

POSITIVE SOLUTIONS OF NONLINEAR FOURTH-ORDER TWO-POINT BOUNDARY VALUE PROBLEM WITH A PARAMETER

NOUREDDINE BOUTERAA, SLIMANE BENAICHA, HABIB DJOURDEM AND MOHAMED ELARBI BENATTIA

ABSTRACT. In this paper, we study the existence and nonexistence of positive solutions of elastic beam equations with a parameter λ for fourth-order two-point boundary value problem

$$\begin{aligned} u^{(4)}(t) &= \lambda f(t, u(t)), \quad t \in (0, 1), \\ u(0) = u'(0) = u'(1) = u'''(1) + \psi(u(1)) &= 0, \end{aligned}$$

where λ is a positive parameter. By using Krasnoselskii's fixed point theorem of cone expansion-compression type we show that there exist $\lambda^* \geq \lambda_* > 0$ such that the beam equation has at least two, one and no positive solutions for $0 < \lambda \leq \lambda_*$, $\lambda_* < \lambda \leq \lambda^*$ and $\lambda > \lambda^*$ respectively.

Mathematics Subject Classification (2010): 34B18; 34B15.

Key words: Elastic beam equation; Nonexistence; Positive solutions; Green's function; Fixed-point theorem of cone expansion and compression type

Article history:

Received 08 January 2017

Received in revised form 02 November 2017

Accepted 21 November 2017

1. INTRODUCTION

In this paper, we are interested in the existence of positive solutions to nonlinear fourth-order two-point boundary value problem (BVP) with a parameter:

$$(1.1) \quad u^{(4)}(t) = \lambda f(t, u(t)), \quad t \in (0, 1),$$

$$(1.2) \quad u(0) = u'(0) = u'(1) = u'''(1) + \psi(u(1)) = 0,$$

where $\lambda \geq 0$ is a parameter, $f \in C([0, 1] \times [0, \infty), [0, \infty))$ and $\psi \in C([0, \infty), [0, \infty))$.

Fourth-order boundary value problems modeling bending equilibria of elastic beams have been considered in several papers, because they have important applications in mechanics and engineering; [1, 10, 11, 16]. Many authors have studied the beam equation under various boundary conditions and by different approaches. For example, Benaicha and Haddouchi [7] studied the existence of solutions for the following fourth-order boundary value problem

$$\begin{cases} u^{(4)}(t) + f(u(t)) = 0, & t \in (0, 1), \\ u'(0) = u'(1) = u''(0) = 0, & u(0) = \int_0^1 a(s) u(s) ds. \end{cases}$$

Some nonlinear elastic beam equations have been studied extensively. For a small sample of such work, we refer the reader to the work of Bai [5], Bai and Wang [6], Bonnano and Bella [8], Infante and Pietramala [17], Ma and Thompson [22], Ma and Xu [23], on elastic beams whose ends are simply supported, the works of Alves et al. [3] and Yang [27] on elastic beam of where one end is embedded and another end

is fastened with sliding clamp, the work of Graef et al. [14, 15] on multi-point boundary value problems and the works of Li [21], Sun and Zhu [25] and Wang et al. [26] with parameters. When the elastic beam equation does not contain parameter λ , the existence of multiple positive solutions and unique positive solution was presented in [9, 18, 20, 22] by variational methods, for some other results on boundary value problems for the beam equation, we refer the reader to [2, 4, 12, 13, 28]. The aim of this paper is to show that the existence and number of positive solutions of BVP (1.1)-(1.2) are affected by the parameter λ .

Inspired and motivated by the works mentioned above, we deal with existence and nonexistence of positive solutions to the BVP (1.1)-(1.2) by using the fixed point theorem together with the properties of the Green's function. The paper is organized as follows. In Section 2, we present that a nontrivial and nonnegative solution of BVP (1.1)-(1.2) is a positive solution. In Section 3, we obtain that some results about existence, multiplicity and nonexistence of positive solutions for BVP(1.1)-(1.2) depend on the parameter λ , and we give an example to illustrate our results.

2. PRELIMINARIES

We shall consider the Banach space $C[0, 1]$ equipped with sup norm $\|u\| = \max_{0 \leq t \leq 1} |u(t)|$ and $C^+[0, 1]$ is the cone of nonnegative functions in $C[0, 1]$.

Definition 2.1. A nonempty closed and convex set $P \subset E$ is called a cone of the Banach space E if it satisfies

- (i) $u \in P, r > 0$ implies $ru \in P$;
- (ii) $u \in P, -u \in P$ implies $u = \theta$, where θ denote the zero element of E .

Definition 2.2. A cone P is said to be normal if there exists a positive number N called the normal constant of P , such that $\theta \leq u \leq v$ implies $\|u\| \leq N \|v\|$.

In arriving our results, we need the following seven preliminary lemmas. The first one is well known.

Lemma 2.3. Let $y \in C[0, 1]$. If $u \in C^4[0, 1]$, then the BVP

$$(2.1) \quad \begin{cases} u^{(4)}(t) = y(t), & 0 \leq t \leq 1, \\ u(0) = u'(0) = u'(1) = u'''(1) + \psi(u(1)) = 0, \end{cases}$$

has a unique solution

$$u(t) = \int_0^1 G(t, s) y(s) ds + \psi(u(1)) \phi(t),$$

where

$$(2.2) \quad G(t, s) = \begin{cases} \frac{1}{12}t^2(6s - 3s^2 - 2t), & 0 \leq t \leq s \leq 1, \\ \frac{1}{12}s^2(6t - 3t^2 - 2s), & 0 \leq s \leq t \leq 1, \end{cases}$$

and

$$(2.3) \quad \phi(t) = \frac{1}{4}t^2 - \frac{1}{6}t^3.$$

Lemma 2.4. For any $(t, s) \in [0, 1] \times [0, 1]$, we have

$$(2.4) \quad \begin{aligned} \frac{1}{12}t^2s^2 \leq G(t, s) \leq \frac{1}{4}s^2(2t - t^2) \quad \text{and} \quad 0 \leq \frac{\partial G}{\partial t}(t, s) \leq \frac{1}{2}(1-t)s^2, \\ 0 \leq \phi(t) \leq \frac{1}{4}(2t - t^2) \quad \text{and} \quad 0 \leq \phi'(t) \leq \frac{1}{2}(1-t). \end{aligned}$$

Proof. Suppose $0 \leq t \leq s \leq 1$. Then

$$\begin{aligned} \frac{1}{12}t^2(6s - 3s^2 - 2t) &\geq \frac{1}{12}t^2(6s - 3s^2 - 2s) \\ &\geq \frac{1}{12}t^2(4s - 3s^2) \\ &\geq \frac{1}{12}t^2(4s^2 - 3s^2) \\ &\geq \frac{1}{12}t^2s^2. \end{aligned}$$

Analogously if $0 \leq s \leq t \leq 1$. Now for showing the upper bound about G we suppose $0 \leq s \leq t \leq 1$ (it is similar in the other case). Then

$$\frac{1}{12}s^2(6t - 3t^2 - 2s) \leq \frac{1}{12}s^2(6t - 3t^2) = \frac{1}{4}s^2(2t - t^2).$$

It is easy to prove that

$$0 \leq \frac{\partial G}{\partial t}(t, s) \leq \frac{1}{2}(1-t)s^2, \quad t, s \in [0, 1] \times [0, 1].$$

Now, it is obvious that $\phi(t) \geq 0$ for $t \in [0, 1]$. Let us check the inequality $\phi(t) \leq \frac{1}{4}(2t - t^2)$, $t \in [0, 1]$. For $t \in [0, 1]$, we have

$$\phi(t) = \frac{t}{4} \left(t - \frac{2t^2}{3} \right) \leq \frac{t}{4} \left(t + \left(t - \frac{t^2}{3} \right) - \frac{2t^2}{3} \right) = \frac{1}{4}(2t - t^2).$$

□

Lemma 2.5. For $y \in C^+[0, 1]$. Then the unique solution $u(t)$ of BVP

$$\begin{cases} u^{(4)}(t) = y(t), & 0 < t < 1, \\ u(0) = u'(0) = u'(1) = u'''(1) = 0. \end{cases}$$

is nonnegative and satisfies

$$u(t) \geq \frac{t^3}{6} \|u\|.$$

Proof. Let $y(t) \in C^+[0, 1]$, then from $G(t, s) \geq 0$. We know $u \in C^+[0, 1]$. Set $u(t_0) = \|u\|$, $t_0 \in (0, 1]$. We first prove that

$$\frac{G(t, s)}{G(t_0, s)} \geq \frac{1}{6}t^3, \quad t, t_0, s \in (0, 1].$$

In fact, we can consider four cases:

(1) if $0 < t, t_0 \leq s \leq 1$, then

$$\frac{G(t, s)}{G(t_0, s)} \geq \frac{t^2(6s - 3s^2 - 2s)}{6s - 3s^2 - 2t_0} \geq \frac{t^2t(4 - 3s)}{6} \geq \frac{t^3}{6}.$$

(2) if $0 < t \leq s \leq t_0 \leq 1$, then

$$\frac{G(t, s)}{G(t_0, s)} \geq \frac{t^2(6s - 3s^2 - 2t)}{6t_0 - 3t_0^2 - 2s} \geq \frac{t^2(6s - 3s^2 - 2s)}{6t_0} \geq \frac{t^2s(4 - 3s)}{6} \geq \frac{t^2(t)}{6} \geq \frac{t^3}{6}.$$

(3) if $0 < s \leq t, t_0 \leq 1$, then

$$\frac{G(t, s)}{G(t_0, s)} \geq \frac{(6t - 3t^2 - 2s)}{6t_0} \geq \frac{(4t - 3t^2)}{6} \geq \frac{t(4 - 3t)}{6} \geq \frac{t}{6} \geq \frac{t^3}{6}.$$

(4) if $0 < t_0 \leq s \leq t \leq 1$, then

$$\frac{G(t, s)}{G(t_0, s)} \geq \frac{t_0^2 (6t - 3t^2 - 2s)}{t_0^2 (6s - 3s^2 - 2t_0)} \geq \frac{(6t - 3t^2 - 2t)}{6} \geq \frac{t(4 - 3t)}{6} \geq \frac{t}{6} \geq \frac{t^3}{6}.$$

Therefore, for $t \in [t_0, 1]$ we have

$$u(t) = \int_0^1 G(t, s) y(s) ds = \int_0^1 \frac{G(t, s)}{G(t_0, s)} G(t_0, s) y(s) ds \geq \int_0^1 \frac{t^3}{6} G(t_0, s) y(s) ds = \frac{t^3}{6} u(t_0) = \frac{t^3}{6} \|u\|.$$

The proof is complete. \square

If we let

$$P = \left\{ u \in C^+[0, 1] : u(t) \geq \frac{t^3}{6} \|u\| \right\},$$

then it is easy to see that P is a cone in $C[0, 1]$. From [3] and [20], it is evident that BVP (1.1)-(1.2) has an integral formulation given by

$$(2.5) \quad u(t) = \lambda \int_0^1 G(t, s) f(s, u(s)) ds + \psi(u(1)) \phi(t),$$

where G and ϕ defined in (2.2) and (2.3).

Now, we define the integral operators $A, B, T_\lambda : P \rightarrow C[0, 1]$ by

$$(\lambda Au)(t) = \lambda \int_0^1 G(t, s) f(u(s)) ds; \quad (Bu)(t) = \psi(u(1)) \left(\frac{1}{4} t^2 - \frac{1}{6} t^3 \right), \quad T_\lambda = \lambda A + B.$$

Lemma 2.6. *Let $y \in C^+[0, 1]$. If $u \in C^4[0, 1]$ satisfies*

$$(2.6) \quad \begin{cases} u^{(4)}(t) = y(t), & 0 \leq t \leq 1, \\ u(0) = u'(0) = u'(1) = 0, \quad u^{(3)}(1) + \psi(u(1)) = 0, \end{cases}$$

then

- (i) $u(t) \geq 0$ for $t \in [0, 1]$;
- (ii) $u'(t) \geq 0$ for $t \in [0, 1]$.

Proof. From Lemma 2.4, we obtain $u(t) \geq 0$ and $u'(t) \geq 0$ for $t \in [0, 1]$. \square

It is clear from (2.5) that solution of the BVP (1.1)-(1.2) are fixed points of $T_\lambda : P \rightarrow P$. In particular u is fixed point of B if and only if u is solution of the following BVP

$$(2.7) \quad \begin{cases} u^{(4)}(t) = 0, & 0 < t < 1, \\ u(0) = u'(0) = u'(1) = u'''(1) + \psi(u(1)) = 0, \end{cases}$$

and u is fixed point of λA if and only if u is solution of the following BVP

$$(2.8) \quad \begin{cases} u^{(4)}(t) = \lambda f(t, u(t)), & 0 < t < 1, \\ u(0) = u'(0) = u'(1) = u'''(1) = 0. \end{cases}$$

Lemma 2.7. $A(P) \subset P$, $B(P) \subset P$ and $T_\lambda(P) \subset P$.

Proof. $u \in P$ implies $u(t) \geq 0$, so $f(t, u(t)) \geq 0$ and $\psi(u(1)) \geq 0$. Moreover, for $u \in P$,

$$(T_\lambda u)^{(4)}(t) = \lambda f(t, u(t)) \geq 0, \quad t \in [0, 1],$$

$$(T_\lambda u)(0) = (T_\lambda u)'(0) = (T_\lambda u)'(1) = 0,$$

$$\text{and } (T_\lambda u)^{(3)}(1) = -\psi(u(1)) \leq 0.$$

To prove $T_\lambda(P) \subset P$, we show that $(T_\lambda u)(t) - \frac{t^3}{6}(T_\lambda u)(1) \geq 0$, for $t \in [0, 1]$. Let $u(t) \in P$, $t \in [0, 1]$, we have

$$\begin{aligned} (T_\lambda u)(t) - \frac{t^3}{6}(T_\lambda u)(1) &\geq \frac{\lambda}{36} t^3 \int_0^1 s^3 f(s, u(s)) ds + \psi(u(1)) \left(\frac{t^2}{4} - \frac{t^3}{6} \right) - \frac{t^3}{6} \left[\left(\frac{1}{12} \psi(u(1)) \right) \right] \\ &\geq \frac{\lambda}{36} t^3 \int_0^1 s^3 f(s, u(s)) ds + \psi(u(1)) \left(\frac{t^2}{4} - \frac{13t^3}{6 \times 12} \right) \\ &\geq \frac{\lambda}{36} t^3 \int_0^1 s^3 f(s, u(s)) ds \geq 0. \end{aligned}$$

Therefore $(T_\lambda u)(t) \geq \frac{t^3}{6}(T_\lambda u)(1)$, for $t \in [0, 1]$. Thus, we obtain $T_\lambda(P) \subset P$. From the above proof, we can show that $A(P) \subset P$, $B(P) \subset P$. \square

If we let

$$K = \{u \in P / u(t) \text{ is nondecreasing}\},$$

then, it is easy to show that $K \subset P$ is also a cone in E , and if $u \in K$, then $\|u\| = u(1)$.

Lemma 2.8. $T_\lambda(P) \subset K$, $A(P) \subset K$ and $B(P) \subset K$.

Proof. It follows from Lemma 2.6 (ii) and Lemma 2.7. \square

Lemma 2.9. (i) $A : K \rightarrow K$ is completely continuous.

(ii) If $\psi(u)$ is nondecreasing, then $B : K \rightarrow K$ is completely continuous.

Proof. We know that $A(P) \subset K$, $B(P) \subset K$. Since the proof is similar we only prove that A is completely continuous. To do this let D be a bounded subset of K . Then there exists a positive constant M_1 such that

$$\|u\| \leq M_1, \quad \forall u \in D.$$

Now, we shall prove that $A(D)$ is relatively compact in K . Suppose that $(y_k)_{k \in \mathbb{N}^*} \subset A(D)$. Then there exist $(x_k)_{k \in \mathbb{N}^*} \subset D$, such that

$$y_k = Ax_k.$$

Let $M_2 = \sup_{0 \leq t \leq 1} |f(t, u(t))|$ for all $(t, u) \in [0, 1] \times [0, M_1]$. For any $k \in \mathbb{N}^*$, by Lemma 2.4, we have

$$\begin{aligned} |y_k(t)| &= |(Ax_k)(t)| = \lambda \left| \int_0^1 G(t, s) f(s, x_k(s)) ds \right| \\ &\leq \lambda M_2 \int_0^1 G(t, s) ds \\ &\leq \frac{1}{4} \lambda M_2 (2t - t^2) \int_0^1 s^2 ds \\ &\leq \frac{\lambda}{6} M_2, \end{aligned}$$

which implies that $(y_k(t))_{k \in \mathbb{N}^*}$ is uniformly bounded.

Now, we show that $A(D)$ is equicontinuous. For any $u \in K$, $n \geq k$, and $t_1, t_2 \in [0, 1]$ with $|t_1 - t_2| < \delta$, we have

$$\begin{aligned} |y_k(t_1) - y_k(t_2)| &= |Au(t_1) - Au(t_2)| \\ &\leq \left| \lambda \int_0^1 (G(t_1, s) - G(t_2, s)) f(s, x_k(s)) ds \right| \\ &\leq \lambda M_2 \int_0^1 |G(t_1, s) - G(t_2, s)| ds. \end{aligned}$$

It follows from the uniform continuity of Green's function G on $[0, 1] \times [0, 1]$, that for any $\varepsilon > 0$, we have

$$|G(t_1, s) - G(t_2, s)| \leq \frac{\varepsilon}{\lambda M_2}, \quad \text{for } t_1, t_2, s \in [0, 1], |t_1 - t_2| < \delta.$$

Then

$$\begin{aligned} |y_k(t_1) - y_k(t_2)| &= |Au(t_1) - Au(t_2)| \\ &\leq \lambda M_2 \int_0^1 |G(t_1, s) - G(t_2, s)| ds \\ &\leq \frac{\lambda M_2 \varepsilon}{\lambda M_2} = \varepsilon. \end{aligned}$$

Therefore, $A(D)$ is equicontinuous. By the Ascoli-Arzela Theorem, we know that A is completely continuous. \square

By Lemmas 2.8 and 2.9, we know that if $u \in P \setminus \{\theta\}$ is solution for BVP (1.1)-(1.2), then u is positive solution for BVP (1.1)-(1.2) and it is obvious from Lemma 2.8 that if $u \in P \setminus \{\theta\}$ is a solution for BVP (1.1)-(1.2) then $u \in K \setminus \{\theta\}$.

3. EXISTENCE AND NONEXISTENCE RESULTS

In this section we will apply a theorem due to Krasnoselskii to study the existence, multiplicity and nonexistence of solutions for BVP (1.1)-(1.2) in $K \setminus \{\theta\}$.

Theorem 3.1. (see [19]) *Let E be a Banach space and $K \subset E$ be a cone in E . Assume Ω_1, Ω_2 are open subset of E with $0 \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$, and let*

$$T : K \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow K$$

be a completely continuous operator such that

(A) $\|Tu\| \leq \|u\|, \forall u \in K \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|, \forall u \in K \cap \partial\Omega_2$; or

(B) $\|Tu\| \geq \|u\|, \forall u \in K \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|, \forall u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

We adopt the following assumptions:

(H_1) $f(t, u)$ is nondecreasing in $u \in [0, \infty)$ for fixed $t \in [0, 1]$.

(H_2) $\psi(u)$ is nondecreasing in $u \in [0, \infty)$.

(H_3) $F_0 = \int_0^1 s^2 f(s, 0) ds > 0$.

(H_4) $\psi(1) < 4$.

(H_5) $f_\infty = \lim_{u \rightarrow \infty} \min_{t \in [\frac{1}{2}, 1]} \frac{f(t, u)}{u} = +\infty$.

(H_6) $\psi_\infty = \lim_{u \rightarrow \infty} \inf \frac{\psi(u)}{u} > 12$.

Set

$$(3.1) \quad \Lambda = \{\lambda > 0 / \text{there exists } u_\lambda \in K \setminus \{\theta\} \text{ such that } T_\lambda u_\lambda = u_\lambda\},$$

and

$$\lambda^* = \sup \Lambda.$$

Lemma 3.2. *Suppose that (H_1) , (H_2) and (H_3) hold. If $\lambda' \in \Lambda$, then $(0, \lambda'] \subset \Lambda$.*

Proof. $\lambda' \in \Lambda$ means that there exists $u_{\lambda'} \in K \setminus \{\theta\}$ such that $T_{\lambda'} u_{\lambda'} = u_{\lambda'}$. Therefore, for any $\lambda \in (0, \lambda']$ we have

$$T_\lambda u_{\lambda'} \leq T_{\lambda'} u_{\lambda'} = u_{\lambda'}$$

Set

$$w_0 = u_{\lambda'}, w_n = T_\lambda w_{n-1}, n = 1, 2, \dots$$

From (H_1) and (H_2) , we obtain

$$w_0(t) \geq w_1(t) \geq \dots \geq w_n(t) \geq \dots \geq \frac{F_0 \lambda}{12} t^2,$$

by Lemma 2.9 and (H_3) , $\{w_n\}$ converges to fixed point of T_λ in $K \setminus \{\theta\}$. Thus $(0, \lambda'] \subset \Lambda$. The proof is complete. \square

Now, we let $\lambda_* = \frac{4 - \psi(1)}{F_1}$, $F_1 = \int_0^1 2s^2 f(s, 1) ds$, $u_0(t) = \frac{F_0 \lambda}{12} t^2$, $v_0(t) = t$, $F_\infty = \lim_{u \rightarrow \infty} \sup_{0 \leq t \leq 1} \max \frac{f(t, u)}{u}$ and $\Psi_\infty = \lim_{u \rightarrow \infty} \sup \frac{\psi(u)}{u}$.

Theorem 3.3. *Suppose that (H_1) , (H_2) and (H_3) hold. If (H_4) holds, then T_λ has minimal and maximal fixed point in $[u_0, v_0]$ for $\lambda \in (0, \lambda_*]$. Moreover, there exists $\lambda^* \geq \lambda_* > 0$ such that T_λ has at least one and has no fixed points in $K \setminus \{\theta\}$ for $0 < \lambda < \lambda^*$ and $\lambda > \lambda^*$, respectively.*

Proof. From $(H_1), (H_2), (H_3), (H_4)$ and (2.4), we have $\lambda_* > 0$. For any $\lambda \in (0, \lambda_*]$, we obtain

$$\begin{aligned}
(T_\lambda u_0)(t) &= \lambda \int_0^1 G(t, s) f(s, u_0(s)) ds + \psi(u_0(1)) \left(\frac{t^2}{4} - \frac{t^3}{6} \right) \\
&\geq \lambda \int_0^1 G(t, s) f(s, u_0(0)) ds + \psi(u_0(1)) \left(\frac{t^2}{4} - \frac{t^3}{6} \right) \\
&\geq \frac{\lambda}{12} t^2 \int_0^1 s^2 f(s, 0) ds + \psi(u_0(1)) \left(\frac{t^2}{4} - \frac{t^3}{6} \right) \\
&\geq \frac{\lambda}{12} t^2 F_0 = u_0(t),
\end{aligned}$$

and

$$\begin{aligned}
(T_\lambda v_0)(t) &= \lambda \int_0^1 G(t, s) f(s, v_0(s)) ds + \psi(v_0(1)) \left(\frac{t^2}{4} - \frac{t^3}{6} \right) \\
&\leq \frac{\lambda_*}{4} (2t - t^2) \int_0^1 s^2 f(s, v_0(1)) ds + \psi(v_0(1)) \left(\frac{t}{4} - \frac{t^2}{6} \right) t \\
&\leq \frac{\lambda_*}{4} (2t - t^2) \int_0^1 s^2 f(s, v_0(1)) ds + \psi(v_0(1)) \left(\frac{t}{4} - \frac{t^2}{6} \right) \\
&\leq \frac{\lambda_*}{4} (2t) \int_0^1 s^2 f(s, v_0(1)) ds + \psi(v_0(1)) \left(\frac{t}{4} \right) \\
&\leq \frac{t}{4} \left[\lambda_* \int_0^1 2s^2 f(s, v_0(1)) ds + \psi(1) \right], \quad v_0(1) = 1 \\
&\leq \frac{t}{4} [\lambda_* F_1 + \psi(1)] \leq v_0(t).
\end{aligned}$$

Set

$$u_n = T_\lambda u_{n-1}, \quad v_n = T_\lambda v_{n-1}, \quad n = 1, 2, \dots,$$

then from $(H_1), (H_2)$, we have

$$(3.2) \quad u_0(t) \leq u_1(t) \leq \dots \leq u_n(t) \leq \dots \leq v_n(t) \dots \leq v_1(t) \leq v_0(t).$$

Lemma 2.9 implies that $\{u_n\}$ and $\{v_n\}$ converge to fixed points u_λ and v_λ of T_λ , respectively.

From (3.2) it is evident that $u_\lambda, v_\lambda \in K \setminus \{\theta\}$ are the minimal fixed point and maximal fixed point of T_λ in $[u_0, v_0]$, respectively.

By the definition of λ^* , there exists a nondecreasing sequence $\{\lambda_n\}_1^{+\infty}$ such that $\lim_{n \rightarrow +\infty} \lambda_n = \lambda^*$. Let $\{u_{\lambda_n}\}_1^{+\infty}$ is bounded subset in K . Then there exists a constant $M > 0$ such that

$$\|u_{\lambda_n}\| \leq M, \quad \text{for } n \in \mathbb{N}^*,$$

which implies that $\{u_{\lambda_n}\}_1^{+\infty}$ is uniformly bounded.

Now, we show that $\{u_{\lambda_n}\}_1^{+\infty}$ is equicontinuous. For any $u_{\lambda_n} \in K$, $n \in \mathbb{N}^*$ and $t_1, t_2 \in [0, 1]$, with

$|t_1 - t_2| < \delta$, we have

$$\begin{aligned} |u_{\lambda_n}(t_1) - u_{\lambda_n}(t_2)| &\leq \lambda^* \int_0^1 |G(t_1, s) - G(t_2, s)| f(s, M) ds \\ &\quad + \frac{1}{2} \psi(M) |t_1 - t_2| \left[\frac{1}{2} |t_1 + t_2| + \frac{1}{3} |t_1^2 + t_1 t_2 + t_2^2| \right] \\ &\leq \lambda^* \int_0^1 |G(t_1, s) - G(t_2, s)| f(s, M) ds + \frac{5}{6} \psi(M) |t_1 - t_2|, \end{aligned}$$

which implies that $\{u_{\lambda_n}\}_1^{+\infty}$ is equicontinuous subset in K . Consequently, by an application of the Arzela-Ascoli Theorem we conclude that $\{u_{\lambda_n}\}_1^{+\infty}$ is a relatively compact set in K . So, there exists a subsequence $\{u_{\lambda_{n_i}}\} \subset \{u_{\lambda_n}\}$ converging to $u^* \in K$. Note that

$$(u_{\lambda_{n_i}})(t) = \lambda_{n_i} \int_0^1 G(t, s) f(s, u_{\lambda_{n_i}}(s)) ds + \psi(u_{\lambda_{n_i}}(1)) \left(\frac{t^2}{4} - \frac{t^3}{6} \right).$$

By taking the limit we have $u^*(t) = (T_{\lambda^*} u^*)(t)$. Therefore T_λ has at least one fixed point for $0 < \lambda < \lambda^*$. Finally, T_λ has no fixed point for $\lambda > \lambda^*$. The proof is complete. \square

Theorem 3.4. *Suppose that (H_1) , (H_2) , (H_3) , (H_4) and (2.4) hold. If $(F_{+\infty} < +\infty, \Psi_\infty < 4)$, then, when $F_\infty > 0$, there exists $\lambda^* \geq \frac{3(4-\Psi_\infty)}{2F_\infty} > 0$ such that T_λ has at least one and has no fixed points in $K \setminus \{\theta\}$ for $0 < \lambda < \lambda^*$ and $\lambda > \lambda^*$, respectively. When $F_\infty = 0$, T_λ has at least one fixed points in $K \setminus \{\theta\}$ for $\lambda > 0$.*

Proof. Since $F_\infty < \infty$, $\Psi_\infty < 4$, for any $0 < \varepsilon < 4 - \Psi_\infty$, there exists $N_0 > 0$ such that $f(t, u) \leq (F_\infty + \varepsilon)u$ and $\psi(u) \leq (\Psi_\infty + \varepsilon)u$ for $u > N_0$, $t \in [0, 1]$. Let $w_0(t) = 4N_0t$ and $\lambda_0 = \frac{3(4-\Psi_\infty-\varepsilon)}{2(F_\infty+\varepsilon)}$, then $\lambda_0 > 0$ and

$$\begin{aligned} (T_{\lambda_0} w_0)(t) &= \lambda_0 \int_0^1 G(t, s) f(s, w_0(s)) ds + \psi(w_0(1)) \left(\frac{t^2}{4} - \frac{t^3}{6} \right) \\ &\leq \frac{\lambda_0}{4} (2t) \int_0^1 s^2 f(s, w_0(s)) ds + \psi(w_0(1)) \left(\frac{t^2}{4} \right) \\ &\leq \frac{\lambda_0}{4} (2t) \int_0^1 s^2 (F_\infty + \varepsilon) w_0(s) ds + w_0(1) (\Psi_\infty + \varepsilon) \left(\frac{t^2}{4} \right) \\ &\leq \frac{\lambda_0}{4} w_0(t) (F_\infty + \varepsilon) (2t) \int_0^1 s^2 ds + (\Psi_\infty + \varepsilon) \left(\frac{t}{4} \right) \\ &\leq \frac{1}{4} t w_0(t) \left[\frac{2\lambda_0}{3} (F_\infty + \varepsilon) + (\Psi_\infty + \varepsilon) \right] \\ &\leq \frac{1}{4} w_0(t) \left[\frac{2\lambda_0}{3} (F_\infty + \varepsilon) + (\Psi_\infty + \varepsilon) \right] \leq w_0(t). \end{aligned}$$

Now, set

$$w_0(t) = 4N_0t, \quad w_n = T_{\lambda_{n-1}} w_{n-1}, \quad n = 1, 2, \dots$$

From (H_1) and (H_2) , we obtain

$$w_0(t) \geq w_1(t) \geq \dots \geq w_n(t) \geq \dots \geq \frac{F_0 \lambda}{12} t^2.$$

Therefore, the sequence $\{w_n\}$ is bounded in $K \setminus \{\theta\}$. By Lemma 2.9 and the definition of λ^* , the operator T_{λ_n} completely continuous. Hence the sequence $\{w_n\}$ is compact in $K \setminus \{\theta\}$, it is also monotone. Then it is uniformly convergent to fixed points u^* of T_{λ_n} in $K \setminus \{\theta\}$. When we pass to the limit we get

$$u^* = T_{\lambda^*} u^*.$$

For $\lambda > \lambda^*$, there exists $\{\lambda_n\}_1^{+\infty}$, with $\lim_{n \rightarrow \infty} \lambda_n = \lambda$, we prove that the problem has no positive solution. Suppose the contrary that the problem has a positive solution u_{λ_n} , then we get

$$\begin{aligned} \|u_{\lambda_n}\| &= (T_{\lambda_n} u_{\lambda_n})(1) \leq \frac{1}{4} \lambda_n \int_0^1 2s^2 f(s, u_{\lambda_n}(s)) ds + \frac{1}{4} \psi(u_{\lambda_n}(1)) \\ &\leq \frac{\lambda_n}{4} \int_0^1 2s^2 (F_\infty + \varepsilon) u_{\lambda_n}(1) ds + u_{\lambda_n}(1) (\Psi_\infty + \varepsilon) \left(\frac{1}{4}\right) \\ &\leq \frac{1}{4} \left[\frac{2\lambda_n}{3} (F_\infty + \varepsilon) + (\Psi_\infty + \varepsilon) \right] \|u_{\lambda_n}\| < \|u_{\lambda_n}\|. \end{aligned}$$

Thus

$$\|u_{\lambda}\| < \|u_{\lambda}\|,$$

which is a contradiction. The proof is complete. \square

Lemma 3.5. *Assume that (H_1) , (H_2) , (H_3) and (H_4) hold and one of (H_5) and (H_6) holds. If Λ is nonempty, then*

- (i) Λ is bounded from above, that is $\lambda^* < +\infty$.
- (ii) $\lambda^* \in \Lambda$.

Proof. (i) Suppose to the contrary that there exists an increasing sequence $\{\lambda_n\}_1^\infty \subset \Lambda$ such that $\lim_{n \rightarrow +\infty} \lambda_n = +\infty$. Set $u_{\lambda_n} \in K \setminus \{\theta\}$ is a fixed point of T_{λ_n} that is,

$$T_{\lambda_n} u_{\lambda_n} = u_{\lambda_n}.$$

There are two cases to be considered.

Case 1. $\{u_{\lambda_n}\}_1^{+\infty}$ is bounded, that is there exists a constant $M > 0$ such that

$$\|u_{\lambda_n}\| \leq M, \text{ for } n = 1, 2, \dots$$

Hence, from (H_1) , (H_2) , (H_3) , and (2.4), we have

$$\begin{aligned} M &\geq \|u_{\lambda_n}\| \geq (T_{\lambda_n} u_{\lambda_n})(1) \\ &\geq \frac{\lambda_n}{12} \int_0^1 s^2 f(s, u_{\lambda_n}(s)) ds \\ &\geq \frac{\lambda_n}{12} \int_0^1 s^2 f(s, 0) ds = \frac{\lambda_n}{12} F_0 \rightarrow +\infty, \end{aligned}$$

which is a contradiction.

Case 2. $\{u_{\lambda_n}\}_1^{+\infty}$ is unbounded, that is there exists subsequence of $\{u_{\lambda_n}\}_1^{+\infty}$ still denoted by $\{u_{\lambda_n}\}_1^{+\infty}$

such that $\lim_{n \rightarrow +\infty} \|u_{\lambda_n}\| = +\infty$. When (H_5) holds, take $L > \frac{288}{\lambda_1}$ there exists $N_1 > 0$ such that $f(t, u) \geq Lu$, for $u \geq N_1$, $t \in [\frac{1}{2}, 1]$. Choose n_1 such that $\|u_{\lambda_{n_1}}\| > 24N_1$. Thus

$$f\left(t, \frac{1}{24} \|u_{\lambda_{n_1}}\|\right) \geq \frac{1}{24} L \|u_{\lambda_{n_1}}\|, \quad t \in \left[\frac{1}{2}, 1\right]$$

Moreover, from (H_1) and the definition of K , we have

$$\begin{aligned} \|u_{\lambda_{n_1}}\| &\geq (T_{\lambda_{n_1}} u_{\lambda_{n_1}})(1) \\ &\geq \frac{\lambda_{n_1}}{12} \int_{\frac{1}{2}}^1 s^2 f(s, u_{\lambda_{n_1}}(s)) ds \\ &\geq \frac{\lambda_{n_1}}{12} \int_{\frac{1}{2}}^1 s^2 f\left(s, \frac{1}{6} \|u_{\lambda_{n_1}}\|\right) ds \\ &> \frac{\lambda_{n_1}}{36 \times 8} L \|u_{\lambda_{n_1}}\| > \|u_{\lambda_{n_1}}\|, \end{aligned}$$

which is contradiction.

When (H_6) holds, choose $\varepsilon > 0$ such that $\frac{1}{12}(\psi_\infty - \varepsilon) > 1$. There exists $N_2 > 1$ such that $\psi(u) \geq (\psi_\infty - \varepsilon)u$, for $u \geq N_2$. Choose n_2 such that $\|u_{\lambda_{n_2}}\| > N_2$. Then

$$\psi(u_{\lambda_{n_2}}(1)) = \psi(\|u_{\lambda_{n_2}}\|) \geq (\psi_\infty - \varepsilon) \|u_{\lambda_{n_2}}\|.$$

Moreover

$$\|u_{\lambda_{n_2}}\| = (T_{\lambda_{n_2}} u_{\lambda_{n_2}})(1) \geq \frac{1}{12} \psi(u_{\lambda_{n_2}}(1)) \geq \frac{1}{12} (\psi_\infty - \varepsilon) \|u_{\lambda_{n_2}}\| > \|u_{\lambda_{n_2}}\|,$$

which is contradiction. Consequently, we find that Λ is bounded from above.

(ii) From the definition of λ^* , there exists a nondecreasing sequence $\{\lambda_n\}_1^\infty$ such that $\lim_{n \rightarrow +\infty} \lambda_n = \lambda^*$. Let $u_{\lambda_n} \in K \setminus \{\theta\}$ be a fixed point of T_{λ_n} . Arguing similarly as above in case 2, we can show that $\{u_{\lambda_n}\}_1^{+\infty}$ is a bounded subset in K , that is there exists a constant $M > 0$. From (H_1) , (H_3) and (H_4) , we have

$$\begin{aligned} \|u_{\lambda_n}\| &= (T_{\lambda_n} u_{\lambda_n})(1) \\ &\leq \frac{\lambda_n}{12} \int_0^1 s^2 f(s, u_{\lambda_n}(s)) ds + \frac{1}{12} \psi(1) \\ &\leq \frac{1}{12} \left(\lambda_n \int_0^1 s^2 f(s, 0) ds + 4 \right) \\ &= \frac{1}{12} (\lambda_n F_0 + 4) \rightarrow \frac{1}{12} (\lambda^* F_0 + 4) = M, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Therefore

$$\|u_{\lambda_n}\| \leq M, \quad n = 1, 2, \dots$$

From the proof of Theorem 3.3 we know that $\{u_{\lambda_n}\}_1^{+\infty}$ is equicontinuous subset in K and by an application of the Arzela-Ascoli Theorem we conclude that $\{u_{\lambda_n}\}_1^\infty$ is a relatively compact set in K . So, there exists a subsequence $\{u_{\lambda_{n_i}}\} \subset \{u_{\lambda_n}\}$ converging to $u^* \in K$. Note that

$$(u_{\lambda_{n_i}})(t) = \lambda_{n_i} \int_0^1 G(t, s) f(s, u_{\lambda_{n_i}}(s)) ds + \psi(u_{\lambda_{n_i}}(1)) \left(\frac{t^2}{4} - \frac{t^3}{6} \right).$$

By taking the limit we have $u^*(t) = (T_{\lambda^*} u^*)(t) \geq \frac{\lambda^*}{12} F_0 t^2$, that is $\lambda^* \in \Lambda$. The proof is complete. \square

Theorem 3.6. *Suppose that (H_1) , (H_2) , (H_3) and (H_4) hold and that one of (H_5) and (H_6) holds. Then there exists $\lambda^* \geq \lambda_* > 0$ such that BVP (1.1)-(1.2) has at least two, one and no positive solutions for $0 < \lambda \leq \lambda_*$, $\lambda_* < \lambda \leq \lambda^*$ and $\lambda > \lambda^*$ respectively.*

Proof. From (H_1) , (H_2) , (H_3) and (H_4) we have $(0, \lambda_*] \subset \Lambda$. So $\lambda^* \geq \lambda_* > 0$. From Lemmas 3.2 and 3.5, we have $(0, \lambda^*] = \Lambda$. Therefore, from the definition of λ^* we only to prove that T_λ has at least two fixed points in $K \setminus \{\theta\}$ for $\lambda \in (0, \lambda_*]$.

Now, given $\lambda \in (0, \lambda_*]$. Theorem 3.3 means that T_λ has at least one fixed point $u_{\lambda,1} \in K \setminus \{\theta\}$ which satisfies $\|u_{\lambda,1}\| \leq 1$.

Let

$$K_1 = \{u \in K : \|u\| < 1\}.$$

Note that $(\frac{3}{2} - t) \leq 1$, for $t \in [0, 1]$, so for $u \in K$ with $\|u\| = 1$, i.e $u \in \partial K_1$, we have

$$\begin{aligned} \|u\| = \|T_\lambda u\| &= (T_\lambda u)(1) \leq \lambda \int_0^1 G(t,s) f(s, u(s)) ds + \frac{1}{12} \psi(u(1)) \\ &\leq \frac{\lambda}{12} \int_0^1 s^2 (3-2s) f(s, u(s)) ds + \frac{1}{12} \psi(u(1)) \\ (3.3) \quad &\leq \frac{\lambda}{12} \int_0^1 s^2 (3-2s) f(s, 1) ds + \frac{1}{12} \psi(u(1)) \\ &\leq \frac{1}{12} \left(\lambda_* \int_0^1 2s^2 \left(\frac{3}{2} - s\right) f(s, 1) ds + \psi(u(1)) \right) \\ &\leq \frac{1}{12} \left(\left(\frac{4 - \psi(1)}{F_1}\right) F_1 + \psi(u(1)) \right) = \frac{1}{3} < \|u\|, \end{aligned}$$

When (H_5) holds, take $L > \frac{288}{\lambda}$ there exists $N_1 > 0$ such that $f(t, u) \geq Lu$, for $u \geq N_1$, $t \in [\frac{1}{2}, 1]$. Set $K_2 = \{u \in K : \|u\| < 24N_1\}$. Then $\overline{K_1} \subset K_2$. If $u \in \partial K_2$, we have

$$\begin{aligned} \|T_\lambda u\| &= (T_\lambda u)(1) \\ &\geq \frac{1}{12} \psi(u(1)) \geq \frac{\lambda}{12} \int_{\frac{1}{2}}^1 s^2 f\left(s, \frac{1}{24} \|u\|\right) ds \\ &\geq \frac{\lambda L}{288} \|u\| > \|u\|. \end{aligned}$$

Consequently, applying Theorem 3.1 that T_λ has a fixed point $u_{\lambda,2} \in \overline{K_2} \setminus K_1$.

When (H_6) holds, choose $\varepsilon > 0$ such that $\frac{1}{12}(\psi_\infty - \varepsilon) > 1$. There exists $N_2 > 1$ such that $\psi(u) \geq (\psi_\infty - \varepsilon)u$, for $u \geq N_2$.

Set $K'_2 = \{u \in K : \|u\| < N_2\}$. Then $\overline{K_1} \subset K'_2$. If $u \in \partial K'_2$, we have

$$\begin{aligned} \|T_\lambda u\| &= (T_\lambda u)(1) \geq \frac{1}{12} \psi(u(1)) \\ &\geq \frac{1}{12} (\psi_\infty - \varepsilon) u(1) \\ &> \|u\|. \end{aligned}$$

Consequently, applying Theorem 3.1 that T_λ has a fixed point $u_{\lambda,2} \in \overline{K'_2} \setminus K_1$.

Equation (3.3) implies that T_λ has no fixed points in ∂K_1 . In conclusion, for $\lambda \in (0, \lambda_*]$, T_λ has at least two fixed points $u_{\lambda,1}$ and $u_{\lambda,2}$ in K with $0 < \|u_{\lambda,1}\| < 1 < \|u_{\lambda,2}\|$. The proof is complete. \square

We present an example to illustrate the applicability of the results shown before.

Example 3.7. Consider the following boundary value problem:

$$(3.4) \quad u^{(4)}(t) = \lambda(t + u + \ln(1 + u)), \quad 0 < t < 1,$$

$$(3.5) \quad u(0) = u'(0) = u'(1) = u'''(1) + \psi(u(1)) = 0,$$

where

$$\psi(u) = \begin{cases} 2 \arcsin u, & 0 \leq u \leq \frac{\sqrt{2}}{2}, \\ \frac{\pi}{4} + \arcsin u, & \frac{\sqrt{2}}{2} \leq u \leq 1, \\ 0, & u > 1. \end{cases}$$

It is easy to compute that

$$F_\infty = 2, \psi_\infty = 0, F_0 = \frac{1}{4}, F_1 = \frac{1}{2} + \frac{2}{3}(1 + \ln 2), \psi(1) = \frac{3\pi}{4} \text{ and } \lambda_* = \frac{48 - 9\pi}{6 + 8(1 + \ln 2)}.$$

So, the assumptions (H_1) , (H_2) , (H_3) and (H_4) are satisfied, it follows from Theorem 3.4 there exists $\lambda^* = 3 \geq \lambda_*$ such that BVP (3.4)-(3.5) has at least one positive solution for $0 < \lambda \leq 3$ and has no positive solution for $\lambda > \lambda^*$.

Acknowledgments. The authors want to thank the anonymous referee for the throughout reading of the manuscript and several suggestions that help us improve the presentation of the paper.

REFERENCES

- [1] A. R. Aftabizadeh, *Existence and uniqueness theorems for fourth-order boundary value problems*, J. Math. Anal. Appl. **116** (1986), 415-426.
- [2] R. P. Agarwal, D. O'Regan and P. J. Wang, *Positive solutions of differential, Difference, and integral equations*, Kluwer Academic, Boston, Ma, 1999.
- [3] E. Alves, T. F. Ma. and M. L. Pelicer, *Monotone positive solutions for a fourth order equation with nonlinear boundary conditions*, Nonlinear anal. **71** (2009), 3834-3841.
- [4] D. R. Anderson and R. I. Avery, *A fourth-order four-point right focal boundary value problem*, Rocky Mountain of Mathematics. **36** (2006), 367-380.
- [5] Z. Bai, *The upper and lower solution method for some fourth order boundary value problem*, Nonlinear Anal. **67** (2007), 1704- 1709.
- [6] Z. Bai and H. Wang, *Positive solutions of some nonlinear fourth order beam equations*, J. Math. Anal. Appl. **270** (2002), 357-368.
- [7] S. Benaicha and F. Hadouchi, *Positive solutions of a nonlinear fourth-order integral boundary value problem*, Annals of West University of Timisoara - Mathematics and Computer Science, **54** (2016), 73- 86.
- [8] G. Bonanno and BD. Bella, *A boundary value problem for fourth-order elastic beam equations*, J. Math. Anal. Appl. **343** (2008), 1166-1176.
- [9] N. Bouteraa and S. Benaicha, *Triple positive solutions of higher-order nonlinear boundary value problem*, J. Comput. Sci . Comp. Math. **7** (2017), 25-30.
- [10] A. Cabada and S. Heikkila, *Uniqueness comparison and existence results for third order functional initial-boundary value problems*, Comput. Math. Appl. **41** (2001), 607-618.
- [11] A. Cabada and S. Tersian, *Existence and multiplicity of solutions to boundary value problems for fourth-order impulsive differential equations*. Bound. Value Probl. **105** (2014).
- [12] J. Davis and J. Henderson, *Uniqueness implies existence for fourth-order Lidstone boundary value problems*, Panamer. Math.J. **8** (1998), 23-35.
- [13] L. H. Erbe and H. Wang, *On he existence of positive solutions of ordinary differential equations*, Proc. Amer. Math. Soc. **120** (1994), 743-748.

- [14] J. R. Graef, C. Qian and B. Yang, *A three point boundary value problem for nonlinear fourth order differential equations*, J. Math. Anal. Appl. **287** (2003), 217-233.
- [15] J. R. Graef and B. Yang, *Existence and nonexistence of positive solutions of fourth order nonlinear boundary value problems*, Appl. Anal. **74** (2000), 201-214.
- [16] C. P. Gupta, *Existence and uniqueness theorem for the bending of an elastic beam equation*, Appl. Anal. **26** (1988), 289-304.
- [17] G. Infante and P. Pietramala, *A cantilever equation with nonlinear boundary conditions*, Electron. J. Qual. Theory Differ. Equ. Spec. Ed. **15** (2009), 1-14.
- [18] N. Kosmatov, *Countably many solutions of a fourth order boundary-value problems*, Electron. J. Qual. Theory Differ. Equ. **12** (2004), 1-15.
- [19] M. A. Krasnosel'skii, *Positive solutions of operator equations*, Noordhoff, Groningen, 1964.
- [20] S. Li and Q. Zhang, *Existence and uniqueness of monotone positive solutions for an elastic beam equation with nonlinear boundary conditions* Comput. Math. Appl. **63** (2012), 1355-1360.
- [21] Y. Li, *Positive solutions of fourth-order boundary value problems with two parameters*, J. Math. Anal. Appl. **281** (2003), 477-484.
- [22] R. Ma and B. Thompson, *Nodal solutions for a nonlinear fourth-order eigenvalue problem*, Acta Math. sin. Engl. Ser. **24** (2008), 27-34.
- [23] R. Ma and L. Xu, *Existence of positive solutions for a nonlinear fourth-order boundary value problem*. Appl. Math. Lett. **23** (2010), 537-543.
- [24] R. Ma and H. Wang, *On the existence of positive solutions of fourth order ordinary differential equations*, Appl. Anal. **59** (1995), 225-231.
- [25] Y. Sun and C. Zhu, *Existence of positive solutions for singular fourth-order three-point boundary value problems*. Adv. Differ. Equ. **51**(2013), 1-13.
- [26] W. Wang, Y. Zheng, H. Yang and J. Wang, *Positive solutions for elastic beam equations with nonlinear boundary conditions and a parameter*, Bound. Value Prob. **80** (2014), 17 pages.
- [27] B. Yang, *Positive solutions for a fourth order boundary-value problem*, Electron. J. Qual. Theory Differ. Equ. **3** (2005), 1-17
- [28] X. Zhang, *Existence and iteration of monotone positive solutions for an elastic beam equation with a corner*, Nonlinear Anal. Real World appl. **10** (2009), 2097-2103.

LABORATORY OF FUNDAMENTAL AND APPLIED MATHEMATICS OF ORAN (LMFAO), UNIVERSITY OF ORAN1, AHMED BENBELLA, ORAN, ALGERIA
E-mail address: `bouteraa-27@hotmail.fr`

LABORATORY OF FUNDAMENTAL AND APPLIED MATHEMATICS OF ORAN (LMFAO), UNIVERSITY OF ORAN1, AHMED BENBELLA, ORAN, ALGERIA
E-mail address: `slimanebenaicha@yahoo.fr`

LABORATORY OF FUNDAMENTAL AND APPLIED MATHEMATICS OF ORAN (LMFAO), UNIVERSITY OF ORAN1, AHMED BENBELLA, ORAN, ALGERIA
E-mail address: `djourdem.habib7@gmail.com`

LABORATORY OF MATHEMATICS AND ITS APPLICATIONS (LAMAP), UNIVERSITY OF ORAN1, AHMED BENBELLA, ORAN, ALGERIA
E-mail address: `mohamed.souaflia@gmail.com`