

# ON THE STANLEY DEPTH OF THE PATH IDEAL OF A CYCLE GRAPH

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**ABSTRACT.** We give tight bounds for the Stanley depth of the quotient ring of the path ideal of a cycle graph. In particular, we prove that it satisfies the Stanley inequality.

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## INTRODUCTION

Let  $K$  be a field and  $S = K[x_1, \dots, x_n]$  the polynomial ring over  $K$ . Let  $M$  be a  $\mathbb{Z}^n$ -graded  $S$ -module. A *Stanley decomposition* of  $M$  is a direct sum  $\mathcal{D} : M = \bigoplus_{i=1}^r m_i K[Z_i]$  as a  $\mathbb{Z}^n$ -graded  $K$ -vector space, where  $m_i \in M$  is homogeneous with respect to  $\mathbb{Z}^n$ -grading,  $Z_i \subset \{x_1, \dots, x_n\}$  such that  $m_i K[Z_i] = \{um_i : u \in K[Z_i]\} \subset M$  is a free  $K[Z_i]$ -submodule of  $M$ . We define  $\text{sdepth}(\mathcal{D}) = \min_{i=1, \dots, r} |Z_i|$  and  $\text{sdepth}(M) = \max\{\text{sdepth}(\mathcal{D}) \mid \mathcal{D} \text{ is a Stanley decomposition of } M\}$ . The number  $\text{sdepth}(M)$  is called the *Stanley depth* of  $M$ .

Herzog, Vladoiu and Zheng show in [10] that  $\text{sdepth}(M)$  can be computed in a finite number of steps if  $M = I/J$ , where  $J \subset I \subset S$  are monomial ideals. In [13], Rinaldo give a computer implementation for this algorithm, in the computer algebra system CoCoA [6]. In [2], J. Apel restated a conjecture firstly given by Stanley in [14], namely that  $\text{sdepth}(M) \geq \text{depth}(M)$  for any  $\mathbb{Z}^n$ -graded  $S$ -module  $M$ . This conjecture proves to be false, in general, for  $M = S/I$  and  $M = J/I$ , where  $0 \neq I \subset J \subset S$  are monomial ideals, see [7]. For a friendly introduction in the thematic of Stanley depth, we refer the reader [11].

Let  $\Delta \subset 2^{[n]}$  be a simplicial complex. A face  $F \in \Delta$  is called a *facet*, if  $F$  is maximal with respect to inclusion. We denote  $\mathcal{F}(\Delta)$  the set of facets of  $\Delta$ . If  $F \in \mathcal{F}(\Delta)$ , we denote  $x_F = \prod_{j \in F} x_j$ . Then the *facet ideal*  $I(\Delta)$  associated to  $\Delta$  is the squarefree monomial ideal  $I = (x_F : F \in \mathcal{F}(\Delta))$  of  $S$ . The facet ideal was studied by Faridi [8] from the **depth** perspective.

The *line graph* of length  $n$ , denoted by  $L_n$ , is a graph with the vertex set  $V = [n]$  and the edge set  $E = \{\{1, 2\}, \{2, 3\}, \dots, \{n-1, n\}\}$ . Let  $\Delta_{n,m}$  be the simplicial complex with the set of facets  $\mathcal{F}(\Delta_{n,m}) = \{\{1, 2, \dots, m\}, \{2, 3, \dots, m+1\}, \dots, \{n-m+1, n-m+2, \dots, n\}\}$ , where  $1 \leq m \leq n$ . We denote  $I_{n,m} = (x_1 x_2 \cdots x_m, x_2 x_3 \cdots x_{m+1}, \dots, x_{n-m+1} x_{n-m+2} \cdots x_n)$ , the associated facet ideal. Note that  $I_{n,m}$  is the  $m$ -path ideal of the graph  $L_n$ , provided with the direction given by  $1 < 2 < \dots < n$ , see [9] for further details.

According to [9, Theorem 1.2], the projective dimension of  $S/I_{n,m}$  is:

$$\text{pd}(S/I_{n,m}) = \begin{cases} \frac{2(n-d)}{m+1}, & n \equiv d \pmod{m+1} \text{ with } 0 \leq d \leq m-1, \\ \frac{2n-m+1}{m+1}, & n \equiv m \pmod{m+1}. \end{cases}$$

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By Auslander-Buchsbaum formula (see [15]), it follows that  $\text{depth}(S/I_{n,m}) = n - \text{pd}(S/I_{n,m})$  and, by a straightforward computation, we can see  $\text{depth}(S/I_{n,m}) = n + 1 - \left\lfloor \frac{n+1}{m+1} \right\rfloor - \left\lceil \frac{n+1}{m+1} \right\rceil =: \varphi(n, m)$ . We proved in [5] that  $\text{sdepth}(S/I_{n,m}) = \varphi(n, m)$ .

The *cycle graph* of length  $n$ , denoted by  $C_n$ , is a graph with the vertex set  $V = [n]$  and the edge set  $E = \{\{1, 2\}, \{2, 3\}, \dots, \{n-1, n\}, \{n, 1\}\}$ . Let  $\bar{\Delta}_{n,m}$  be the simplicial complex with the set of facets  $\mathcal{F}(\bar{\Delta}_{n,m}) = \{\{1, 2, \dots, m\}, \{2, 3, \dots, m+1\}, \dots, \{n-m+1, n-m+2, \dots, n\}, \{n-m+2, \dots, n, 1\}, \dots, \{n, 1, \dots, m-1\}\}$ . We denote  $J_{n,m} = (x_1x_2 \cdots x_m, x_2x_3 \cdots x_{m+1}, \dots, x_{n-m+1}x_{n-m+2} \cdots x_n, \dots, x_nx_1 \cdots x_{m-1})$ , the associated facet ideal. Note that  $J_{n,m}$  is the  $m$ -path ideal of the graph  $C_n$ .

Let  $p = \left\lfloor \frac{n}{m+1} \right\rfloor$  and  $d = n - (m+1)p$ . According to [1, Corollary 5.5],

$$\text{pd}(S/J_{n,m}) = \begin{cases} 2p + 1, & d \neq 0, \\ 2p, & d = 0. \end{cases}$$

By Auslander-Buchsbaum formula, it follows that  $\text{depth}(S/J_{n,m}) = n - \text{pd}(S/J_{n,m}) = n - \left\lfloor \frac{n}{m+1} \right\rfloor - \left\lceil \frac{n}{m+1} \right\rceil = \varphi(n-1, m)$ . Our main result is Theorem 1.4, in which we prove that  $\varphi(n, m) \geq \text{sdepth}(S/J_{n,m}) \geq \varphi(n-1, m)$ . We also prove that,  $\text{sdepth}(J_{n,m}/I_{n,m}) = \text{depth}(J_{n,m}/I_{n,m}) = \varphi(n-1, m) + m - 1$ , see Proposition 1.6. These results generalize [4, Theorem 1.9] and [4, Proposition 1.10].

## 1. MAIN RESULTS

First, we recall the well known Depth Lemma, see for instance [15, Lemma 1.3.9].

**Lemma 1.1.** (*Depth Lemma*) *If  $0 \rightarrow U \rightarrow M \rightarrow N \rightarrow 0$  is a short exact sequence of modules over a local ring  $S$ , or a Noetherian graded ring with  $S_0$  local, then*

- a)  $\text{depth } M \geq \min\{\text{depth } N, \text{depth } U\}$ .
- b)  $\text{depth } U \geq \min\{\text{depth } M, \text{depth } N + 1\}$ .
- c)  $\text{depth } N \geq \min\{\text{depth } U - 1, \text{depth } M\}$ .

In [12], Asia Rauf proved the analog of Lemma 1.1(a) for  $\text{sdepth}$ :

**Lemma 1.2.** *Let  $0 \rightarrow U \rightarrow M \rightarrow N \rightarrow 0$  be a short exact sequence of  $\mathbb{Z}^n$ -graded  $S$ -modules. Then:  $\text{sdepth}(M) \geq \min\{\text{sdepth}(U), \text{sdepth}(N)\}$ .*

The following result is well known. However, we present an original proof.

**Lemma 1.3.** *Let  $I \subset S$  be a nonzero proper monomial ideal. Then,  $I$  is principal if and only if  $\text{sdepth}(S/I) = n - 1$ .*

*Proof.* Assume  $\text{sdepth}(S/I) = n - 1$  and let  $S/I = \bigoplus_{i=1}^r u_i K[Z_i]$  be a Stanley decomposition with  $|Z_i| = n - 1$  for all  $i$ , and  $u_i \in S$  monomials. Since  $1 \notin I$ , we may assume that  $u_1 = 1$ . Let  $x_{j_1}$  be the variable which is not in  $Z_1$ . If  $x_{j_1} \in I$ , since  $S/(x_{j_1}) = K[Z_1]$  and  $K[Z_1] \subset S/I$ , then  $I = (x_{j_1})$ . Otherwise, we may assume that  $u_2 = x_{j_1}$ .

Let  $x_{j_2}$  be the variable which is not in  $Z_2$ . If  $x_{j_1}x_{j_2} \in I$ , then, one can easily see that  $I = (x_{j_1}x_{j_2})$ . If  $x_{j_1}x_{j_2} \notin I$ , then we may assume  $u_3 = x_{j_1}x_{j_2}$  and so on. Thus, we have  $u_i = x_{j_1} \cdots x_{j_{i-1}}$ , for all  $1 \leq i \leq r + 1$ , where  $x_{j_i}$  is the variable which is not in  $Z_i$ . Moreover,  $I = (u_{r+1})$ , and therefore  $I$  is principal.

In order to prove the other implication, assume that  $I = (u)$  and write  $u = \prod_{i=1}^r x_{j_i}$ . We let  $u_i = \prod_{k=1}^{i-1} x_{j_k}$  and  $Z_i = \{x_1, \dots, x_n\} \setminus \{x_{j_i}\}$ , for all  $1 \leq i \leq r$ . Then,  $S/I = \bigoplus_{i=1}^r u_i K[Z_i]$  is a Stanley decomposition with  $|Z_i| = n - 1$  for all  $i$ . Therefore  $\text{sdepth}(S/I) = n - 1$ .  $\square$

Our main result, is the following theorem.

**Theorem 1.4.**  $\varphi(n, m) \geq \text{sdepth}(S/J_{n,m}) \geq \text{depth}(S/J_{n,m}) = \varphi(n-1, m)$ .

*Proof.* If  $n = m$ , then  $J_{n,n} = (x_1 \dots x_n)$  is a principal ideal, and, according to Lemma 1.3 we are done. Also, if  $m = 1$ , then  $J_{n,1} = (x_1, \dots, x_n)$  and so there is nothing to prove, since  $S/J_{n,1} = K$ . The case  $m = 2$  follows from [4, Proposition 1.8] and [4, Theorem 1.9].

Assume  $n > m \geq 3$ . If  $n = m + 1$ , then we consider the short exact sequence

$$0 \rightarrow S/(J_{n,n-1} : x_n) \rightarrow S/J_{n,n-1} \rightarrow S/(J_{n,n-1}, x_n) \rightarrow 0.$$

Note that  $(J_{n,n-1} : x_n) = (x_1 \dots x_{n-2}, x_2 \dots x_{n-1}, x_3 \dots x_{n-1}x_1, \dots, x_{n-1}x_1 \dots x_{n-3}) \cong J_{n-1,n-2}S$ . Therefore, by induction hypothesis and [10, Lemma 3.6],

$$\text{sdepth}(S/(J_{n,n-1} : x_n)) = \text{depth}(S/(J_{n,n-1} : x_n)) = 1 + \varphi(n-2, n-2) = n-2.$$

Also,  $(J_{n,n-1}, x_n) = (x_1 \dots x_{n-1}, x_n)$  and thus  $S/(J_{n,n-1}, x_n) \cong K[x_1, \dots, x_{n-1}]/(x_1 \dots x_{n-1})$ . Therefore, by Lemma 1.3, we have  $\text{sdepth}(S/(J_{n,n-1}, x_n)) = n-2 = \text{depth}(S/(J_{n,n-1}, x_n))$ .

Now, assume  $n > m + 1 > 3$ . We consider the ideals  $L_0 = J_{n,m}$ ,  $L_{k+1} = (L_k : x_{n-k})$  and  $U_k = (L_k, x_{n-k})$ , for  $0 \leq k \leq m-2$ . Note that

$$L_{m-1} = (J_{n,m} : x_{n-m+2} \dots x_n) = (x_1, x_2 \dots x_{m+1}, \dots, x_{n-2m+1} \dots x_{n-m}, x_{n-m+1}).$$

If  $n-2m \leq 2$ , then  $L_{m-1} = (x_1, x_{n-m+1})$  and thus  $\text{sdepth}(S/L_{m-1}) = \text{depth}(S/L_{m-1}) = n-2 = \varphi(n, m)$ , since  $\lfloor \frac{n+1}{m+1} \rfloor = 1$  and  $\lceil \frac{n+1}{m+1} \rceil = 2$ .

If  $n-2m > 2$ , then  $S/L_{m-1} \cong K[x_2, \dots, x_{n-m}, x_{n-m+2}, \dots, x_n]/(x_2 \dots x_{m+1}, \dots, x_{n-2m+1} \dots x_{n-m})$  and therefore, by [10, Lemma 3.6] and [5, Theorem 1.3], we have  $\text{sdepth}(S/L_{m-1}) = \text{depth}(S/L_{m-1}) = n-1 - \lfloor \frac{n-m}{m+1} \rfloor - \lceil \frac{n-m}{m+1} \rceil = \varphi(n, m)$ . On the other hand, for example by [3, Proposition 2.7],  $\text{sdepth}(S/L_{m-1}) \geq \text{sdepth}(S/J_{n,m})$ . Thus,  $\text{sdepth}(S/J_{n,m}) \leq \varphi(n, m)$ .

For any  $0 < k < m$ , we have  $L_k = (x_1 \dots x_{m-k}, x_2 \dots x_{m+1}, \dots, x_{n-m-k} \dots x_{n-k-1}, x_{n-m+1} \dots x_{n-k}, x_{n-m+2} \dots x_{n-k}x_1, \dots, x_{n-k}x_1 \dots x_{m-k-1})$ . Therefore,  $U_k = (x_1 \dots x_{m-k}, x_2 \dots x_{m+1}, \dots, x_{n-m-k} \dots x_{n-k-1}, x_{n-k})$ , for  $k \leq m-2$ . We consider two cases:

(i) If  $n-m-k < 2$  and  $0 \leq k \leq m-2$ , then  $U_k = (x_1 \dots x_{m-k}, x_{n-k})$  and therefore  $\text{sdepth}(S/U_k) = \text{depth}(S/U_k) = n-2 = \varphi(n, m)$ , since  $\lfloor \frac{n+1}{m+1} \rfloor = 1$  and  $\lceil \frac{n+1}{m+1} \rceil = 2$ .

(ii) If  $n-m-k \geq 2$ , then, for any  $0 \leq j \leq k \leq m-2$ , we consider the ideals  $V_{k,j} := (x_1 \dots x_{m-j}, x_2 \dots x_{m+1}, \dots, x_{n-m-k} \dots x_{n-k-1})$  in  $S_k := K[x_1, \dots, x_{n-k-1}]$ . Note that  $S/U_k \cong (S_k/V_{k,k})[x_{n-k+1}, \dots, x_n]$  and thus, by [10, Lemma 3.6],  $\text{depth}(S/U_k) = \text{depth}(S_k/V_{k,k}) + k$  and  $\text{sdepth}(S/U_k) = \text{sdepth}(S_k/V_{k,k}) + k$ .

For any  $0 \leq j < k \leq m-2$ , we claim that  $V_{k,j}/V_{k,j+1}$  is isomorphic to

$$(K[x_{m-j+2}, \dots, x_{n-k-1}]/(x_{m-j+2} \dots x_{2m-j+1}, \dots, x_{n-m-k} \dots x_{n-k-1}))[x_1, \dots, x_{m-j}].$$

Indeed, if  $u \in V_{k,j} \setminus V_{k,j+1}$  is a monomial, then  $x_1 \dots x_{m-j} \nmid u$  and  $x_{m-j+1} \nmid u$ . Also,  $x_{m-j+2} \dots x_{2m-j+1} \nmid u$ ,  $\dots$ ,  $x_{n-m-k} \dots x_{n-k-1} \nmid u$ . Denoting  $v = u/(x_1 \dots x_{m-j})$ , we can write  $v = v'v''$ , with  $v' \in K[x_{m-j+2}, \dots, x_{n-k-1}] \setminus (x_{m-j+2} \dots x_{2m-j+1}, \dots, x_{n-m-k} \dots x_{n-k-1})$  and  $v'' \in K[x_1, \dots, x_{m-j}]$ .

By [10, Lemma 3.6] and [5, Theorem 1.3],  $\text{sdepth}(V_{k,j}/V_{k,j+1}) = \text{depth}(V_{k,j}/V_{k,j+1}) = m-j + \varphi(n-k-m+j-2, m) = n-k-1 - \lfloor \frac{n-m-1-k+j}{m+1} \rfloor - \lceil \frac{n-m-1-k+j}{m+1} \rceil = n-k+1 - \lfloor \frac{n-k+j}{m+1} \rfloor - \lceil \frac{n-k+j}{m+1} \rceil \geq \varphi(n, m) - k$ .

On the other hand,  $V_{k,0} = I_{n-k-1,m}$  for any  $0 \leq k \leq m-2$  and therefore, by [5, Theorem 1.3],  $\text{sdepth}(S/V_{k,0}) = \text{depth}(S/V_{k,0}) = \varphi(n-k-1, m) = n-k - \lfloor \frac{n-k}{m+1} \rfloor - \lceil \frac{n-k}{m+1} \rceil \geq \varphi(n, m) - k$ , for any  $k \geq 1$ . From the short exact sequences  $0 \rightarrow V_{k,j}/V_{k,j+1} \rightarrow S/V_{k,j+1} \rightarrow S/V_{k,j} \rightarrow 0$ ,  $0 \leq j < k$ , Lemma 1.1 and Lemma 1.2, it follows that  $\text{sdepth}(S/V_{k,j+1}) \geq \text{depth}(S/V_{k,j+1}) = \varphi(n, m) - k$ , for all  $0 \leq j < k \leq m-2$ . Thus  $\text{sdepth}(S/U_k) \geq \text{depth}(S/U_k) \geq \varphi(n, m)$ , for all  $0 < k \leq m-2$ . On the other hand,  $\text{sdepth}(S/V_{0,0}) = \text{depth}(S/V_{0,0}) = \varphi(n-1, m)$ , and thus  $\text{sdepth}(S/U_0) = \text{depth}(S/U_0) = \varphi(n-1, m)$ .

Now, we consider short exact sequences

$$0 \rightarrow S/L_{k+1} \rightarrow S/L_k \rightarrow S/U_k \rightarrow 0, \text{ for } 0 \leq k < m.$$

By Lemma 1.1 and Lemma 1.2 we get  $\text{sdepth}(S/L_k) \geq \text{depth}(S/L_k) = \varphi(n, m)$ , for any  $0 < k \leq m - 2$ , and  $\text{sdepth}(S/L_0) \geq \text{depth}(S/L_0) = \varphi(n - 1, m)$ .  $\square$

**Corollary 1.5.** *If  $\lfloor \frac{n+1}{m+1} \rfloor = \lfloor \frac{n}{m+1} \rfloor$  and  $\lceil \frac{n+1}{m+1} \rceil = \lceil \frac{n}{m+1} \rceil$ , then*

$$\text{sdepth}(S/J_{n,m}) = \text{depth}(S/J_{n,m}) = \varphi(n, m).$$

**Proposition 1.6.**  $\text{sdepth}(J_{n,m}/I_{n,m}) \geq \text{depth}(J_{n,m}/I_{n,m}) = \varphi(n - 1, m) + m - 1$ .

*Proof.* We claim that  $J_{n,m}/I_{n,m}$  is isomorphic to

$$\begin{aligned} & x_{n-m+2} \cdots x_n x_1 \left( \frac{K[x_2, \dots, x_{n-m}]}{(x_2 \cdots x_m, x_3 \cdots x_{m+2}, \dots, x_{n-2m+1} \cdots x_{n-m})} \right) [x_{n-m+2}, \dots, x_n, x_1] \oplus \\ & \oplus x_{n-m+3} \cdots x_n x_1 x_2 \left( \frac{K[x_3, \dots, x_{n-m+1}]}{(x_3 \cdots x_m, x_4 \cdots x_{m+3}, \dots, x_{n-2m+2} \cdots x_{n-m+1})} \right) [x_{n-m+3}, \dots, x_n, x_1, x_2] \oplus \\ & \cdots \oplus x_n x_1 \cdots x_{m-1} \left( \frac{K[x_m, \dots, x_{n-2}]}{(x_m, x_{m+1} \cdots x_{2m}, \dots, x_{n-m-1} \cdots x_{n-2})} \right) [x_n, x_1, \dots, x_{m-1}]. \end{aligned}$$

Indeed, let  $u \in J_{n,m} \setminus I_{n,m}$  be a monomial. If  $x_{n-m+2} \cdots x_n x_1 | u$ , then  $x_{n-m+1} \nmid u$  and  $x_2 \cdots x_m \nmid u$ . It follows that:

$$u \in x_{n-m+2} \cdots x_n x_1 \left( \frac{K[x_2, \dots, x_{n-m}]}{(x_2 \cdots x_m, x_3 \cdots x_{m+2}, \dots, x_{n-2m+1} \cdots x_{n-m})} \right) [x_{n-m+2}, \dots, x_n, x_1].$$

If  $x_{n-m+2} \cdots x_n x_1 \nmid u$  and  $x_{n-m+3} \cdots x_n x_1 x_2 | u$  then  $x_{n-m+2} \nmid u$  and  $x_3 \cdots x_m \nmid u$ . Thus:

$$u \in x_{n-m+3} \cdots x_n x_1 x_2 \left( \frac{K[x_3, \dots, x_{n-m+1}]}{(x_3 \cdots x_m, x_4 \cdots x_{m+3}, \dots, x_{n-2m+2} \cdots x_{n-m+1})} \right) [x_{n-m+3}, \dots, x_n, x_1, x_2].$$

Finally, if  $x_{n-m+2} \cdots x_n x_1 \nmid u$ ,  $\dots$ ,  $x_{n-1} x_n x_1 \cdots x_{m-2} \nmid u$  and  $x_n x_1 \cdots x_{m-1} | u$ , then it follows that  $x_{n-1} \nmid u$  and  $x_m \nmid u$ . Therefore:

$$u \notin x_n x_1 \cdots x_{m-1} \left( \frac{K[x_m, \dots, x_{n-2}]}{(x_m, x_{m+1} \cdots x_{2m}, \dots, x_{n-m-1} \cdots x_{n-2})} \right) [x_n, x_1, \dots, x_{m-1}].$$

As in the proof of Theorem 3.1 (see the computations for  $V_{k,j}$ 's), by applying Lemma 1.1 and Lemma 1.2, it follows that  $\text{sdepth}(J_{n,m}/I_{n,m}) \geq \text{depth}(J_{n,m}/I_{n,m}) = \varphi(n - m - 2, m) + m = \varphi(n - 1, m) + m - 1$ , as required.  $\square$

Inspired by [4, Conjecture 1.12] and computer experiments [6], we propose the following:

**Conjecture 1.7.** *For any  $n \geq 3(m + 1) + 1$ , we have  $\text{sdepth}(S/J_{n,m}) = \varphi(n, m)$ .*

#### REFERENCES

- [1] A. Alilooee, S. Faridi, On the resolution of path ideals of cycles, *Commun. Algebra*, **43** (12), (2015), 5413-5433 .
- [2] J. Apel, On a conjecture of R. P. Stanley; Part II - Quotients Modulo Monomial Ideals, *J. of Alg. Comb.* **17**, (2003), 57-74.
- [3] M. Cimpoeas, Several inequalities regarding Stanley depth, *Rom. J. Math. Comput. Sci.* **2**(1), (2012), 28-40.
- [4] M. Cimpoeas, On the Stanley depth of edge ideals of line and cyclic graphs, *Rom. J. Math. Comput. Sci.* **5**(1), (2015), 70-75.
- [5] M. Cimpoeas, Stanley depth of the path ideal associated to a line graph, <http://arxiv.org/pdf/1508.07540v2.pdf>, to appear in *Math. Rep., Buchar*.
- [6] CoCoATeam, *CoCoA: a system for doing Computations in Commutative Algebra*, Available at <http://cocoa.dima.unige.it>
- [7] A. M. Duval, B. Goeckneker, C. J. Klivans, J. L. Martine, A non-partitionable Cohen-Macaulay simplicial complex, <http://arxiv.org/pdf/1504.04279>

- [8] S. Faridi, The facet ideal of a simplicial complex, *Manuscr. Math.*, **109** (2002), 159-174.
- [9] Jing He, Adam Van Tuyl, Algebraic properties of the path ideal of a tree, *Commun. Algebra* **38** (2010), no. 5, 1725-742.
- [10] J. Herzog, M. Vladoiu, X. Zheng, How to compute the Stanley depth of a monomial ideal, *J. Algebra* **322** (9), (2009), 3151-3169.
- [11] J. Herzog, *A survey on Stanley depth*, In Monomial Ideals, Computations and Applications, Springer, (2013), 3-45.
- [12] A. Rauf, Depth and sdepth of multigraded module, *Commun. Algebra*, **38** (2), (2010), 773-784.
- [13] G. Rinaldo, An algorithm to compute the Stanley depth of monomial ideals, *Matematiche*, **LXIII** (ii), (2008), 243-256.
- [14] R. P. Stanley, Linear Diophantine equations and local cohomology, *Invent. Math.* **68**, (1982), 175-193.
- [15] R. H. Villarreal, *Monomial algebras*. Monographs and Textbooks in Pure and Applied Mathematics, **238**, Marcel Dekker, Inc., New York, (2001).

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