

# ON A GENERALIZED GAUSS CONVERGENCE CRITERION

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**ABSTRACT.** In this paper we combine the well known Raabe-Duhamel, Kummer, Bertrand . . . criterions of convergence for series with positive terms and we obtain a new one which is more powerful than those cited before. Even the famous Gauss criterion, which was in fact our starting point, is a consequence of this new convergence test.

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## 1. PRELIMINARY AND FIRST RESULTS

It is well known that the sequence  $((1 + \frac{1}{n})^n)_{n \geq 1}$  (resp.  $((1 + \frac{1}{n})^{n+1})_{n \geq 1}$ ) increases (respectively decreases) to the real number  $e = 2,718\dots$

Sometimes is useful to rephrase this assertion in a more powerful form:

**Lemma 1.** *The function  $f$  (resp.  $g$ ) defined by*

$$f(x) = \left(1 + \frac{1}{x}\right)^x \quad (\text{resp. } g(x) = \left(1 + \frac{1}{x}\right)^{x+1}), \quad x \in (0, \infty)$$

*is increasing (resp. decreasing) to the real number  $e$  when  $x$  tends to  $\infty$ .*

*Notations.* The function  $E_1 : (-\infty, \infty) \rightarrow (0, \infty)$  given by  $E_1(x) = e^x$  being increasing, bijective and continuous (in fact  $E_1 \in \mathcal{C}^\infty(\mathbb{R})$ ), the functions  $E_p$ ,  $p \in \mathbb{N}^*$ , inductively defined by

$$E_{p+1}(x) = E_1(E_p(x)) = E_p(E_1(x)) = \underbrace{(E_1 \circ E_1 \circ E_1 \circ \dots \circ E_1)}_{p+1 \text{ times}}(x)$$

belong also to the class  $\mathcal{C}^\infty$  and we have

$$\begin{aligned} E_1(\mathbb{R}) &= (0, \infty), \quad E_2(\mathbb{R}) = E_1(0, \infty) = (1, \infty), \\ E_3(\mathbb{R}) &= E_1(1, \infty) = (e, \infty), \quad E_4(\mathbb{R}) = E_1(e, \infty) = (e^e, \infty), \\ E_5(\mathbb{R}) &= E_1(e^e, \infty) = (e^{e^e}, \infty) \dots \end{aligned}$$

Let us denote by  $A_1, A_2, \dots, A_p, \dots$  the elements of  $\mathbb{R}_+$  given by  $A_1 = 0$ ,  $A_2 = 1 = E_1(A_1)$ ,  $A_3 = e = E_1(A_2)$ ,  $A_4 = e^e = E_1(A_3) \dots A_{p+1} = E_1(A_p) \dots$

Obviously we have  $0 = A_1 < A_2 < A_3 < A_4 < \dots < A_p < A_{p+1} < \dots$  and the functions  $E_1, E_2, \dots, E_p \dots$  defined on the interval  $(-\infty, \infty)$  belong to the class  $\mathcal{C}^\infty(\mathbb{R})$ , they are strictly increasing and  $E_1(-\infty, \infty) = (0, \infty) = (A_1, \infty)$ ,  $E_2(-\infty, \infty) = (1, \infty) = (A_2, \infty)$ ,  $E_3(-\infty, \infty) = (A_3, \infty)$ ,  $\dots, E_p(-\infty, \infty) = (A_p, \infty)$ .

One may show inductively that

$$E_p'(x) = E_1(x)E_2(x)\dots E_p(x), \quad \forall p \geq 1$$

and therefore, if we denote by  $l_p$  the inverse of the function  $E_p$ ,  $l_p : (A_p, \infty) \rightarrow (-\infty, \infty)$ , the function  $l_p$  belongs to the class  $C^\infty(A_p, \infty)$  and we have

$$E_q \circ E_p = E_p \circ E_q = E_{p+q}, \quad E_p^{-1} \circ E_q^{-1} = E_q^{-1} \circ E_p^{-1} = E_{p+q}^{-1}.$$

Hence, for any  $t \in E_{p+q}(-\infty, \infty) = (A_{p+q}, \infty)$  we have

$$\begin{aligned} E_{p+q}^{-1}(t) &= E_p^{-1} \circ E_q^{-1}(t) = E_q^{-1} \circ E_p^{-1}(t); \\ l_{p+q}(t) &= (l_p \circ l_q)(t) = (l_q \circ l_p)(t); \\ E_q(l_{q+p}(t)) &= E_q(l_q(l_p(t))) = l_p(t) \end{aligned}$$

and therefore

$$\begin{aligned} t \in (A_{p+q}, \infty) &\Rightarrow l'_{p+q}(t) = \frac{1}{E'_{p+q}(l_{p+q}(t))} = \frac{1}{(E_1 E_2 \cdots E_{p+q})(l_{p+q}(t))} = \\ &= \frac{1}{E_1(l_{p+q}(t)) E_2(l_{p+q}(t)) \cdots E_{p+q}(l_{p+q}(t))} = \frac{1}{l_{p+q-1}(t) \cdot l_{p+q-2}(t) \cdots l_1(t) \cdot t}; \\ l'_n(t) &= \frac{1}{t \cdot l_1(t) \cdot l_2(t) \cdots l_{n-1}(t)}, \quad \forall t \in (A_n, \infty). \end{aligned}$$

We remark also that for any  $p \geq 1$ ,  $p \in \mathbb{N}$  we have

$$l_p(A_{p+1}, \infty) = (0, \infty) \quad \text{and} \quad l_p(A_{p+2}, \infty) = (1, \infty).$$

We shall denote by  $\Delta_k$ ,  $k \in \mathbb{N}^*$ , the function defined on  $(A_k, \infty)$  given by

$$\Delta_k(x) = l_k(x+1) - l_k(x).$$

**Lemma 2.** a) For any  $x \in [e, \infty) = [A_3, \infty)$  and any  $y \geq x$  we have

$$\frac{l_1(y)}{l_1(x)} \leq \frac{y}{x}.$$

b) For any  $x \in [A_{k+2}, \infty)$  and any  $y \geq x$  we have

$$\frac{l_k(y)}{l_k(x)} \leq \frac{y}{x}.$$

*Proof.* a) If we denote  $u = l_1(x)$ ,  $v = l_1(y)$  we have  $1 \leq u \leq v$  and therefore

$$\frac{l_1(y)}{l_1(x)} = \frac{v}{u} = 1 + \frac{v-u}{u} \leq 1 + (v-u) \leq e^{v-u} = \frac{e^v}{e^u} = \frac{y}{x}.$$

b) The inequality may be done inductively. For  $k = 1$  the assertion b) is just the assertion a). We suppose that for  $k \geq 1$ ,  $x \in [A_{k+2}, \infty)$ ,  $y \geq x$  we have

$$\frac{l_k(y)}{l_k(x)} \leq \frac{y}{x}.$$

If  $x \in [A_{k+3}, \infty)$  and  $y \geq x$  we have  $l_1(x) \in [A_{k+2}, \infty) \subset [A_3, \infty)$ ,  $l_1(y) \geq l_1(x)$  and therefore by the hypothesis we have

$$\frac{l_k(l_1(y))}{l_k(l_1(x))} \leq \frac{l_1(y)}{l_1(x)} \leq \frac{y}{x}. \quad \square$$

**Lemma 3.** For any  $k \in \mathbb{N}^*$  and any  $x \in [A_{k+1}, \infty)$  we have

$$0 \leq \frac{(x+1)l_1(x+1)l_2(x+1) \cdots l_{k-1}(x+1)\Delta_{k-1}(x)}{l_{k-1}(x)} - 1 \leq \frac{1}{x} \cdot 2^k.$$

*Proof.* Taking  $k \in \mathbb{N}^*$ ,  $x \geq A_{k+1}$  and applying Lagrange Theorem we deduce the existence of a real number  $x' \in (x, x+1)$  such that

$$\Delta_{k-1}(x) = l_{k-1}(x+1) - l_{k-1}(x) = \frac{1}{x'l_1(x')l_2(x') \cdots l_{k-2}(x')}.$$

If we denote

$$F_{k-1}(x) = \frac{(x+1)l_1(x+1)l_2(x+1) \cdots l_{k-1}(x+1)\Delta_{k-1}(x)}{l_{k-1}(x)} - 1$$

we have

$$F_{k-1}(x) = \frac{x+1}{x'} \cdot \frac{l_1(x+1)}{l_1(x')} \cdot \frac{l_2(x+1)}{l_2(x')} \cdots \frac{l_{k-2}(x+1)}{l_{k-2}(x')} \cdot \frac{l_{k-1}(x+1)}{l_{k-1}(x)} - 1$$

Since the functions  $l_1, l_2, \dots, l_{k-1}$  are positive and increasing on the interval  $[A_{k+1}, \infty)$  we deduce that the function  $F_{k-1}$  is positive on the interval  $[A_{k+1}, \infty)$  and moreover we have

$$0 \leq F_{k-1}(x) \leq \frac{x+1}{x} \cdot \frac{l_1(x+1)}{l_1(x)} \cdot \frac{l_2(x+1)}{l_2(x)} \cdots \frac{l_{k-1}(x+1)}{l_{k-1}(x)} - 1.$$

We apply now Lemma 2 and we obtain

$$\frac{x+1}{x} = 1 + \frac{1}{x}, \quad \frac{l_1(x+1)}{l_1(x)} \leq 1 + \frac{1}{x}, \dots, \quad \frac{l_{k-1}(x+1)}{l_{k-1}(x)} \leq 1 + \frac{1}{x},$$

$$F_{k-1}(x) \leq \left(1 + \frac{1}{x}\right)^k - 1 = \sum_{j=1}^k C_k^j \cdot \left(\frac{1}{x}\right)^j \leq \frac{1}{x} \sum_{j=1}^k C_k^j < \frac{1}{x} \cdot 2^k. \quad \square$$

We remember now, under a convenient form, the well known Raabe-Duhamel and Gauss criterions of convergence (or divergence) for the series with positive terms (see [1] or [2]).

From now on  $\sum a_n$  will be a series of real numbers such that  $a_n > 0$  for all  $n \in \mathbb{N}$ .

**Raabe-Duhamel divergence criterion.** If  $\frac{a_n}{a_{n+1}} \leq 1 + \frac{1}{n}$ , for  $n$  sufficiently large, then the series  $\sum a_n$  is divergent.

**Raabe-Duhamel convergence criterion.** If  $\alpha \in \mathbb{R}$ ,  $\alpha > 1$  and for  $n$  sufficiently large we have  $\frac{a_n}{a_{n+1}} \geq 1 + \frac{\alpha}{n}$  then the series  $\sum a_n$  is convergent.

**Gauss divergence criterion.** If there exist  $\alpha \in (1, \infty)$  and a (positive) real number  $M$  such that  $\frac{a_n}{a_{n+1}} \leq 1 + \frac{1}{n} + \frac{M}{n^\alpha}$ , for  $n$  sufficiently large, then the series  $\sum a_n$  is divergent.

**Gauss convergence criterion.** If there exist  $r \in (1, \infty)$ ,  $\alpha \in (1, \infty)$  and  $M$  a (negative) real number such that  $\frac{a_n}{a_{n+1}} \geq 1 + \frac{r}{n} + \frac{M}{n^\alpha}$  for  $n$  sufficiently large, then the series  $\sum a_n$  is convergent.

**Kummer divergence criterion.** If  $(k_n)$  is a sequence of real numbers,  $k_n > 0$ , for all  $n \in \mathbb{N}$  such that the series  $\sum_n \frac{1}{k_n}$  is divergent and we have

$$k_n \cdot \frac{a_n}{a_{n+1}} - k_{n+1} \leq 0$$

for  $n$  sufficiently large, then the series  $\sum_n a_n$  is divergent.



**Theorem**  $CT_p$  ( $p$ -convergence criterion). *If there exists  $\alpha > 1$  such that*

$$\frac{a_n}{a_{n+1}} \geq 1 + \frac{1}{n} + \frac{1}{nl_1(n)} + \dots + \frac{1}{nl_1(n)l_2(n) \cdot \dots \cdot l_{p-1}(n)} + \frac{\alpha}{nl_1(n)l_2(n) \cdot \dots \cdot l_p(n)},$$

for  $n$  sufficiently large, then the series  $\sum a_n$  is convergent.

*Proof.* Using the same sequence  $(k_n)_n$  we shall use again Kummer (convergence) criterion. We shall try to show that for any  $n \in \mathbb{N}$ , sufficiently large, we have

$$k_n \cdot \frac{a_n}{a_{n+1}} - k_{n+1} \geq r > 0, \quad \text{where } r = \frac{\alpha - 1}{2}.$$

From the calculus performed in the proof of Theorem  $DT_p$  we have

$$\begin{aligned} k_n \cdot \frac{a_n}{a_{n+1}} - k_{n+1} &\geq [(n+1)(l_1(n) - l_1(n+1)) + 1](l_2 l_3 \cdot \dots \cdot l_p)(n) + \\ &+ \sum_{s=2}^{p-1} [(n+1)(l_1 l_2 \cdot \dots \cdot l_{s-1})(n+1)(l_s(n) - l_s(n+1)) + 1] \cdot (l_{s+1} l_{s+2} \cdot \dots \cdot l_p)(n) + \\ &+ [(n+1)(l_1 l_2 \cdot \dots \cdot l_{p-1})(n+1)(l_p(n) - l_p(n+1)) + 1] + 2r. \end{aligned}$$

To finish the proof it will be sufficient to show that

$$\lim_{n \rightarrow \infty} [(n+1)(l_1 l_2 \cdot \dots \cdot l_{s-1})(n+1)(l_s(n) - l_s(n+1)) + 1] \cdot (l_{s+1} l_{s+2} \cdot \dots \cdot l_p)(n) = 0$$

for  $s \in \{2, 3, \dots, p-1\}$  and

$$\begin{aligned} \lim_{n \rightarrow \infty} [(n+1)(l_1(n) - l_1(n+1)) + 1](l_2 l_3 \cdot \dots \cdot l_p)(n) &= 0 = \\ &= \lim_{n \rightarrow \infty} [(n+1)(l_1 \cdot \dots \cdot l_{p-1})(n+1)(l_p(n) - l_p(n+1)) + 1]. \end{aligned}$$

For our purpose we shall use the above Lemma 1 and Lemma 3. We have

$$\begin{aligned} 0 &\leq (n+1)(l_1 l_2 \cdot \dots \cdot l_{s-1})(n+1)(l_s(n+1) - l_s(n)) - 1 = \\ &= (n+1)(l_1 l_2 \cdot \dots \cdot l_{s-1})(n+1) \cdot \left( l_1 \left( \frac{l_{s-1}(n+1)}{l_{s-1}(n)} \right) \right) - 1 = \\ &= \frac{(n+1)(l_1 l_2 \cdot \dots \cdot l_{s-1})(n+1) \Delta_{s-1}(n)}{l_{s-1}(n)} \cdot l_1 \left[ \left( 1 + \frac{\Delta_{s-1}(n)}{l_{s-1}(n)} \right)^{\frac{l_{s-1}(n)}{\Delta_{s-1}(n)}} \right] - 1 \leq \\ &\leq \left( \frac{(n+1)(l_1 l_2 \cdot \dots \cdot l_{s-1})(n+1) \Delta_{s-1}(n)}{l_{s-1}(n)} - 1 \right) l_1 \left[ \left( 1 + \frac{\Delta_{s-1}(n)}{l_{s-1}(n)} \right)^{\frac{l_{s-1}(n)}{\Delta_{s-1}(n)}} \right] \leq \\ &\leq \frac{1}{n} \cdot 2^s \end{aligned}$$

and therefore, if  $n \geq A_{p+1}$ , the following inequality holds

$$\begin{aligned} 0 &\leq [(n+1)(l_1 l_2 \cdot \dots \cdot l_{s-1})(n+1)(l_s(n+1) - l_s(n)) - 1](l_{s+1} \cdot \dots \cdot l_p)(n) \leq \\ &\leq 2^s \cdot \frac{(l_{s+1} \cdot \dots \cdot l_p)(n)}{n}. \end{aligned}$$

Hence

$$\lim_{n \rightarrow \infty} [(n+1)(l_1 l_2 \cdot \dots \cdot l_{s-1})(n+1)(l_s(n+1) - l_s(n)) - 1](l_{s+1} \cdot \dots \cdot l_p)(n) = 0.$$

In a similar way one can show the assertions

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} [(n+1)(l_1(n) - l_1(n+1)) + 1](l_2 l_3 \cdot \dots \cdot l_p)(n), \\ \lim_{n \rightarrow \infty} [(n+1)(l_1 l_2 \cdot \dots \cdot l_{p-1})(n+1)(l_p(n) - l_p(n+1)) - 1] &= 0. \end{aligned}$$

Hence  $k_n \cdot \frac{a_n}{a_{n+1}} - k_{n+1} > r$  for  $n$  sufficiently large. □

**Theorem  $LT_p$ .** *If there exists the following limit*

$$l := \lim_{n \rightarrow \infty} n(l_1 l_2 \cdots l_p)(n) \left( \frac{a_n}{a_{n+1}} - 1 - \frac{1}{n} - \frac{1}{nl_1(n)} - \frac{1}{n(l_1 l_2)(n)} - \cdots \right. \\ \left. \cdots - \frac{1}{n(l_1 l_2 \cdots l_{p-1})(n)} \right)$$

then the series  $\sum_n a_n$  converges for  $l > 1$  and diverges for  $l < 1$ .

### 3. EXAMPLES AND COUNTEREXAMPLES

In this part we establish some relations between different convergence (or divergence) criteria for series with positive terms.

*Notations.* If  $C', C''$  are two criteria of convergence (or divergence) for series  $\sum a_n$ , with  $a_n > 0$  we say that  $C''$  is stronger than  $C'$  and we write  $C' \leq C''$ , if the conditions in which  $C''$  acts are automatically fulfilled whenever the conditions in which  $C'$  acts are fulfilled.

**Proposition.** *If we denote by  $CR, CG$  (respectively  $DR, DG$ ) the Raabe convergence, Gauss convergence (respectively Raabe divergence, Gauss divergence) criteria then we have*

$$CR < CG < CT_1 < CT_2 < CT_3 < \cdots < CT_p < CT_{p+1} < \cdots \\ DR < DG < DT_1 < DT_2 < DT_3 < \cdots < DT_p < DT_{p+1} < \cdots$$

*Proof.* From the inequalities

$$1 + \frac{1}{n} \leq 1 + \frac{1}{n} + \frac{M}{n^\alpha} \leq 1 + \frac{1}{n} + \frac{1}{nl_1(n)} \leq 1 + \frac{1}{n} + \frac{1}{nl_1(n)} + \frac{1}{n(l_1 l_2)(n)} \leq \cdots \\ \cdots \leq 1 + \frac{1}{n} + \frac{1}{nl_1(n)} + \frac{1}{n(l_1 l_2)(n)} + \frac{1}{n(l_1 l_2 l_3)(n)} + \cdots + \frac{1}{n(l_1 l_2 \cdots l_p)(n)}$$

for any  $\alpha > 1$ ,  $M > 0$  and  $n$  sufficiently large,  $n \geq A_{p+1}$ , we deduce that

$$DR \leq DG \leq DT_1 \leq DT_2 \leq \cdots \leq DT_p.$$

Some examples will show that these inequalities are strict.

If  $b_1, b_2, \dots, b_k$  are real numbers we shall denote by  $\prod_{i \leq k} b_i$  their product  $b_1 \cdot b_2 \cdots b_k$ .

Let us take the sequence  $(a_n)_n$  in  $\mathbb{R}_+$  given by

$$a_n = \frac{((n-1)!)^2}{\prod_{k \leq n-1} (1+k+k^2)}.$$

We have

$$\frac{a_n}{a_{n+1}} = \frac{n^2 + n + 1}{n^2} = 1 + \frac{1}{n} + \frac{1}{n^2}.$$

$DG$  criterion decides the divergence of the series  $\sum a_n$  but  $DR$  criterion doesn't it. Hence  $DR < DG$ . Let now  $b_n \in \mathbb{R}_+$  given by

$$b_n = \frac{(n-1)!(\ln(2))(\ln(3)) \cdots (\ln(n-1))}{(1+3 \ln 2)(1+4 \ln 3) \cdots (1+n \ln(n-1))}, \quad n \geq 3.$$

We have

$$\frac{b_n}{b_{n+1}} = \frac{1 + (n+1) \ln(n)}{n \ln(n)} = 1 + \frac{1}{n} + \frac{1}{n \ln(n)}.$$

By  $DT_1$  - criterion the series  $\sum_{n \geq 3} b_n$  is divergent. In the same time if we take  $\alpha > 1$  and  $M > 0$  we can not have the inequalities

$$\frac{b_n}{b_{n+1}} \leq 1 + \frac{1}{n} + \frac{M}{n^\alpha}$$

at least for a sufficiently large  $n$  because the inequality  $\frac{1}{n \ln(n)} \leq \frac{M}{n^\alpha}$  fails for  $n$  sufficiently large.

Hence  $DG$  - criterion does not decide the nature of the series  $\sum b_n$  i.e.  $DG < DT_1$ .

Let now  $p \in \mathbb{N}$ ,  $p \geq 2$  and  $n_0 \in \mathbb{N}$  be the smallest natural number greater than  $A_{p+1}$ , defined in Section 1. We consider a sequence  $(b_n)_{n \geq n_0}$  inductively defined by

$$b_{n_0} = 1,$$

$$b_{n+1} = b_n \cdot \frac{n(l_1 l_2 \cdot \dots \cdot l_p)(n)}{1 + (n+1)(l_1 l_2 \cdot \dots \cdot l_p)(n) + (l_2 l_3 \cdot \dots \cdot l_p)(n) + (l_3 l_4 \cdot \dots \cdot l_p)(n) + \dots + l_p(n)}.$$

Obviously  $b_n > 0$  and

$$\frac{b_n}{b_{n+1}} = 1 + \frac{1}{n} + \frac{1}{n l_1(n)} + \dots + \frac{1}{n(l_1 l_2 \cdot \dots \cdot l_{p-1})(n)} + \frac{1}{n(l_1 l_2 \cdot \dots \cdot l_p)(n)}.$$

Using  $DT_p$  - criterion we decide that the series  $\sum_{n \geq n_0} b_n$  is divergent. Since

$$1 + \frac{1}{n} + \frac{1}{n l_1(n)} + \dots + \frac{1}{n(l_1 l_2 \cdot \dots \cdot l_{p-1})(n)} < \frac{b_n}{b_{n+1}}, \quad \forall n \geq n_0,$$

the  $DT_{p-1}$  - criterion does not decide the nature of the series  $\sum_{n \geq n_0} b_n$ , i.e.  $DT_{p-1} < DT_p$ . From the preceding considerations we get  $DR < DG < DT_1 < DT_2 < \dots < DT_p$ .

We show now that  $CR < CG < CT_1 < CT_2 < \dots < CT_p$ .

The relations  $CR \leq CG \leq CT_1$  are obvious.

For an arbitrary  $p \in \mathbb{N}$ ,  $p \geq 1$  we consider a series  $\sum a_n$  for which there exists  $\alpha > 1$  such that

$$\frac{a_n}{a_{n+1}} \geq 1 + \frac{1}{n} + \frac{1}{n l_1(n)} + \frac{1}{n(l_1 l_2)(n)} + \dots + \frac{1}{n(l_1 l_2 \cdot \dots \cdot l_{p-1})(n)} + \frac{\alpha}{n(l_1 l_2 \cdot \dots \cdot l_p)(n)}.$$

Since

$$\liminf_{n \rightarrow \infty} n(l_1 l_2 \cdot \dots \cdot l_{p+1}) \left( \frac{a_n}{a_{n+1}} - 1 - \frac{1}{n} - \frac{1}{n l_1(n)} - \dots - \frac{1}{n(l_1 \cdot \dots \cdot l_p)(n)} \right) \geq (\alpha - 1)\infty > 2,$$

we have for  $n$  sufficiently large

$$\frac{a_n}{a_{n+1}} \geq 1 + \frac{1}{n} + \frac{1}{n l_1(n)} + \dots + \frac{1}{n(l_1 \cdot \dots \cdot l_p)(n)} + \frac{2}{n(l_1 \cdot \dots \cdot l_{p+1})(n)}.$$

Hence  $CT_p \leq CT_{p+1}$ ,  $CR \leq CG \leq CT_1 \leq CT_2 \leq \dots \leq CT_p$ .

The fact that we have strict inequalities before, may be shown by some examples. More precisely we consider  $p \in \mathbb{N}$ ,  $p > 1$  and for any  $n \geq [A_{p+1}] + 1 = n_0$  let  $b_n \in \mathbb{R}_+$  given by

$$b_{n_0} = 1,$$

$$b_{n+1} = b_n \cdot \frac{n(l_1 l_2 \cdot \dots \cdot l_p)(n)}{2 + l_p(n) + (l_{p-1} l_p)(n) + \dots + (l_2 l_3 \cdot \dots \cdot l_p)(n) + (n+1)(l_1 l_2 \cdot \dots \cdot l_p)(n)}.$$

We have

$$\frac{b_n}{b_{n+1}} = 1 + \frac{1}{n} + \frac{1}{n l_1(n)} + \frac{1}{n(l_1 l_2)(n)} + \dots + \frac{1}{n(l_1 l_2 \cdot \dots \cdot l_{p-1})(n)} + \frac{2}{n(l_1 \cdot \dots \cdot l_p)(n)}$$

and therefore using  $CT_p$  - criterion the series  $\sum b_n$  is convergent. In the same time there is no  $\alpha > 1$  such that

$$\frac{b_n}{b_{n+1}} \geq 1 + \frac{1}{n} + \frac{1}{nl_1(n)} + \frac{1}{n(l_1l_2)(n)} + \dots + \frac{1}{n(l_1l_2 \cdot \dots \cdot l_{p-2})(n)} + \frac{\alpha}{n(l_1l_2 \cdot \dots \cdot l_{p-1})(n)}$$

at least for  $n$  sufficiently large. Hence  $CT_{p-1}$  - criterion does not decide the convergence of the series  $\sum b_n$  i.e.  $CT_{p-1} < CT_p$ .

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