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A ZETA-BARNES FUNCTION ASSOCIATED TO GRADED MODULES

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Abstract. Let K be a field and let $S = \bigoplus_{n\geq 0} S_n$ be a positively graded K-algebra. Given $M = \bigoplus_{n\geq 0} M_n$, a finitely generated graded S-module, and $w > 0$, we introduce the function $\zeta_M(z, w) := \sum_{n=0}^{\infty} \frac{H(M,n)}{(n+w)^z}$, where $H(M, n) := \dim_K M_n$, $n \geq 0$, is the Hilbert function of M, and we study the relations between the algebraic properties of M and the analytic properties of $\zeta_M(z, w)$. In particular, in the standard graded case, we prove that the multiplicity of M is $e(M) = (m - 1)! \lim_{w\searrow 0} Res_{z=m}\zeta_M(z, w)$.

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Introduction

Let K be a field and let S be a positively graded K-algebra. Let M be a finitely generated S-module of dimension $m \geq 0$. Given a real number $w > 0$, we consider the *zeta-Barnes type* (see [3]) function

$$\zeta_M(z, w) := \sum_{n=0}^{\infty} \frac{H(M, n)}{(n + w)^z},$$

where $H(M, n) := \dim_K M_n$, $n \geq 0$, is the *Hilbert function* of M. According to a Theorem of Serre, see for instance [5, Theorem 4.4.3], there exists a positive integer D such that

$$H(M, n) = d_{M,m-1}(n)n^m + \cdots + d_{M,1}(n)n + d_{M,0}(n),\ (\forall)n \gg 0,$$

where $d_{M,j}(n + D) = d_{M,j}(n)$, $(\forall)n \geq 0$. In Theorem 1.1 we show that

$$\zeta_M(z, w) = \theta_M(z, w) + D^{-z} \sum_{k=0}^{m-1} \sum_{j=0}^{D-1} d_{M,k}(j + \alpha(M)) \sum_{\ell=0}^{k} \binom{k}{\ell} (-w)^{\ell} D^{k-\ell} \zeta(z - k + \ell, \frac{j + \alpha(M) + w}{D}),$$

where $\alpha(M) := \min\{n_0 \ :\ H(M, n) = q_M(n),\ (\forall)n \geq n_0\}$, $\theta_M(z, w) := \sum_{n=0}^{\alpha(M)-1} \frac{H(M,n)}{(n+w)^z}$ and $\zeta(z, w) = \sum_{n=0}^{\infty} \frac{1}{(n+w)^z}$ is the *Hurwitz-zeta* function. Consequently, $\zeta_M(z, w)$ is a meromorphic function on the complex plane with the poles in the set $\{1, 2, \dots, m\}$ which are simple with residues

$$Res_{z=k+1}\zeta_M(z, w) = \frac{1}{D} \sum_{\ell=k}^{m-1} \binom{\ell}{k} (-w)^{\ell-k} \sum_{j=0}^{D-1} d_{M,k}(j),\ 0 \leq k \leq m - 1.$$

Other properties of $\zeta_M(z, w)$ are given in Proposition 1.1, 1.2 and Corollary 1.3, 1.4.

We also consider the function $\zeta_M(z) := \lim_{w\searrow 0}(\zeta_M(z, w) - H(M, 0)w^{-z})$. In Proposition 1.5 we compute $\zeta_M(z)$ and its residues. In Proposition 1.6 we prove that S is Gorenstein if and only if $\zeta_{\omega_S}(z, w) = \zeta_S(z, w - a(S))$, where S is Cohen-Macaulay with the canonical module ω_S.

In the second section, we apply the results obtained in the first section in the case when $S = K[x_1,\ldots,x_r]$ is the ring of polynomials with $\deg(x_i) = a_i$, $1 \leq i \leq r$. Given a graded S-module M, we compute the residues of $\zeta_M(z,w)$ and $\zeta_M(z)$ in terms of the graded Betti numbers of M and the Bernoulli-Barnes polynomial associated to $(a_1,\ldots,a_r)$, see Corollary 2.2.

In the third section, we consider the standard graded case and we prove that the multiplicity of M, is

$$e(M) = (m-1)! \lim_{w \searrow 0} Res_{z=m}\zeta_M(z,w),$$

see Corollary 3.3. In the fourth section, we outline the non-graded case and we give a formula for the multiplicity of the module with respect to an ideal, see Proposition 4.1.

1. Graded modules over positively graded K-algebras

Let K be a field and let S be a positively graded K-algebra, that is

$$S := \bigoplus_{n\geq 0} S_n, S_0 = K,$$

and S is finitely generated over K. Assume $S = K[u_1,\ldots,u_r]$, where $u_i \in S$ are homogeneous elements of $\deg(u_i) = a_i$. Let

$$M = \bigoplus_{n\in\mathbb{N}} M_n$$

be a finitely generated graded S-module with the Krull dimension $m := \dim(M)$. The *Hilbert function* of M is

$$H(M,-):\mathbb{N}\to\mathbb{N},\ H(M,n) := \dim_K(M_n),\ n\in\mathbb{N}.$$

The *Hilbert series* of M is

$$H_M(t) := \sum_{n=0}^{\infty} H(M,n)t^n \in \mathbb{Z}[[t]].$$

According to the Hibert-Serre's Theorem [1, Theorem 11.1] and [5, Exercise 4.4.11]

$$H_M(t) = \frac{h_M(t)}{(1-t^{a_1})\cdots(1-t^{a_r})},$$

where $h_M(t) \in \mathbb{Z}[t]$. According to Serre's Theorem [5, Theorem 4.4.3] and [5, Exercise 4.4.11] there exists a quasi-polynomial $q_M(n)$ of degree $m-1$ with the period $D := \operatorname{lcm}(a_1,\ldots,a_r)$ such that

$$H(M,n) = q_M(n) = d_{M,m-1}(n)n^{m-1} + \cdots + d_{M,1}(n)n + d_{M,0}(n),\ (\forall)n \gg 0, \tag{1.1}$$

where $d_{M,k}(n+D) = d_{M,k}(n)$ for any $n \geq 0$ and $0 \leq k \leq m-1$. We denote

$$\alpha(M) := \min\{n_0\ :\ H(M,n) = q_M(n),\ (\forall)n \geq n_0\}. \tag{1.2}$$

Let $w > 0$ be a real number. We denote

$$\zeta_M(z,w) := \sum_{n\geq 0} \frac{H(M,n)}{(n+w)^z},\ z\in\mathbb{C}, \tag{1.3}$$

and we call it the *Zeta-Barnes type function* associated to M and w. We also denote

$$\theta_M(z,w) := \sum_{n=0}^{\alpha(M)-1} \frac{H(M,n)}{(n+w)^z},\ z\in\mathbb{C}. \tag{1.4}$$

The function $\theta_M(z,w)$ is entire. Moreover, M is Artinian if and only if $\zeta_M(z,w) = \theta_M(z,w)$. Also, $\alpha(M) = 0$ if and only if $\theta_M(z,w) = 0$.

Theorem 1.1. *We have that*

$$\zeta_M(z,w)=\theta_M(z,w)+D^{-z}\sum_{k=0}^{m-1}\sum_{j=0}^{D-1}d_{M,k}(j+\alpha(M))\sum_{\ell=0}^{k}\binom{k}{\ell}(-w)^{\ell}D^{k-\ell}\zeta(z-k+\ell,\frac{j+\alpha(M)+w}{D}),$$

where $\zeta(z,w)=\sum_{n=0}^{\infty}\frac{1}{(n+w)^z}$ *is the* Hurwitz-zeta *function.*

Moreover, $\zeta_M(z,w)$ *is a meromorphic function on* $\mathbb{C}$ *with the poles in the set* $\{1,2,\ldots,m\}$ *which are simple with residues*

$$R_M(w,k+1):=Res_{z=k+1}\zeta_M(z,w)=\frac{1}{D}\sum_{\ell=k}^{m-1}\binom{\ell}{k}(-w)^{\ell-k}\sum_{j=0}^{D-1}d_{M,k}(j),\ 0\leq k\leq m-1.$$

Proof. The proof follows the line of the proof of [6, Proposition 3.2]. According to (1.1), (1.2), (1.3) and (1.4), we have

$$\zeta_M(z,w)=\theta_M(z,w)+\sum_{n=\alpha(M)}^{\infty}\frac{q_M(n)}{(n+w)^z}=\theta_M(z,w)+\sum_{k=0}^{m-1}\sum_{n=\alpha(M)}^{\infty}\frac{d_{M,k}(n)n^k}{(n+w)^z}. \tag{1.5}$$

For any $0\leq k\leq m-1$, we write

$$n^k=(n+w-w)^k=\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}(n+w)^{k-\ell}w^{\ell}. \tag{1.6}$$

By (1.5) and (1.6) and the fact that $d_{M,k}(n+D)=d_{M,k}(n)$, $(\forall)n,k$, it follows that

$$\zeta_M(z,w)=\theta_M(z,w)+\sum_{k=0}^{m-1}\sum_{n=\alpha(M)}^{\infty}d_{M,k}(n)\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}w^{\ell}\frac{1}{(n+w)^{z-k+\ell}}=\theta_M(z,w)+$$

$$+\sum_{k=0}^{m-1}\sum_{j=0}^{D-1}d_{M,k}(j+\alpha(M))\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}w^{\ell}\sum_{t=0}^{\infty}\frac{1}{(j+tD+\alpha(M)+w)^{z-k+\ell}}. \tag{1.7}$$

On the other hand,

$$\sum_{t=0}^{\infty}\frac{1}{(j+tD+\alpha(M)+w)^{z-k+\ell}}=\sum_{t=0}^{\infty}\frac{D^{-z+k-\ell}}{(t+\frac{j+\alpha(M)+w}{D})^{z-k+\ell}}=D^{-z+k-\ell}\zeta(z-k+\ell,\frac{j+\alpha(M)+w}{D}). \tag{1.8}$$

Replacing (1.8) in (1.7) we get the required result.

The last assertion is a consequence of the fact that the Hurwitz-zeta function $\zeta(z-k,w)$ is a meromorphic function and has a simple pole at $k+1$ with the residue 1 and, also, $\theta_M(z,w)$ is an entire function. □

Proposition 1.1. *Let* $0\to U\to M\to N\to 0$ *be a graded short exact sequence of* S*-modules. Then*

$$\zeta_M(z,w)=\zeta_U(z,w)+\zeta_N(z,w).$$

Proof. It follows from $H(M,n)=H(U,n)+H(N,n)$, $n\geq 0$, and (1.3). □

Proposition 1.2. *For any* $k\geq 0$*, it holds that* $\zeta_{M(-k)}(z,w)=\zeta_M(z,w+k)$.

Proof. Since $M(-k)_n=M_{n-k}$, it follows that $H(M(-k),n)=0$ for all $0\leq n<k$ and $H(M(-k),n)=H(M,n-k)$, for all $n\geq k$. Consequently, by (1.3), we get

$$\zeta_{M(-k)}(z,w)=\sum_{n=0}^{\infty}\frac{H(M(-k),n)}{(n+w)^z}=\sum_{n=k}^{\infty}\frac{H(M,n-k)}{(n+w)^z}=\sum_{n=0}^{\infty}\frac{H(M,n)}{(n+k+w)^z}=\zeta_M(z,w+k).$$

□

Corollary 1.3. *If $f \in S_k$ is regular on M, then*

$$\zeta_{\frac{M}{fM}}(z,w) = \zeta_M(z,w) - \zeta_M(z,w+k).$$

Proof. We consider the short exact sequence

$$0 \to M(-k) \xrightarrow{\cdot f} M \to \frac{M}{fM} \to 0.$$

The conclusion follows from Proposition 1.1 and Proposition 1.2. □

Corollary 1.4. *If $f_1,\ldots,f_p \in S$ is a regular sequence on M, consisting of homogeneous elements with* $\deg(f_i) = k_i$*, then*

$$\zeta_{\frac{M}{(f_1,\ldots,f_p)M}}(z) = \zeta_M(z,w) + \sum_{\ell=1}^{p}(-1)^\ell \sum_{1\le i_1<\cdots<i_\ell\le p} \zeta_M(z,w+k_{i_1}+\ldots+k_{i_\ell}).$$

Proof. It follows from Corollary 1.3, using induction on $k \geq 1$. □

Let

$$\zeta_M(z) := \lim_{w\searrow 0}(\zeta_M(z,w) - H(M,0)w^{-z}) = \sum_{n=1}^{\infty}\frac{H(M,n)}{n^z}. \tag{1.9}$$

Note that $\zeta_M(z)$ codify all the information about the Hilbert function of M with the exception of $H(M,0)$. Let

$$\theta_M(z) := \sum_{n=1}^{\alpha(M)-1}\frac{H(M,n)}{n^z}. \tag{1.10}$$

Note that $\theta_M(z)$ is an entire function. Also, if $\alpha(M) \leq 1$ then $\theta_M(z)$ is identically zero.

Proposition 1.5. *We have that*

$$\zeta_M(z) = \theta_M(z) + \sum_{k=0}^{m-1}\frac{1}{D^{z-k}}\sum_{j=0}^{D-1} d_{M,k}(j+\alpha(M))\zeta(z-k,\frac{j+\alpha(M)+1}{D}).$$

The function $\zeta_M(z)$ is meromorphic with poles at most in the set $\{1,\ldots,m\}$ which are all simple with residues

$$R_M(k+1) := Res_{z=k+1}\zeta_M(z) = \frac{1}{D}\sum_{j=0}^{D} d_{M,k}(j),\ 0 \leq k \leq m-1.$$

Proof. The proof is similar to the proof of Theorem 1.1, therefore we will omite it. Also, the result could be derived from the proof of [6, Proposition 3.4(i)]. □

Let $k \geq 1$ be an integer and let

$$M(k) := \bigoplus_{n=-k}^{\infty} M_{n+k}.$$

Given a real number $w > k$, we consider the function

$$\zeta_{M(k)}(z,w) := \sum_{n=-k}^{\infty}\frac{H(M,n+k)}{(n+w)^z} = \sum_{n=0}^{\infty}\frac{H(M,n)}{(n+w-k)^z} = \zeta_M(z,w-k). \tag{1.11}$$

Let $a(S) := \deg(H_S(t))$ be the a-invariant of S. Assume S is Gorenstein. Then, according to [5, Proposition 3.6.11], the canonical module of S, ω_S is isomorphic to $S(a(S))$. Consequently, we get $\zeta_{\omega_S}(z,w) = \zeta_S(z,w-a(S))$, where $w > \max\{0,a(s)\}$.

Proposition 1.6. *Let S be a Cohen-Macaulay domain with the canonical module ω_S. Then S is Gorenstein if and only if $\zeta_{\omega_S}(z,w) = \zeta_S(z,w-a(S))$.*

Proof. Note that $\zeta_{\omega_S}(z,w) = \zeta_S(z, w - a(S))$ is equivalent to $H_{\omega_S}(t) = t^{a(S)}H_S(t)$. Hence, according to [5, Theorem 4.4.5(2)], this is equivalent to S is Gorenstein. $\square$

Remark 1.7. Assume that $S = K[x_1,\ldots,x_r]$ is the ring of polynomials with $\deg(x_i) = a_i$, $1 \leq i \leq r$. The Hilbert series of S is

$$H_S(t) = \frac{1}{(1-t^{a_1})\cdots(1-t^{a_r})},$$

hence $a(S) = -(a_1 + \cdots + a_r)$. It is well known that S is Gorenstein, therefore

$$\omega_S \cong S(a(S)) = S(-a_1 - \cdots - a_r).$$

It follows that

$$\zeta_{\omega_S}(z,w) = \zeta_S(z, w + a_1 + \cdots + a_r),\ (\forall) w > 0.$$

In the next section we will discuss the case of graded modules over S.

2. Graded modules over the ring of polynomials.

Let $\mathbf{a} = (a_1,\ldots,a_r)$ be a sequence of positive integers. In the following, $S = K[x_1,\ldots,x_r]$ is the ring of polynomials in r indeterminates, with $\deg(x_i) = a_i$, $1 \leq i \leq r$. The *restricted partition function* associated to $\mathbf{a}$ is $p_{\mathbf{a}} : \mathbb{N} \to \mathbb{N}$,

$$p_{\mathbf{a}}(n) := \text{the number of integer solutions } (x_1,\ldots,x_r) \text{ of } \sum_{i=1}^{r} a_i x_i = n \text{ with } x_i \geq 0.$$

For a kindly introduction on the restricted partition function we reffer to [2]. One can easily see that $p_{\mathbf{a}}(n) = H(S,n)$, $(\forall) n \geq 1$, hence

$$\zeta_S(z,w) = \zeta_{\mathbf{a}}(z,w) := \sum_{n=0}^{\infty} \frac{p_{\mathbf{a}}(n)}{(n+w)^z} \tag{2.1}$$

is the *Zeta-Barnes function* associated to the sequence $\mathbf{a}$. We also have

$$\zeta_S(z) = \zeta_{\mathbf{a}}(z) := \lim_{w \searrow 0}(\zeta_{\mathbf{a}}(z,w) - w^z) = \sum_{n=1}^{\infty} \frac{p_{\mathbf{a}}(n)}{n^z}. \tag{2.2}$$

See [6] for further details on the properties of the function $\zeta_{\mathbf{a}}(z)$.

Proposition 2.1. *Let M be a finitely generated graded S-module. Then:*

(1) $\zeta_M(z,w) := \sum_{i=0}^{p}(-1)^i \sum_{j\geq i} \beta_{ij}(M)\zeta_{\mathbf{a}}(z, w+j)$, *where* $\beta_{ij}(M) := \dim_K(Tor_i(M,K))_j$ *are the* graded Betti numbers *of M and p is the* projective dimension *of M.*

(2) $\zeta_M(z) = \sum_{i=0}^{p}(-1)^i \sum_{j\geq \max\{i,1\}} \beta_{ij}(M)\zeta_{\mathbf{a}}(z,j) + \beta_{00}(M)\zeta_{\mathbf{a}}(z)$.

Proof. (1) Let

$$\mathbf{F} : 0 \to F_p \to \cdots \to F_1 \to F_0 \to M \to 0, \tag{2.3}$$

be the minimal free resolution of M. We have that $F_i = \bigoplus_{j\geq 0} S(-j)^{\beta_{ij}}$. By (2.1), Proposition 1.1 and Proposition 1.2, it follows that

$$\zeta_{F_i}(z,w) = \sum_{j\geq 0} \beta_{ij}\zeta_{\mathbf{a}}(z, w+j).$$

The result follows from Proposition 1.1 applied several times to the exact sequence (2.3). (2) By (2.1), it follows that

$$\lim_{w\searrow 0} \zeta_{\mathbf{a}}(z, j+w) = \zeta_{\mathbf{a}}(z,j),\ (\forall) j \geq 1. \tag{2.4}$$

Using (2.2), (2.4) and (1) we get the required result. $\square$

The Bernoulli numbers B_ℓ are defined by

$$\frac{z}{e^z-1}=\sum_{\ell=0}^{\infty} B_j \frac{z^\ell}{\ell!},$$

$B_0=1$, $B_1=-\frac{1}{2}$, $B_2=\frac{1}{6}$, $B_4=-\frac{1}{30}$ and $B_n=0$ if $n\geq 3$ is odd. For $k>0$ we have the Faulhaber's identity

$$1^k+2^k+\cdots+n^k=\frac{1}{k+1}\sum_{\ell=0}^{k}\binom{k+1}{\ell}B_\ell n^{1+k-\ell}.$$

The Bernoulli-Barnes polynomials $B_\ell(x;a_1,\ldots,a_r)$ are defined by

$$\frac{z^r e^{xz}}{(e^{a_1 z}-1)\cdots(e^{a_r z}-r)}=\sum_{\ell=0}^{\infty} B_\ell(x;a_1,\ldots,a_r)\frac{z^\ell}{\ell!}.$$

According to formula (3.9) in Ruijsenaars [8],

$$Res_{z=\ell}\zeta_{\mathbf{a}}(z,w)=\frac{(-1)^{r-\ell}}{(\ell-1)!(r-\ell)!}B_{r-\ell}(w;a_1,\ldots,a_r),\ 1\leq \ell\leq r. \tag{2.5}$$

The Bernoulli-Barnes numbers are defined by

$$B_\ell(a_1,\ldots,a_r):=B_\ell(0;a_1,\ldots,a_r).$$

The Bernoulli-Barnes numbers and the Bernoulli numbers are related by

$$B_\ell(a_1,\ldots,a_r)=\sum_{i_1+\cdots+i_r=\ell}\binom{\ell}{i_1,\ldots,i_r}B_{i_1}\cdots B_{i_r}a_1^{i_1-1}\cdots a_r^{i_r-1},$$

see Bayad and Beck [4, Page 2] for further details. According to [6, Theorem 3.10],

$$Res_{z=\ell}\zeta_{\mathbf{a}}(z)=\frac{(-1)^{r-\ell}}{(\ell-1)!(r-\ell)!}B_{r-\ell}(a_1,\ldots,a_r),\ 1\leq \ell\leq r. \tag{2.6}$$

Note that (2.6) can be deduced from (2.5).

Corollary 2.2. *Let M be a finitely generated graded S-module and $w>0$. Then*

(1) $R_M(w,\ell)=\sum_{i=0}^{p}\sum_{j\geq 0}\beta_{ij}(M)\frac{(-1)^{i+r-\ell}}{(\ell-1)!(r-\ell)!}B_{r-\ell}(w+j;a_1,\ldots,a_r),\ 1\leq\ell\leq r.$

(2) $R_M(\ell)=\sum_{i=0}^{p}\sum_{j\geq 0}\beta_{ij}(M)\frac{(-1)^{i+r-\ell}}{(\ell-1)!(r-\ell)!}B_{r-\ell}(j;a_1,\ldots,a_r),\ 1\leq\ell\leq r.$.

Proof. The results follow from Proposition 2.1 and the formulas (2.5) and (2.6). □

Example 2.3. Let $\mathbf{a}=(a_1,\ldots,a_r)$ be a sequence of positive integers, $D=\operatorname{lcm}(a_1,\ldots,a_r)$. We consider the ideal $I=(x_1^{\frac{D}{a_1}},\ldots,x_r^{\frac{D}{a_r}})\subset S$. Note that I is an Artinian complete intersection monomial ideal generated in degree D, w.r.t. the $\mathbf{a}$-grading. According to (2.2) and Corollary 1.4, we have

$$\zeta_{S/I}(z,w)=\theta_{S/I}(z,w)=\sum_{j=0}^{r}(-1)^j\binom{r}{j}\zeta_{\mathbf{a}}(z,w+Dj). \tag{2.7}$$

On the other hand, one can easily check that

$$H_{S/I}(t)=\frac{(1-t^D)^r}{(1-t^{a_1})\cdots(1-t^{a_r})}=(1+t^{a_1}+\cdots+t^{a_1(\frac{D}{a_1}-1)})\cdots(1+t^{a_r}+\cdots+t^{a_r(\frac{D}{a_r}-1)})$$

is a reciprocal polynomial of degree $Dr-a_1-\cdots-a_r$. The coefficient of t^n in $H_{S/I}(t)$ equals to

$$f_{\mathbf{a}}(n)=\#\{(x_1,\ldots,x_r)\in\mathbb{Z}^r\ :\ a_1x_1+\cdots+a_rx_r=n,\ 0\leq x_1<\frac{D}{a_1}-1,\ldots,0\leq x_r<\frac{D}{a_r}-1\}.$$

By (2.7) it follows that

$$\sum_{n=0}^{Dr-a_1-\cdots-a_r} f_{\mathbf{a}}(n)(n+w)^{-z} = \sum_{j=0}^{r}(-1)^j\binom{r}{j}\zeta_{\mathbf{a}}(z,w+Dj).$$

See Rødseth and Sellers [7] for further details on the coefficients $f_{\mathbf{a}}(n)$.

Example 2.4. Let $S = K[x_1, x_2]$ with $\deg(x_1) = 2$, $\deg(x_2) = 3$. Let $\mathbf{a} = (2, 3)$. The polynomial $f = x_1^3 - x_2^2 \in S$ is homogeneous of degree 6. Let $R = S/(f)$. R has the minimal graded free resolution

$$0 \to S(-6) \stackrel{\cdot f}{\to} S \to R \to 0 \tag{2.8}$$

It follows that the non-zero Betti numbers of R are $\beta_{00}(R) = 1$ and $\beta_{16}(R) = 1$. Let $w > 0$. According to (2.1) and Corollary 1.3 (or (2.8) and Proposition 2.1(1)) we have

$$\zeta_R(z,w) = \zeta_{\mathbf{a}}(z,w) - \zeta_{\mathbf{a}}(z,w+6) = \sum_{n=0}^{\infty}\frac{p_{\mathbf{a}}(n)}{(n+w)^z} - \sum_{n=0}^{\infty}\frac{p_{\mathbf{a}}(n)}{(n+w+6)^z} =$$

$$= \sum_{n=0}^{5}\frac{p_{\mathbf{a}}(n)}{(n+w)^z} + \sum_{n=6}^{\infty}\frac{p_{\mathbf{a}}(n)-p_{\mathbf{a}}(n-6)}{(n+w)^z} = \frac{1}{w^z} + \sum_{n=2}^{\infty}\frac{1}{(n+w)^z} = \frac{1}{w^z} + \zeta(z,w+2).$$

In particular, the Hilbert series of R is

$$H_R(t) = 1 + \sum_{n=2}^{\infty} t^n = 1 + \frac{t^2}{1-t} = \frac{t^2-t+1}{1-t},$$

hence $\alpha(R) = a(R) = 1$. It follows that $\theta_R(z,w) = \frac{1}{w^z}$. Also,

$$\zeta_R(z) = \lim_{w\searrow 0}(\zeta_R(z,w) - \frac{1}{w^z}) = \zeta(z,2) \text{ and } \theta_R(z) = 0.$$

3. The standard graded case

Let S be a standard graded K-algebra, that is $S = \bigoplus_{n\geq 0} S_n$, $S_0 = K$ and $S = K[S_1]$. Let M be a finitely generated graded S-module. According to the Hilbert-Serre's Theorem, it holds that

$$H_M(t) = \frac{h_M(t)}{(t-1)^m}, \tag{3.1}$$

where $h_M \in \mathbb{Z}[t]$, $m = \dim(M)$ and $h_M(1) \neq 0$. Also, there exists a polynomial $P_M(t) \in \mathbb{Z}[t]$ of degree $m-1$, such that

$$H(M,n) = P_M(n),\ (\forall)n \gg 0,$$

which is called the *Hilbert polynomial* of M.

The number $e(M) := h_M(1)$ is called the *multiplicity* of the module M.

Proposition 3.1. *If $P_M(t) = d_{M,m-1}t^{m-1} + \cdots + d_{M,1}t + d_{M,0}$ is the Hilbert polynomial of M, then*

$$\zeta_M(z,w) = \theta_M(z,w) + \sum_{k=0}^{m-1} d_{M,k}\sum_{\ell=0}^{k}\binom{k}{\ell}(-w)^{\ell}\zeta(z-k+\ell,\alpha(M)+w)$$

is a meromorphic function on $\mathbb{C}$ with the poles in the set $\{1, 2, \ldots, m\}$ which are simple with residues

$$R_M(w,k+1) := Res_{z=k+1}\zeta_M(z,w) = \sum_{\ell=k}^{m-1}\binom{\ell}{k}(-w)^{\ell-k}d_{M,\ell},\ 0 \leq k \leq m-1.$$

Proof. It is the particular case of Theorem 1.1 for $\mathbf{a} = (1, \ldots, 1)$. □

Proposition 3.2. *We have that*

$$\zeta_M(z) = \theta_M(z) + \sum_{k=0}^{m-1} d_{M,k}\zeta(z-k+\ell, \alpha(M)+1)$$

is a meromorphic function on $\mathbb{C}$ *with the poles in the set* $\{1,2,\ldots,m\}$ *which are simple with residues*

$$R_M(\ell+1) := Res_{z=\ell+1}\zeta_M(z) = d_{M,\ell}.$$

Proof. It is the particular case of Proposition 1.5 for $\mathbf{a} = (1,\ldots,1)$. □

If $\dim M \geq 1$, then we can write

$$P_M(t) = \sum_{k=0}^{m-1} (-1)^k e_k(M) \binom{t+m-1-k}{m-1-k}. \tag{3.2}$$

According to [5, Proposition 4.1.9], we have

$$e_k(M) = \frac{h_M^{(k)}(t)}{k!},\ (\forall) 0 \leq k \leq m-1. \tag{3.3}$$

Corollary 3.3. *If* $m = \dim M \geq 1$*, then*

$$e(M) = e_0(M) = (m-1)! d_{M,m-1} = (m-1)! R_M(m).$$

Proof. It follows from (3.2), (3.3) and Proposition 3.2. □

The *higher iterated Hilbert functions* $H_i(M,n)$, $i \in \mathbb{N}$, of a finitely generated S-module M are defined recursively as follows:

$$H_0(M,n) := H(M,n), \text{ and } H_i(M,n) = \sum_{j=0}^{n} H_{i-1}(M,n),\ i \geq 1. \tag{3.4}$$

The functions $H_i(M,n)$ are of polynomial type of degree $m+i-1$, hence

$$H_i(M,n) = P_i(M,n) := d^i_{M,m+i-1} n^{m+i-1} + \cdots + d^i_{M,1} n + d^i_{M,0},\ (\forall) n \gg 0. \tag{3.5}$$

We define the *higher Zeta-Barnes type functions* associated to M as follows:

$$\zeta^i_M(z,w) := \sum_{n=0}^{\infty} \frac{H_i(M,n)}{(n+w)^z},\ i \geq 0. \tag{3.6}$$

and

$$\zeta^i_M(z) = \lim_{w \searrow 0} (\zeta^i_M(z,w) - H(M,0) w^{-z}),\ i \geq 0. \tag{3.7}$$

Let

$$\alpha^i(M) := \min\{n_0 \in \mathbb{N} \ :\ H_i(M,n) = P_i(M,n), (\forall) n \geq n_0\}.$$

We define

$$\theta^i_M(z,w) = \sum_{n=0}^{\alpha^i(M)-1} \frac{H_i(M,n)}{(n+w)^z} \text{ and } \theta^i_M(z) = \sum_{n=1}^{\alpha^i(M)-1} \frac{H_i(M,n)}{n^z}.$$

Proposition 3.4. *With the above notations:*

(1) $\zeta^i_M(z,w) = \theta^i_M(z,w) + \sum_{k=0}^{m+i-1} d^i_{M,k} \sum_{\ell=0}^{k} \binom{k}{\ell} (-w)^\ell \zeta(z-k+\ell, \alpha^i(M)+w)$ *is a meromorphic function on* $\mathbb{C}$ *with the poles in the set* $\{1,2,\ldots,m+i\}$ *which are simple with residues*

$$R^i_M(w,k+1) := Res_{z=k+1}\zeta_M(z,w) = \sum_{\ell=k}^{m+i-1} \binom{\ell}{k} (-w)^{\ell-k} d^i_{M,\ell},\ 0 \leq k \leq m+i-1.$$

(2) $\zeta^i_M(z)=\theta^i_M(z)+\sum_{k=0}^{m+i-1}d^i_{M,k}\zeta(z-k+\ell,\alpha^i(M)+1)$ *is a meromorphic function on* $\mathbb{C}$ *with the poles in the set* $\{1,2,\ldots,m+i\}$ *which are simple with residues*

$$R^i_M(k+1):=Res_{z=k+1}\zeta_M(z)=d^i_{M,k},\ 0\leq k\leq m+i-1.$$

Proof. Is similar to Proposition 3.1 and Proposition 3.2. □

Corollary 3.5. *We have that* $e(M)=m!R^1_M(m+1)$.

Proof. According to [5, Remark 4.1.6], $H_1(M,n)=d^1_{M,m}n^m+\cdots+d^1_{M,1}n+d^1_{M,0}$, $(\forall)n\gg 0$, and $e(M)=m!d^1_{M,m}$. Now, apply Proposition 3.4(2). □

Remark 3.6. Let $S=K[x_1,\ldots,x_r]$ and $I\subset S$ a graded ideal. We say that S/I has a *pure resolution* of type $(d_1,\ldots,d_p)$ if its minimal resolution is

$$0\to S(-d_p)^{\beta_p}\to\cdots\to S(-d_1)^{\beta_1}\to S\to S/I\to 0,$$

where p is the projective dimension of S/I, $d_1<d_2<\cdots<d_p$ and $\beta_i=\sum_{j\geq 0}\beta_{ij}(S/I)$, $1\leq i\leq p$, are the Betti numbers of S/I. According to Corollary 3.3, $e(S/I)=R_{S/I}(m)$, where $m=\dim(S/I)$. On the other hand, according to Corollary 2.2(2), we have

$$R_{S/I}(m)=\sum_{i=0}^{p}\beta_i\frac{(-1)^{r-m}}{(m-1)!(r-m)!}B_{r-m}(d_i;1,1,\ldots,1). \tag{3.8}$$

Suppose S/I is Cohen-Macaulay and has a pure resolution of type $(d_1,\ldots,d_p)$. According to [5, Theorem 4.1.15],

$$\beta_i=(-1)^{i+1}\prod_{j\neq i}\frac{d_j}{d_j-d_i}\text{ and }e(S/I)=\frac{d_1d_2\cdots d_p}{p!}. \tag{3.9}$$

The Ausländer-Buchsbaum formula [5, Theorem 1.3.3] implies $p=r-m$, hence (3.8) and (3.9) give the identity:

$$\sum_{i=0}^{p}(-1)^{i+1}\prod_{j\neq i}\frac{d_j}{d_j-d_i}B_p(d_i;1,1,\ldots,1)=(m-1)!(-1)^p d_1d_2\cdots d_p.$$

4. The non-graded case

Let $(S,\mathfrak{m},K)$ be a Noetherian local ring, where $\mathfrak{m}$ is the maximal ideal of S and $K=S/\mathfrak{m}$ is the residue field. Let M be a finitely generated S-module, with $m=\dim(M)$, and let $I\subset S$ be an ideal such that $\mathfrak{m}^nM\subset IM$ for some $n\geq 1$. The associated graded ring is

$$\operatorname{gr}_I(S)=\bigoplus_{n\geq 0}\frac{I^n}{I^{n+1}}=\frac{S}{I}\oplus\frac{I}{I^2}\oplus\cdots.$$

The associated graded module of M, with respect to I, is

$$\operatorname{gr}_I(M):=\bigoplus_{n\geq 0}\frac{I^nM}{I^{n+1}M},$$

which has a structure of a $\operatorname{gr}_I(S)$-module. According to [5, Theorem 4.5.6], it holds that

$$\dim(\operatorname{gr}_I(M))=\dim(M)=m.$$

The *Hilbert-Samuel function* of M, w.r.t. I, is

$$\chi_M(n):=H_1(\operatorname{gr}_I(M),n)=\sum_{i=0}^{n}H(\operatorname{gr}_I(M),i)=\dim_K\frac{M}{I^{n+1}M},\ (\forall)n\geq 0.$$

The *multiplicity* of M with respect to I is $e(M,I) := e(\mathrm{gr}_I(M))$. For $n \gg 0$, according to [5, Remark 4.1.6], we have that

$$\chi_M(n) = \frac{e(M,I)}{m!}n^m + \text{ terms in lower powers of } n. \tag{4.1}$$

We consider the functions

$$\zeta^i_{M,I}(z,w) := \zeta^i_{\mathrm{gr}_I(M)}(z,w) \text{ and } \zeta^i_{M,I}(z) := \zeta^i_{\mathrm{gr}_I(M)}(z),\ i \geq 0. \tag{4.2}$$

Proposition 4.1. *It holds that*

$$e(M,I) = m! Res_{z=m+1}\zeta^1_{M,I}(z).$$

Proof. This follows from (4.1), (4.2) and Corollary 3.5. □

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