γ-hyperideals, we give several characterizations of Γ-semihyperrings.

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1 Introduction

In 1964, Nobusawa introduced Γ-rings as a generalization of ternary rings. Barnes [4] weakened slightly the conditions in the definition of Γ-ring in the sense of Nobusawa. Barnes [4], Luh [27] and Kyuno [21] studied the structure of Γ-rings and obtained various generalizations analogous to corresponding parts in ring theory. The concept of Γ-semigroups was introduced by Sen and Saha [30, 31] as a generalization of semigroups and ternary semigroups. Then the notion of Γ-semirings introduced by Rao [28].

Algebraic hyperstructures represent a natural extension of classical algebraic structures and they were introduced by the French mathematician Marty [22] . Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Since then, hundreds of papers and several books have been written on this topic, see [5, 6, 12, 33]. In

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[2, 3, 20], Davvaz et. al. introduced the notion of a Γ-semihypergroup as a generalization of a semihypergroup. Many classical notions of semigroups and semihypergroups have been extended to Γ-semihypergroups and a lot of results on Γ-semihypergroups are obtained. In [17, 18, 19], Davvaz et. al. studied the notion of a Γ-semihyperring as a generalization of semiring, a generalization of a semihyperring and a generalization of a Γ-semiring.

After the introduction of fuzzy sets by Zadeh [34], reconsideration of the concept of classical mathematics began. In 1971, Rosenfeld [29] introduced fuzzy sets in the context of group theory and formulated the concept of a fuzzy subgroup of a group. The concept of a fuzzy ideal of a ring was introduced by Liu [26]. The study of fuzzy hyperstructures is an interesting research topic of fuzzy sets. In [9], Davvaz introduced the concept of fuzzy H_v-ideals of H_v-rings. Then this concept was studied in the depth in several papers, for example see [7, 8, 11, 14, 15, 16, 23, 24, 25]. In [13], Davvaz and Leoreanu studied the notion of a fuzzy Γ-hyperideal of a Γ-semihypergroup. Now, in this paper, we define the notion of a fuzzy (prime) Γ-hyperideal of a Γ-semihyperring.

2 Γ-semihyperrings

Let H be a non-empty set and let $\mathcal{P}^*(H)$ be the set of all non-empty subsets of H. A *hyperoperation* on H is a map $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$ and the couple $(H, \circ)$ is called a *hypergroupoid*. If A and B are non-empty subsets of H and $x \in H$, then we denote

$$A \circ B = \bigcup_{a\in A,\, b\in B} a \circ b, \quad x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\}.$$

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for all x, y, z of H we have $(x \circ y) \circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u\in x\circ y} u \circ z = \bigcup_{v\in y\circ z} x \circ v.$$

Let $(S, \cdot)$ be an ordinary semigroup and let P be a subset of S. We define

$$x \circ y = x \cdot P \cdot y, \text{ for all } x, y \in S.$$

Then, $(S, \circ)$ is a semihypergroup.

A *semihyperring* is an algebraic structure $(R, +, \cdot)$ which satisfies the following properties;

(1) $(R, +)$ is a *commutative semihypergroup*, that is,

(i) $(x + y) + z = x + (y + z)$,

(ii) $x + y = y + x$ for all $x, y, z \in R$.

(2) $(R, \cdot)$ is a semihypergroup that is $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y \in R$.

(3) The multiplication is distributive with respect to hyperoperation + that is $x \cdot (y + z) = x \cdot y + x \cdot z$, $(x + y) \cdot z = x \cdot z + y \cdot z$

(4) The element $0 \in R$ is an *absorbing element* that is $x \cdot 0 = 0 \cdot x = 0$ for all $x \in R$. A semihyperring $(R, +)$ is called *commutative* if and only if $a \cdot b = b \cdot a$ for all

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$a, b \in R$. Vougiouklis in [32] and Davvaz in [10] studied the notion of semihyperrings in a general form, i.e., both the sum and product are hyperoperations, also see [1].

A semihyperring $(R, +)$ with identity $1_R \in R$ means that $1_R \cdot x = x \cdot 1_R = x$ for all $x \in R$. An element $x \in R$ is called *unit* if there exists $y \in R$ such that $1_R = x \cdot y = y \cdot x$. Here $U(R)$ is the set of all unit elements. A nonempty subset S of a semihyperring $(R, +, \cdot)$ is called a *sub-semihyperring* if $a + b \subseteq S$ and $a \cdot b \subseteq S$ for all $a, b \in S$. A *left hyperideal* of a semihyperring R is a non-empty subset I of R satisfying

(i) $x + y \subseteq I$ for all $x, y \in I$.

(ii) $x \cdot a \subseteq I$ for all $a \in I$ and $x \in R$.

The concept of Γ-semihyperring is introduced and studied by Dehkordi and Davvaz [17]. We recall the following definition from [17].

Definition 2.1. Let R be a commutative semihypergroup and Γ be a commutative group. Then R is called a Γ-*semihyperring* if there exists a map $R \times \Gamma \times R \longrightarrow \mathcal{P}^*(R)$ (image to be denoted by a α b for $a, b \in R$ and $\alpha \in \Gamma$) and $\mathcal{P}^*(R)$ is the set of all non-empty subsets of R satisfying the following conditions:

(i) $a\alpha(b + c) = a\alpha b + a\alpha c$,

(ii) $(a + b)\alpha c = a\alpha c + b\alpha c$,

(iii) $a(\alpha + \beta)c = a\alpha c + a\beta c$,

(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$.

In the above definition if R is a semigroup, then R is called a *multiplicative* Γ-*semihyperring*. A Γ-semihyperring R is called *commutative* if $x\alpha y = y\alpha x$ for every $x, y \in R$ and $\alpha \in \Gamma$. We say that R is a Γ-semihyperring with zero, if there exists $0 \in R$ such that $a \in a + 0$ and $0 \in 0\alpha a$, $0 \in a\alpha 0$ for all $a \in R$ and $\alpha \in \Gamma$. Let A and B be two non-empty subsets of Γ-semihyperring R and $x \in R$. We define

$$A + B = \{x \mid x \in a + b \ \ a \in A,\ b \in B\}$$

and

$$A\Gamma B = \{x \mid x \in a\alpha b \ a \in A,\ b \in B,\ \alpha \in \Gamma\}.$$

A non-empty subset R_1 of Γ-semihyperring R is called a *sub* Γ-*semihyperring* if it is closed with respect to the multiplication and addition. In other words, a non-empty subset R_1 of Γ-semihyperring R is a sub Γ-semihypergroup if $R_1 + R_1 \subseteq R_1$ and $R_1 \Gamma R_1 \subseteq R_1$.

Example 1. [17] Let $R = \mathbb{Z}_4$ and $\Gamma = \{\overline{0}, \overline{2}\}$. Then R is a multiplicative Γ-semihyperring with the following hyperoperation:

$$x\alpha y = \{\overline{0}, \overline{2}\},$$

where $x, y \in R$ and $\alpha \in \Gamma$.

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Example 2. [17] Let R be a ring, $\{A_g\}_{g\in R}$ be a family of disjoint non-empty sets and $(\Gamma,+)$ be a subgroup of $(R,+)$. Then $S=\bigcup_{g\in R} A_g$ is a Γ-semihyperring with the following hyperoperations:

$$x\oplus y = A_{g_1+g_2},\ x\alpha y = A_{g_1\alpha g_2},$$

where $x\in A_{g_1}$ and $y\in A_{g_2}$.

Definition 2.2. A non-empty subset I of Γ-semihyperring R is a *right* (*left*) Γ*-hyperideal* of R if I is a subhypergroup of $(R,+)$ and $I\Gamma R\subseteq I$ $(R\Gamma I\subseteq I)$, and is an ideal of R if both a right and left ideal.

Remark 1. Note that in Definition 2.1, if R is a commutative semigroup and there exists a map $R\times\Gamma\times R\longrightarrow R$ which satisfies the conditions of Definition 2.1, then R is called a Γ-semiring.

The notion of Γ-semiring was introduced by Rao [28] as a generalization of Γ-rings, rings and semirings.

Example 3. Let R be an additive semigroup of all $m\times n$ matrices over the set of all non negative rational numbers and let Γ be the additive semigroup of all $n\times m$ matrices over the set of all non-negative integers. Then, R is a Γ-semiring with matrix multiplication as the ternary operation.

Example 4. Let $\mathbb{Q}^+$ be the set of all non-zero rational numbers and let Γ be the set of all positive integers. Let $a,b\in\mathbb{Q}^+$ and $\alpha\in\Gamma$. If we define the map $R\times\Gamma\times R\longrightarrow R$ by $a\alpha b\mapsto |a|\alpha b$, then $\mathbb{Q}^+$ is a Γ-semiring.

Example 5. Consider the semigroup of positive integers $R=(\mathbb{Z}^+,+)$ and the semigroup of even positive integers $\Gamma=(2\mathbb{Z}^+,+)$. Then R is a Γ-semiring.

3 Fuzzy Γ-hyperideals of Γ-semihyperrings

In this section, we define the notion of a fuzzy Γ-hyperideal of a Γ-semihyperring and study some properties of it.

Definition 3.1. Let R be a Γ-semihyperring and μ be a fuzzy subset of R. Then

(1) μ is called a *fuzzy left* Γ*-hyperideal* of R if

$$\min\{\mu(x),\mu(y)\}\le \inf_{z\in x+y}\{\mu(z)\},\ \forall x,y\in R,$$
$$\mu(y)\le \inf_{z\in x\gamma y}\{\mu(z)\},\ \forall x,y\in R,\ \forall\gamma\in\Gamma.$$

(2) μ is called a *fuzzy right* Γ*-hyperideal* of R if

$$\min\{\mu(x),\mu(y)\}\le \inf_{z\in x+y}\{\mu(z)\},\ \forall x,y\in R,$$
$$\mu(x)\le \inf_{z\in x\gamma y}\{\mu(z)\},\ \forall x,y\in R,\ \forall\gamma\in\Gamma.$$

(3) μ is called a *fuzzy* Γ*-hyperideal* of R if it is both a fuzzy left Γ-hyperideal and a fuzzy right Γ-hyperideal of R.

EXAMPLE 6. Let $(R, +, \cdot)$ be a semihyperring and Γ be a hyperideal of R. We define the map $R \times \Gamma \times R \longrightarrow \mathcal{P}^*(R)$ by $(x, \gamma, y) \mapsto \{a \in R \mid a \in x \cdot \gamma \cdot y\}$. Then, R is a Γ-semihyperring. Now, we define the fuzzy subset μ of R by

$$\mu(x) = \begin{cases} \frac{2}{3} & \text{if } x \in \Gamma \\ 0 & \text{otherwise.} \end{cases}$$

Then, μ is a Γ-hyperideal of R.

Definition 3.2. Let R be a Γ-semiring and μ be a fuzzy subset of R. Then

(1) μ is called a *fuzzy left* Γ*-ideal* of R if

$$\min\{\mu(x), \mu(y)\} \leq \mu(x + y), \ \forall x, y \in R,$$
$$\mu(y) \leq \mu(x\gamma y), \ \forall x, y \in R, \ \forall \gamma \in \Gamma.$$

(2) μ is called a *fuzzy right* Γ*-ideal* of R if

$$\min\{\mu(x), \mu(y)\} \leq \mu(x + y), \ \forall x, y \in R,$$
$$\mu(x) \leq \mu(x\gamma y), \ \forall x, y \in R, \ \forall \gamma \in \Gamma.$$

(3) μ is called a *fuzzy* Γ*-ideal* of R if it is both a fuzzy left Γ-ideal and a fuzzy right Γ-ideal of R.

EXAMPLE 7. Let G and H be two additive abelian groups, $R = Hom(G, H)$ and $\Gamma = Hom(H, G)$. Then, it is easy to see that R is a Γ-semiring with the pointwise addition and composition of homomorphisms.We define a fuzzy subset μ by

$$\mu(\varphi) = \begin{cases} \frac{4}{5} & \text{if } \varphi = 0 \\ \frac{1}{3} & \text{if } \varphi \neq 0. \end{cases}$$

Then, μ is a Γ-ideal of R.

Let I be a non-empty subset of a Γ-semihyperring R and χ_I be the characteristic function of I. Then I is a left Γ-hyperideal (right Γ-hyperideal, Γ-hyperideal) of R if and only if χ_I is a fuzzy left Γ-hyperideal (respectively, fuzzy right Γ-hyperideal, fuzzy Γ-hyperideal) of R.

Lemma 3.3. *If* $\{\mu_i\}_{i\in\Lambda}$ *is a collection of fuzzy* Γ*-hyperideals* R*, then* $\bigcap\limits_{i\in\Lambda} \mu_i$ *and* $\bigcup\limits_{i\in\Lambda} \mu_i$ *are fuzzy* Γ*-hyperideal of* R*, too.*

Proof. For all $a, b \in R$ and $\alpha \in \Gamma$, we have
(a)

$$\begin{aligned}\inf_{z\in a+b}\left\{\left(\bigcap_{i\in\Lambda}\mu_i\right)(z)\right\} &= \inf_{z\in a+b}\left\{\inf_{i\in\Lambda}\{\mu_i(z)\}\right\} \geq \inf_{i\in\Lambda}\{\min\{\mu_i(a),\mu_i(b)\}\}\\ &= \min\left\{\inf_{i\in\Lambda}\{\mu_i(a)\},\inf_{i\in\Lambda}\{\mu_i(b)\}\right\}\\ &= \min\left\{\left(\bigcap_{i\in\Lambda}\mu_i\right)(a),\left(\bigcap_{i\in\Lambda}\mu_i\right)(b)\right\},\end{aligned}$$

$$\begin{aligned}\inf_{z\in a\gamma b}\left\{\left(\bigcap_{i\in\Lambda}\mu_i\right)(z)\right\} &= \inf_{z\in a\gamma b}\left\{\inf_{i\in\Lambda}\{\mu_i(z)\}\right\} = \inf_{i\in\Lambda}\left\{\inf_{z\in a\gamma b}\{\mu_i(z)\}\right\}\\ &\geq \inf_{i\in\Lambda}\{\mu_i(b)\}\\ &= \left(\bigcap_{i\in\Lambda}\mu_i\right)(b).\end{aligned}$$

Similarly, we can prove that

$$\inf_{z\in a\gamma b}\left\{\left(\bigcap_{i\in\Lambda}\mu_i\right)(z)\right\} \geq \left(\bigcap_{i\in\Lambda}\mu_i\right)(a).$$

(b)

$$\begin{aligned}\inf_{z\in a+b}\left\{\left(\bigcup_{i\in\Lambda}\mu_i\right)(z)\right\} &= \inf_{z\in a+b}\left\{\sup_{i\in\Lambda}\{\mu_i(z)\}\right\} \geq \sup_{i\in\Lambda}\{\min\{\mu_i(a),\mu_i(b)\}\}\\ &= \min\left\{\sup_{i\in\Lambda}\{\mu_i(a)\},\sup_{i\in\Lambda}\{\mu_i(b)\}\right\}\\ &= \min\left\{\left(\bigcup_{i\in\Lambda}\mu_i\right)(a),\left(\bigcup_{i\in\Lambda}\mu_i\right)(b)\right\},\end{aligned}$$

$$\begin{aligned}\inf_{z\in a\gamma b}\left\{\left(\bigcup_{i\in\Lambda}\mu_i\right)(z)\right\} &= \inf_{z\in a\gamma b}\left\{\sup_{i\in\Lambda}\{\mu_i(z)\}\right\} = \sup_{i\in\Lambda}\left\{\inf_{z\in a\gamma b}\{\mu_i(z)\}\right\}\\ &\geq \sup_{i\in\Lambda}\{\mu_i(b)\}\\ &= \left(\bigcup_{i\in\Lambda}\mu_i\right)(b).\end{aligned}$$

Similarly, we can prove

$$\inf_{z\in a\gamma b}\left\{\left(\bigcup_{i\in\Lambda}\mu_i\right)(z)\right\} \geq \left(\bigcup_{i\in\Lambda}\mu_i\right)(a).$$

□

We recall the *t-level cut of* μ or *the level subset with respect to* μ as follows:

$$U(\mu; t) = \{x \in R \mid \mu(x) \geq t\}.$$

Theorem 3.4. *A fuzzy subset* μ *of a* Γ*-semihyperring* R *is a fuzzy* Γ*-hyperideal of* R *if and only if for every* $t \in [0,1]$*, the set* $U(\mu; t)$ *is a* Γ*-hyperideal of* R *when it is a non-empty set.*

Proof. Suppose that μ is a fuzzy Γ-hyperideal of R. For every x, y in $U(\mu; t)$, we have $\mu(x) \geq t$ and $\mu(y) \geq t$. Hence, $\min\{\mu(x), \mu(y)\} \geq t$ and so $\inf_{z \in x+y}\{\mu(z)\} \geq t$. Therefore, for every $z \in x + y$ we get $\mu(z) \geq t$ which implies that $z \in U(\mu; t)$. Thus, $x + y \subseteq U(\mu; t)$.

Now, we show that $U(\mu; t)\Gamma R \subseteq U(\mu; t)$. Assume that $x \in U(\mu; t)$, $\gamma \in \Gamma$ and $r \in R$ are arbitrary elements. Since $x \in U(\mu; t)$, $\mu(x) \geq t$. Then, $t \leq \mu(x) \leq \inf_{z \in x\gamma r}\{\mu(z)\}$ which implies that for every $z \in x\gamma r$, $\mu(z) \geq t$. Hence, $z \in U(\mu; t)$. Therefore, $x\gamma r \subseteq U(\mu; t)$. Similarly, we can see $R\Gamma U(\mu; t) \subseteq U(\mu; t)$.

Conversely, suppose that for every $0 \leq t \leq 1$, $U(\mu; t)$ $(\neq \emptyset)$ is a Γ-hyperideal of R. For every x, y in R, we can write $\mu(x) \geq t_0$ and $\mu(y) \geq t_0$, where $t_0 = \min\{\mu(x), \mu(y)\}$. Then, $x \in U(\mu; t_0)$ and $y \in U(\mu; t_0)$. Since $U(\mu; t_0)$ is a Γ-hyperideal, we obtain $x + y \subseteq U(\mu; t_0)$. Therefore, for every $z \in x + y$ we have $\mu(z) \geq t_0$ implying that $\inf_{z \in x+y}\{\mu(z)\} \geq t_0$ and so $\min\{\mu(x), \mu(y)\} \leq \inf_{z \in x+y}\{\mu(\alpha)\}$ and in this way the first condition of the definition is verified.

Now, suppose that $x, y \in R$ and $\gamma \in \Gamma$ are arbitrary elements such that $\mu(x) = s_0$. Then, $x \in U(\mu; s_0)$. Since $U(\mu; s_0)$ is a Γ-hyperideal, we obtain $x\gamma y \subseteq U(\mu; s_0)$. Hence, for every $z \in x\gamma y$, we have $z \in U(\mu; s_0)$ so $\mu(z) \geq s_0$. Therefore, $\inf_{z \in x\gamma y}\{\mu(z)\} \geq \mu(x)$. Similarly, we can see $\inf_{z \in x\gamma y}\{\mu(z)\} \geq \mu(y)$. □

Proposition 3.5. *Let* R *be a* Γ*-semihyperring and* μ *be a fuzzy* Γ*-hyperideal of* R*.*

(1) If a *is a fixed element of* R*, then the set* $\mu^a = \{x \in R \mid \mu(x) \geq \mu(a)\}$ *is a* Γ*-hyperideal.*

(2) The set $U = \{x \in R \mid \mu(x) = (\mu(0)\}$ *is a* Γ*-hyperideal of* R*.*

Proof. It is straightforward. □

4 Fuzzy prime Γ-hyperideals of Γ-semihyperrings

Definition 4.1. Let R be a Γ-semihyperring and P be a proper right (left) Γ-hyperideal of R. Then P is called a *prime right (left)* Γ*-hyperideal* if for every $x, y \in R$, $x\Gamma R\Gamma y \subseteq P$ implies that $x \in P$ or $y \in P$.

Example 8. [18]. Let S be the Γ-semihyperring defined in Example 2. If P is a prime Γ-hyperideal of R such that $\Gamma \cap P = \emptyset$, then $S_P = \bigcup_{g \in P} A_g$ is a prime Γ-hyperideal of Γ-semihyperring S.

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Proposition 4.2. *[18]. Let R be a Γ-semihyperring with zero and P be a right ideal of R. Then the following are equivalent:*

(1) P is a prime Γ-hyperideal,

(2) $I\Gamma J \subseteq P$ implies that $I \subseteq P$ or $J \subseteq P$, where I and J are two left (right) Γ-hyperideals of R.

Definition 4.3. Let R be a Γ-semihyperring and θ, σ be two fuzzy subsets of R. Then the *sum* $\theta \oplus \sigma$, the *product* $\theta\Gamma\sigma$ and the *composition* $\theta \circ \sigma$ are defined by

$$(\theta \oplus \sigma)(z) = \begin{cases} \sup\limits_{z \in x+y} \{\min\{\theta(x), \sigma(y)\}\} & \text{for } x, y \in R \\ 0 & \text{otherwise.} \end{cases}$$

$$(\theta\Gamma\sigma)(z) = \begin{cases} \sup\limits_{z \in x\gamma y} \{\min\{\theta(x), \sigma(y)\}\} & \text{for } x, y \in R \text{ and } \gamma \in \Gamma \\ 0 & \text{otherwise.} \end{cases}$$

$$(\theta \circ \sigma)(z) = \begin{cases} \sup\{\min\limits_{i}\{\min\{\theta(x_i), \sigma(y_i)\}\}\}, & 1 \leq i \leq n, z \in \sum\limits_{i=1}^{n} x_i\gamma_i y_i, x_i, y_i \in R, \gamma_i \in \Gamma \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4.4. *Let R be a Γ-semihyperring and μ be a fuzzy Γ-hyperideal of R. Then*

$$\min\{\mu(x_1), \ldots, \mu(x_n)\} \leq \inf_{z \in x_1 + \ldots + x_n} \{\mu(z)\},$$

for all $x_1, x_2, \ldots, x_n \in R$.

Proof. The validity of this lemma is proved by mathematical induction. □

Proposition 4.5. *Let R be a Γ-semihyperring and θ, σ be two fuzzy Γ-hyperideals of R. Then*

$$\theta\Gamma\sigma \subseteq \theta \circ \sigma \subseteq \theta \cap \sigma.$$

Proof. By using definitions, it is easy to see that $\theta\Gamma\sigma \subseteq \theta \circ \sigma$. Now, suppose that $x \in \sum\limits_{i=1}^{n} x_i\gamma_i y_i$, where $x_i, y_i \in R$ for $1 \leq i \leq n$ and $\gamma_i \in \Gamma$. Hence, there exist $a_i \in x_i\gamma_i y_i$ (for $1 \leq i \leq n$) such that $x \in \sum\limits_{i=1}^{n} a_i$. So, by Lemma 4.4, we obtain

$$\begin{aligned} \theta(x) &\geq \min\{\theta(a_1), \ldots, \theta(a_n)\} \\ &\geq \min\{\inf_{z_1 \in x_1\gamma_1 y_1}\{\theta(z_1)\}, \ldots, \inf_{z_n \in x_n\gamma_n y_n}\{\theta(z_n)\}\} \\ &\geq \min\{\theta(x_1), \ldots, \theta(x_n)\}. \end{aligned}$$

Similarly, we obtain $\sigma(x) \geq \min\{\sigma(y_1), \ldots, \sigma(y_n)\}$. Therefore, we have

$$\begin{aligned} (\theta \cap \sigma)(x) &= \min\{\theta(x), \sigma(x)\} \\ &\geq \min\{\min\{\theta(x_1), \ldots, \theta(x_n)\},\ \min\{\sigma(y_1), \ldots, \sigma(y_n)\}\} \\ &= \min_i\{\min\{\theta(x_i), \sigma(y_i)\}\}. \end{aligned}$$

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Thus, $(\theta\cap\sigma)(x)\geq \sup\{\min_i\{\min\{\theta(x_i),\sigma(y_i)\}\}\}$, where $1\leq i\leq n$ and $x\in\sum_{i=1}^{n}x_i\gamma_i y_i$, $x_i,y_i\in R,\gamma_i\in\Gamma$. Therefore, $\theta\circ\sigma\subseteq\theta\cap\sigma$. □

Definition 4.6. A non-constant fuzzy Γ-hyperideal μ of a Γ-semihypergroup S is called a *fuzzy prime* Γ*-hyperideal* of S if for any two fuzzy Γ-hyperideals θ,σ of S,

$$\theta\Gamma\sigma\subseteq\mu \text{ implies } \theta\subseteq\mu \text{ or } \sigma\subseteq\mu.$$

Example 9. The following Γ-semihyperring is given in [19]. Let $R=\{a,b,c,d\}$, $\Gamma=\mathbb{Z}_2$ and $\alpha=\overline{0},\ \beta=\overline{1}$. Then R is a Γ-semihyperring with the following hyperoperations:

$\oplus$	a	b	c	d
a	{a,b}	{a,b}	{c,d}	{c,d}
b	{a,b}	{a,b}	{c,d}	{c,d}
c	{c,d}	{c,d}	{a,b}	{a,b}
d	{c,d}	{c,d}	{c,d}	{a,b}

β	a	b	c	d
a	{a,b}	{a,b}	{a,b}	{a,b}
b	{a,b}	{a,b}	{a,b}	{a,b}
c	{a,b}	{a,b}	{c,d}	{c,d}
d	{a,b}	{a,b}	{c,d}	{c,d}

For every $x,y\in R$ we define $x\alpha y=\{a,b\}$. Now, we define the fuzzy subset μ of R as follows:

$$\mu(x)=\begin{cases}\frac{5}{7} & \text{if } x=a \text{ or } x=b\\ \frac{1}{5} & \text{if } x=c \text{ or } x=d.\end{cases}$$

Then, μ is a prime Γ-hyperideal of R.

The proof of the following theorem is similar to the proof of Theorem 6.3 in [13]. Here, we give the proof for the sake of completeness.

Theorem 4.7. *Let I be a Γ-hyperideal of a Γ-semihyperring R, $t\in[0,1)$ and μ be the fuzzy subset of R defined by*

$$\mu(x)=\begin{cases}1 & \text{if } x\in I\\ t & \text{if } x\notin I.\end{cases}$$

Then μ is a fuzzy prime Γ-hyperideal of R if and only if I is a prime Γ-hyperideal of R.

Proof. Suppose that I is prime. Clearly, μ is a fuzzy Γ-hyperideal of R. Let θ,σ be fuzzy Γ-hyperideals of R such that $\theta\Gamma\sigma\subseteq\mu$, $\theta\not\subseteq\mu$ and $\sigma\not\subseteq\mu$. Thus, there exist $x,y\in R$ such that $\mu(x)<\theta(x)$ and $\mu(y)<\theta(y)$. Since I is prime, we obtain $x\Gamma R\Gamma y\not\subseteq I$. So there exist $r\in R$ and $\gamma_1,\gamma_2\in\Gamma$ such that $x\gamma_1 r\gamma_2 y\not\subseteq I$. Now, there exists $z\in x\gamma_1 r\gamma_2 y$ such that $\mu(z)=t$. Therefore, we obtain

$$\begin{aligned} t=\mu(z) &\geq (\theta\Gamma\sigma)(z)\\ &\geq \min\{\theta(x),\sigma(u)\}\ \ (\text{where } z\in x\gamma y)\\ &\geq \min\{\theta(x),\sigma(y)\}\\ &> \min\{\mu(x),\mu(y)\}=t \end{aligned}$$

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This is a contradiction. Thus, μ is prime.

Conversely, suppose that μ is a fuzzy prime Γ-hyperideal and A, B be two Γ-hyperideals of R such that $A\Gamma B \subseteq I$, $A \not\subseteq I$ and $B \not\subseteq I$. So there exist $a \in A \setminus I$ and $b \in B \setminus I$. Now, we define fuzzy subsets θ, σ of R as follows:

$$\theta(x) = \begin{cases} 1 & \text{if } x \in A \\ t & \text{if } x \notin A. \end{cases} \quad \text{and} \quad \sigma(x) = \begin{cases} 1 & \text{if } x \in B \\ t & \text{if } x \notin B. \end{cases} .$$

Then θ, σ are fuzzy Γ-hyperideals of R such that $\theta\Gamma\sigma \subseteq \mu$. On the other hand, we have $\theta(a) = 1 > t = \mu(a)$, which implies that $\theta \not\subseteq \mu$. Similarly, we obtain $\sigma \not\subseteq \mu$. This is a contradiction. Thus, I is prime. □

Let I be a Γ-hyperideal of a Γ-semihyperring R. Then, the characteristic function χ_I of I is a fuzzy prime Γ-hyperideal of R if I is prime.

Theorem 4.8. *Let R be a Γ-semihyperring and μ be a non-empty subset of R. Then the following statements are equivalent:*

(1) μ is a fuzzy prime Γ-hyperideal of R.

(2) For any $t \in [0, 1]$, the level subset $U(\mu; t)$ is a prime Γ-hyperideal of R when it is non-empty.

Proof. The proof is similar to the proof of Theorem 3.4. □

5 Image and inverse image of homomorphisms

Definition 5.1. Let R and R' be Γ and Γ'-semihyperrings, respectively, $\varphi : R \longrightarrow R'$ and $f : \Gamma \longrightarrow \Gamma'$ be two maps. Then, (φ, f) is called a (Γ, Γ')-*homomorphism* if

(1) $\varphi(x + y) = \{\varphi(t) \mid t \in x + y\} \subseteq \varphi(x) + \varphi(y)$,

(2) $\varphi(x\alpha y) = \{\varphi(t) \mid t \in x\alpha y\} \subseteq \varphi(x)f(\alpha)\varphi(y)$,

(3) $f(x + y) = f(x) + f(y)$.

In the above definition if $\varphi(x + y) = \varphi(x) + \varphi(y)$ and $\varphi(x\alpha y) = \varphi(x)f(\alpha)\varphi(y)$ then (φ, f) is called a *strong* (Γ, Γ')-*homomorphism*. An ordered set (φ, f) is called an *epimorphism*, if $\varphi : R \longrightarrow R'$ and $f : \Gamma \longrightarrow \Gamma'$ are surjective and is called a (Γ, Γ')-*isomorphism* if $\varphi : R \longrightarrow R'$ and $f : \Gamma \longrightarrow \Gamma'$ are bijective. In the following examples we consider homomorphisms between Γ-semihyperrings.

Example 10. [18]. Let $A_0 = [0, 1]$, $A_1 = (1, 2)$, $A_2 = [2, 3)$ and $A_n = [n, n+1)$ for every $n \in \mathbb{Z}$, $n \notin \{0, 1\}$. Suppose that $R_1 = R_2 = \bigcup_{n\in\mathbb{Z}} A_n$. Then R_1 is a $2\mathbb{Z}$-semihyperring and R_2 is a $\mathbb{Z}$-semihyperring. Suppose that $x \in R_1$. Then, there exists $n \in \mathbb{Z}$ such that $x \in A_n$. We define $\varphi : R_1 \longrightarrow R_2$ and $f : 2\mathbb{Z} \longrightarrow \mathbb{Z}$ as follow:

$$\varphi(x) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases} \quad \text{and} \quad f = 0.$$

It is easy to see that (φ, f) is a $(2\mathbb{Z}, \mathbb{Z})$-homomorphism.

EXAMPLE 11. [18]. Let $A_n = [n, n+1)$, for every $n \in \mathbb{Z}$ and $R = \bigcup_{n\in\mathbb{Z}} A_n$. Then for every $x \in R$ there exists $n \in \mathbb{Z}$ such that $x \in A_n$. We define $\varphi : R \longrightarrow R$ and $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ as follow:

$$\varphi(x) = n \text{ and } f(x) = x, \text{ where } x \in A_n.$$

It is easy to see that (φ, f) is a $(\mathbb{Z}, \mathbb{Z})$-homomorphism.

Let φ be a mapping from a set X to a set Y. Let μ be a fuzzy subset of X and λ be a fuzzy subset of Y. Then the *inverse image* $\varphi^{-1}(\lambda)$ of λ is the fuzzy subset of X defined by $\varphi^{-1}(\lambda)(x) = \lambda(\varphi(x))$ for all $x \in X$. The *image* $\varphi(\mu)$ of μ is the fuzzy subset of Y defined by

$$\varphi(\mu)(y) = \begin{cases} \sup\{\mu(t) \mid t \in \varphi^{-1}(y)\} & \text{if } \varphi^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

for all $y \in Y$. It is not difficult to see that the following assertions hold:

(1) If $\{\lambda_i\}_{i\in I}$ be a family of fuzzy subsets of Y, then

$$\varphi^{-1}\left(\bigcup_{i\in I}\lambda_i\right) = \bigcup_{i\in I}\varphi^{-1}(\lambda_i) \text{ and } \varphi^{-1}\left(\bigcap_{i\in I}\lambda_i\right) = \bigcap_{i\in I}\varphi^{-1}(\lambda_i).$$

(2) If μ is a fuzzy subset of X, then $\mu \subseteq \varphi^{-1}(\varphi(\mu))$. Moreover, if φ is one to one, then $\varphi^{-1}(\varphi(\mu)) = \mu$.

(3) If λ is a fuzzy subset of Y, then $\varphi(\varphi^{-1}(\lambda)) \subseteq \lambda$. Moreover, if φ is onto, then $\varphi(\varphi^{-1}(\lambda)) = \lambda$.

Proposition 5.2. *Let R be a Γ-semihyperring and R' be a Γ'-semihyperring. Let (φ, f) be a strong (Γ, Γ')-homomorphism from R to R'. Then*

(1) If λ is a fuzzy Γ-hyperideal of R', then $\varphi^{-1}(\lambda)$ is a fuzzy Γ-hyperideal of R, too.

(2) If (φ, f) is an epimorphism and μ is a fuzzy Γ-hyperideal of S, then $\varphi(\mu)$ is a fuzzy Γ-hyperideal of R', too.

Proof. (1) Suppose that $x, y \in S$ and $\gamma \in \Gamma$. Then we have

$$\begin{aligned} \inf_{z\in x+y}\left\{\varphi^{-1}(\lambda)(z)\right\} &= \inf_{z\in x+y}\{\lambda(\varphi(z))\} \\ &\geq \inf_{\varphi(z)\in\varphi(x+y)}\{\lambda(\varphi(z))\} \\ &\geq \inf_{\varphi(z)\in\varphi(x)+\varphi(y)}\{\lambda(\varphi(z))\} \\ &\geq \min\{\lambda(\varphi(x)), \lambda(\varphi(y))\} \\ &= \min\{\varphi^{-1}(\lambda)(x), \varphi^{-1}(\lambda)(y)\}, \end{aligned}$$

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and

$$\begin{aligned}\inf_{z\in x\gamma y}\left\{\varphi^{-1}(\lambda)(z)\right\} &= \inf_{z\in x\gamma y}\{\lambda(\varphi(z))\} \\ &\geq \inf_{\varphi(z)\in\varphi(x\gamma y)}\{\lambda(\varphi(z))\} \\ &\geq \inf_{\varphi(z)\in\varphi(x)f(\gamma)\varphi(y)}\{\lambda(\varphi(z))\} \\ &\geq \lambda(\varphi(y)) \\ &= \varphi^{-1}(\lambda)(y).\end{aligned}$$

Therefore, $\varphi^{-1}(\lambda)$ is a fuzzy Γ-hyperideal of R.

(2) Suppose that $x', y' \in R'$ and $\gamma' \in \Gamma'$. Then there exist $x, y \in R$ and $\gamma \in \Gamma$ such that $\varphi(x) = x',\ \varphi(y) = y'$ and $f(\gamma) = \gamma'$. Now, we have

$$\begin{aligned}\inf_{z'\in x'\gamma' y'}\{\varphi(\mu)(z')\} &= \inf_{z'\in x'\gamma' y'}\left\{\sup_{c\in\varphi^{-1}(z')}\{\mu(c)\}\right\} \\ &\geq \inf_{z'\in x'\gamma' y'}\{\mu(z)\} \text{ (if we let } \varphi(z) = z') \\ &= \inf_{\varphi(z)\in\varphi(x)f(\gamma)\varphi(y)}\{\mu(z)\} \\ &= \inf_{\varphi(z)\in\varphi(x\gamma y)}\{\mu(z)\} \\ &= \inf_{z\in x\gamma y}\{\mu(z)\} \\ &\geq \mu(y).\end{aligned}$$

So, for every $y \in \varphi^{-1}(\dot{y})$, we have

$$\inf_{z'\in x'\gamma' y'}\{\varphi(\mu)(z')\} \geq \mu(y)$$

which implies that

$$\inf_{z'\in x'\gamma' y'}\{\varphi(\mu)(z')\} \geq \sup_{y\in\varphi^{-1}(y')}\{\mu(y)\}$$

and so

$$\inf_{z'\in x'\gamma' y'}\{\varphi(\mu)(z')\} \geq \varphi(\mu)(y).$$

Similarly, we have

$$\inf_{z'\in x'+y'}\{\varphi(\mu)(z')\} \geq \min\{\varphi(\mu)(x'), \varphi(\mu)(y')\}.$$

Therefore, $\varphi(\mu)$ is a fuzzy Γ-hyperideal of R'. □

Proposition 5.3. *Let R be a Γ-semihyperring and R' be a Γ'-semihyperring. Let (φ, f) be a strong (Γ, Γ')-homomorphism from R onto R'. If λ is a fuzzy prime Γ-hyperideal of R', then $\varphi^{-1}(\lambda)$ is a fuzzy prime Γ-hyperideal of R, too.*

Proof. The proof is similar to the proof of Proposition 5.2. □

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