ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2012, VOLUME 2, p.62-72

A GENERALIZATION OF LOCAL HOMOLOGY FUNCTORS

Mohammad H. Bijan-Zadeh and Karim Moslehi*

Abstract

Let M and N be modules over a commutative ring R with non-zero identity. In this paper, for an ideal I of R, we introduce generalized local homology modules $U_i^I(M,N)$ as follows: $U_i^I(M,N) = \varprojlim_t \operatorname{Tor}_i^R(M/I^tM, N)$ for all $i \geqslant 0$. Also we study finiteness and vanishing properties of $U_i^I(M,N)$ which show that it is of generalized local cohomology in some sense.

Introduction

Local cohomology was first defined and studied by Grothendieck [7]. Let R be a commutative ring with non-zero identity and M be an R-module. For an ideal I of R, the i-th local cohomology modules with support in I is defined as follows:

$$H_I^i(M) = \varinjlim_{t \geqslant 0} \operatorname{Ext}_R^i\left(R/I^t, M\right).$$

Now, consider the inverse system $\{M/I^tM\}_{t\geqslant 1}$ together with natural maps

$$\pi_{k,t} : \frac{M}{I^kM} \longrightarrow \frac{M}{I^tM}$$

for all $k, t \in \mathbb{N}$ with $k \geqslant t$. (We shall use $\mathbb{N}_0$ (respectively $\mathbb{N}$) to denote the set of non-negative (respectively positive) integers). Put $\Lambda_I(M) := \varprojlim_{t\in\mathbb{N}} M/I^tM$ which is the I-adic completion of M with respect to I. For $i \in \mathbb{N}_0$, we denote the left derived functor of Λ_I by L_i^I. This functor is called the i-th local homology functor with respect to I. On the other hand, a natural generalization of local cohomology modules was introduced by Herzog [8] as follows. For a pair of

2010 *Mathematics Subject Classifications:* 13C10, 13C11, 13C15, 13D05, 13D30.

Key words and Phrases: Local homology, Local cohomology, Koszul complex, Total complex, Matlis duality.

*Corresponding author

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2012, VOLUME 2, p.62-72

R-module (M, N) the i-th generalized local cohomology module of (M, N) with respect to I is the R-module

$$H_I^i(M, N) = \varinjlim_{t \in \mathbb{N}} \mathrm{Ext}_R^i \left(M/I^t M, N\right).$$

Clearly whenever $M = R$, the generalized local cohomology module $H_I^i(R, N)$ is the ordinary local cohomology module $H_I^i(N)$. In this paper, we introduce a natural generalization of local homology functor L_i^I. For $i \in \mathbb{N}_0$, we defined generalized local homology module $U_i^I(M, N)$ of pair (M, N) with respect to I as follows:

$$U_i^I(M, N) = \varprojlim_{t \in \mathbb{N}} \mathrm{Tor}_i^I(M/I^t M, N).$$

Whenever $M = R$, for simplicity of notation we denote $U_i^I(R, N)$ by $U_i^I(N)$.

In this paper we study the finiteness and vanishing properties of generalized local homology modules in several cases. Also we provide a description of generalized local homology in terms of total complexes of Koszul complex.

1 Preliminary results

In this section we recall some basic properties of local homology functor.

Theorem 1.1. ([3, 2.4]) *Let M be an Artinian R-module. Then $L_0^I(M) \cong \Lambda_I(M)$.*

Corollary 1.2. ([3, 2.5]) *Let M be an R-module. Then the following statements are equivalent:*

(i) $IM = M$;

(ii) $L_0^I(M) = 0$;

(iii) $\Lambda_I(M) = 0$.

Theorem 1.3. ([3, 4.1]) *Let M be an Artinian R-module. Then*

$$U_i^I(M) = L_i^I(M)$$

for all $i \in \mathbb{N}_0$.

Remark 1.4. Let M, N and P be R-modules. Then

(i) Using an argument similar that they used in [6, 1.1], it can been seen that there are epimorphisms

$$\Phi_i : L_i^I(M \otimes_R N) \longrightarrow U_i^I(M, N)$$

for all $i \in \mathbb{N}_0$.

(ii) If M is flat, then

$$U_i^I(M, N) \cong U_i^I(M \otimes_R N)$$

and

$$U_i^I(P \otimes_R M, N) \cong U_i^I(P, M \otimes_R N)$$

for all $i \in \mathbb{N}_0$ by [11, 11.53].

(iii) Suppose that $M \otimes_R N$ is a finitely generated R-module. Since Λ_I is an exact functor on the category of finitely generated R-modules, one can show that

$$U_0^I(M, N) \cong \Lambda_I(M \otimes_R N) \cong L_0^I(M \otimes_R N)$$

and $L_i^I(M \otimes_R N) = 0$, for all $i \in \mathbb{N}$. Thus, in view of (i), $U_i^I(M, N) = 0$, for all $i \in \mathbb{N}$.

2 Generalized local homology and change of rings

By the same argument as in the proof of Theorem 3.3 in [3], but by replacing the module R/I^t with the module M/I^tM we have the following theorem.

Theorem 2.1. (i) *For each $i \in \mathbb{N}_0$, the local homology modules $U_i^I(M, N)$ are I-separated; that is*

$$\bigcap_{s>0} I^s U_i^I(M, N) = 0.$$

(ii) *If R is local with the unique maximal ideal $\mathfrak{m}$ and M a finitely generated R-module, then, for each $i \in \mathbb{N}_0$,*

$$U_i^I(M, D(N)) \cong D\left(H_I^i(M, N)\right),$$

where $D(-) := \mathrm{Hom}_R(-, E(R/\mathfrak{m}))$ is the Matlis dual functor with respect to the injective hull of $R/\mathfrak{m}$.

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2012, VOLUME 2, p.62-72

Remark 2.2. If we define

$$D(M,N) = \mathrm{Hom}_R\left(M, E\left(N/\mathfrak{m}N\right)\right)$$

we may restate the second part of the above theorem as follows

$$U_i^I\left(M, D(N,P)\right) \cong D\left(H_I^i(M,N), P\right).$$

Corollary 2.3. (i) *$U_i^I\left(M, D(N)\right) = 0$ if and only if $H_I^i(M,N) = 0$.*

(ii) *If N is an Artinian R-module, then $U_i^I(M,N) = 0$ if and only if $H_I^i\left(M, D(N)\right) = 0$.*

Proof. It is well know that $N = 0$ if and only if $D(N) = 0$. The assertion is now immediate from [9, 1.6(5)]. □

In the following Theorem, we assume that $f : R \longrightarrow R'$ is a flat homomorphism of rings and that M an R'-module. Also, for an ideal I of R, we denote its extension to R' by I^e.

Theorem 2.4. *Let N be an R-module. Then we have the following isomorphism of R-modules*

$$U_i^I(M,N) \cong U_i^{I^e}\left(M, N \otimes_R R'\right)$$

for all $i \in \mathbb{N}_0$.

Proof. It is clear that $(I^t)^e = (I^e)^t$ and $I^tM = (I^e)^tM$. Hence, in view of [11, 11.64],

$$\begin{aligned} U_i^I(M,N) &= \varprojlim_t \mathrm{Tor}_i^R\left(M/I^tM, N\right) \\ &\cong \varprojlim_t \mathrm{Tor}_i^{R'}\left(M/(I^e)^tM, N \otimes_R R'\right) \\ &= U_i^{I^e}\left(M, N \otimes_R R'\right) \end{aligned}$$

for all $i \in \mathbb{N}_0$. □

A (non-zero) Noetherian ring having only finitely many maximal ideals is called a *semi-local* ring.

Corollary 2.5. *Suppose that R is a semi-local ring. Let M and N be R-modules such that N is an Artinian R-module. Then*

$$U_i^I(M,N) \cong U_i^{\widehat{I}}\left(\widehat{M},N\right) \qquad (as\ R-modules)$$

for all $i \in \mathbb{N}_0$ where $\widehat{}$ is the completion functor with respect to the Jacobson radical of R.

Proof. By [9, 3.14], we have $N \otimes_R \widehat{R} \cong N$ and we know that $\widehat{R}$ is a flat R-module (see [5, 2.5.15]). Thus, by Remark 1.4 (ii),

$$U_i^I(M,N) = U_i^I\left(M, N \otimes_R \widehat{R}\right) \cong U_i^I\left(\widehat{M},N\right) \cong U_i^{\widehat{I}}\left(\widehat{M},N\right)$$

for all $i \in \mathbb{N}_0$. □

3 Noetherianness and Artinianness of generalized local homology modules

From now on we suppose that R is a Noetherian local ring with a unique maximal ideal $\mathfrak{m}$.

Theorem 3.1. *Let M be a finitely generated R-module and N an Artinian R-module. Then $U_i^{\mathfrak{m}}(M,N)$ is finitely generated, and so it is Noetherian, for all $i \in \mathbb{N}_0$.*

Proof. By Corollary 2.5 we may assume that $(R,\mathfrak{m})$ is a complete local ring. Since $D(N)$ is finitely generated, by [4, 2.2] $H_{\mathfrak{m}}^i(M, D(N))$ is Artinian. Also, by using Theorem 2.1 (ii), $U_i^{\mathfrak{m}}(M,N)$ is a finitely generated R-module. □

Remark 3.2. (i) Let

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$$

be an exact sequence of R-modules. Then, by [8, Satz 1.1.6], we have the long exact sequence

$$0 \longrightarrow H_I^0(M,N') \longrightarrow H_I^0(M,N) \longrightarrow H_I^0(M,N'') \longrightarrow H_I^1(M,N') \longrightarrow \cdots .$$

(ii) Since the functor D(-) is exact, by using part (i) and Theorem 2.1 (ii), we can see that if

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$$

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2012, VOLUME 2, p.62-72

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2012, VOLUME 2, p.62-72

is an exact sequence of Artinian modules, then there exists an exact sequence

$$\cdots \longrightarrow U_1^I(M,N'') \longrightarrow U_0^I(M,N') \longrightarrow U_0^I(M,N) \longrightarrow U_0^I(M,N'') \longrightarrow 0.$$

(iii) If N is an Artinian R-module, then $U_j^{\mathfrak{m}}(N)$ is a finitely generated R-module for all $j \in \mathbb{N}_0$, by [3, 4.6]. Also if M is a finitely generated, $M \otimes_R U_j^{\mathfrak{m}}(N)$ is a finitely generated R-module. Thus

$$U_i^I\left(M, U_j^{\mathfrak{m}}(N)\right) \cong \begin{cases} \Lambda_I\left(M \otimes_R U_j^{\mathfrak{m}}(N)\right) & ; \quad if \quad i=0\ , j \geq 0, \\ 0 & ; \quad if \quad i>0\ , j \geq 0 \end{cases}$$

by Remark 1.4 (iii).

Theorem 3.3. *Let M be a finitely generated R-module and N an Artinian R-module. Then*

$$U_i^{\mathfrak{m}}\left(M, \bigcap_{t>0} \mathfrak{m}^t N\right) \cong \begin{cases} 0 & ; \quad if \quad i=0, \\ U_i^{\mathfrak{m}}(M,N) & ; \quad if \quad i \geq 1. \end{cases}$$

Proof. Since N is an Artinian, there exists a positive integer n such that $\bigcap_{t>0} \mathfrak{m}^t N = \mathfrak{m}^n N$ and also $\Lambda_{\mathfrak{m}}(N) \cong N/\mathfrak{m}^n N$, Therefore we have the following short exact sequence of Artinian modules

$$0 \longrightarrow \bigcap_{t>0} \mathfrak{m}^t N \longrightarrow N \longrightarrow \Lambda_{\mathfrak{m}}(N) \longrightarrow 0.$$

By Remark 3.2 (ii), we get a long exact sequence

$$\begin{aligned} \cdots \quad &\longrightarrow \quad U_{i+1}^{\mathfrak{m}}\left(M, \Lambda_{\mathfrak{m}}(N)\right) \longrightarrow U_i^{\mathfrak{m}}\left(M, \bigcap_{t>0} \mathfrak{m}^t N\right) \longrightarrow U_i^{\mathfrak{m}}(M,N) \\ &\longrightarrow \quad U_i^{\mathfrak{m}}\left(M, \Lambda_{\mathfrak{m}}(N)\right) \longrightarrow \cdots \\ &\longrightarrow \quad U_0^{\mathfrak{m}}\left(M, \bigcap_{t>0} \mathfrak{m}^t N\right) \longrightarrow U_0^{\mathfrak{m}}(M,N) \longrightarrow U_0^{\mathfrak{m}}\left(M, \Lambda_{\mathfrak{m}}(N)\right) \longrightarrow 0. \end{aligned}$$

Since the R-modules $M \otimes_R N$ and $M \otimes_R \Lambda_{\mathfrak{m}}(N)$ are Artinian (see [9, 2.13]), it is easy to check that $U_0^{\mathfrak{m}}(M,N) \cong U_0^{\mathfrak{m}}\left(M, \Lambda_{\mathfrak{m}}(N)\right)$. Therefore the result follows from Remark 3.2 (iii). □

Theorem 3.4. *Let M be a finitely generated R-module and N an Artinian R-module. Then, for a positive integer s, the following statements are equivalent:*

(i) $U_i^{\mathfrak{m}}(M,N)$ *is Artinian, for all* $i < s$;

(ii) $\mathfrak{m} \subseteq \mathrm{Rad}\,(\mathrm{Ann}_R\,(U_i^{\mathfrak{m}}(M,N)))$, *for all* $i < s$.

Proof. (i)⇒(ii). Suppose that $i < s$. Since $U_i^{\mathfrak{m}}(M,N)$ is Artinian for all $i < s$, we have that $\mathfrak{m}^n U_i^{\mathfrak{m}}(M,N) = 0$ for some positive integer n, by Theorem 2.1 (i). Therefore $\mathfrak{m} \subseteq \mathrm{Rad}\,(\mathrm{Ann}_R\,(U_i^{\mathfrak{m}}(M,N)))$.

(ii)⇒(i). We use induction on s. When $s = 1$, since M is finitely generated and N is Artinian, $U_0^{\mathfrak{m}}(M,N)$ is Artinian. Suppose that $s > 1$. By Theorem 3.3, we can replace N by $\bigcap_{t>0} \mathfrak{m}^t N$. The last module is just equal to $\mathfrak{m}^n N$ for sufficiently large n. Therefore we may assume that $\mathfrak{m}N = N$. Since N is Artinian, there is an element x in $\mathfrak{m}$ such that $xN = N$ (see [2, 1.1(i)]). Thus by the hypothesis, there exists positive integer t such that $x^t U_i^{\mathfrak{m}}(M,N) = 0$ for all $i < s$. Then the short exact sequence

$$0 \longrightarrow 0 :_N x^t \longrightarrow N \xrightarrow{x^t} N \longrightarrow 0$$

implies an exact sequence

$$0 \longrightarrow U_{i+1}^{\mathfrak{m}}(M,N) \longrightarrow U_i^{\mathfrak{m}}\left(M, 0 :_N x^t\right) \longrightarrow U_i^{\mathfrak{m}}(M,N) \longrightarrow 0$$

for all $i < s-1$. It follows that $\mathfrak{m} \subseteq \mathrm{Rad}\left(\mathrm{Ann}_R\left(U_i^{\mathfrak{m}}(M, 0 :_N x^t)\right)\right)$, and by inductive hypothesis $U_i^{\mathfrak{m}}\left(M, 0 :_N x^t\right)$ is Artinian for all $i < s-1$. Thus $U_i^{\mathfrak{m}}(M,N)$ is Artinian for all $i < s$. This finishes the inductive step. □

Remark 3.5. We note that in the implication (i) ⇒ (ii) in the proof of Theorem 3.4 we need not assume that M is finitely generated and N Artinian.

4 Vanishing, non-vanishing results

We begin this section with the definition of coregular sequence which is a dual of regular sequence "in some sense".

Definition 4.1. (a) We say that an element $a \in R$ is M*-coregular* if $aM = M$.

(b) The sequence $a_1, a_2, \cdots, a_n$ of R is called an M*-coregular sequence* if

(i) $\mathrm{Ann}_M(a_1, \cdots, a_n) \neq 0$;

(ii) a_i is an $\mathrm{Ann}_M(a_1, \cdots, a_{i-1})$-coregular element, for all $i = 1, 2, \cdots, n$.

(c) Let M and N be R-modules, where M is finitely generated and N is Artinian. We call the length of any maximal N-coregular sequence contained in $\mathrm{Ann}_R(M)$ the $\mathrm{Cograde}_N(M)$. We note that this is well-defined by [9, 3.10].

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2012, VOLUME 2, p.62-72

Theorem 4.2. *Let M be a finitely generated and N an Artinian R-modules. Then*

$$\text{Cograde}_N\,(M/IM) = \inf\left\{i : U_i^I(M,N) \neq 0\right\}.$$

Proof. It is well-known that $\text{grade}_{D(N)}(M/IM)$ is the least integer i such that $H_I^i(M,D(N)) \neq 0$ (see for example [1, 5.5]). Also since N is Artinian, $D(D(N)) \cong N$, by [9, 1.6(5)]. Hence

$$\begin{aligned}
\text{Cograde}_N(M/IM) &= \inf\left\{i : \text{Tor}_i^R(M/IM,N) \neq 0\right\} \\
&= \inf\left\{i : \text{Tor}_i^R(M/IM,D(D(N))) \neq 0\right\} \\
&= \inf\left\{i : D(\text{Ext}_R^i(M/IM,D(N))) \neq 0\right\} \\
&= \inf\left\{i : \text{Ext}_R^i(M/IM,D(N)) \neq 0\right\} \\
&= \text{grade}_{D(N)}(M/IM) \\
&= \inf\left\{i : H_I^i(M,D(N)) \neq 0\right\} \\
&= \inf\left\{i : U_i^I(M,N) \neq 0\right\},
\end{aligned}$$

by [9, 3.11] and Corollary 2.3 (ii). □

Remark 4.3. Note that whenever R is not necessarily local, for all $i < \text{Cograde}_N(M/IM)$, we have that $U_i^I(M,N) = 0$.

Now we recall the concept of *Krull dimension* of an Artinian module, denote by Kdim$M$, due to Roberts [10]: let M be an Artinian R-module. When $M = 0$ we put Kdim$M = -1$. Then by induction, for any ordinal α, we put Kdim$M = \alpha$ when (i) Kdim$M < \alpha$ is false, and (ii) for every ascending chain $M_0 \subseteq M_1 \subseteq \ldots$ of submodules of M, there exists a positive integer m_0 such that $\text{Kdim}(M_{m+1}/M_m) < \alpha$ for all $m > m_0$. Thus M is non-zero and Noetherian if and only if Kdim$M = 0$.

Theorem 4.4. *Let M be a finitely generated and N an Artinian R-module with d:=Kdim$N$. Then, for each $i > d$,*

(i) $U_i^{\mathfrak{m}}(M,N) = 0$, *and*

(ii) *if there exists an element $x \in I$ which is N-coregular,* $U_i^I(M,N) = 0$.

Proof. (i) We use induction on d. If $d = 0$, then N is Noetherian, and hence $M \otimes_R N$ is finitely generated. Thus, for each $i > 0$, $U_i^{\mathfrak{m}}(M,N) = 0$ by Remark 1.4 (iii). Let $d > 0$. Note that Kdim$N \geq \text{Kdim}\left(\bigcap_{t>0} \mathfrak{m}^t N\right)$. Therefore by using the arguments similar that we use in the proof of Theorem 3.4, without loss

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2012, VOLUME 2, p.62-72

of generality, we may assume that there exists an element $x \in \mathfrak{m}$ such that $xN = N$. Thus the short exact sequence

$$0 \longrightarrow 0 :_N x \longrightarrow N \xrightarrow{x} N \longrightarrow 0$$

induces the long exact sequence

$$\cdots \longrightarrow U_i^{\mathfrak{m}}(M, 0 :_N x) \longrightarrow U_i^{\mathfrak{m}}(M, N) \xrightarrow{x} U_i^{\mathfrak{m}}(M, N) \longrightarrow U_{i-1}^{\mathfrak{m}}(M, 0 :_N x) \longrightarrow \cdots .$$

By the proof of Proposition 4 in [10], we have that $\mathrm{Kdim}(0 :_N x) \leq \mathrm{Kdim} N - 1$. Hence, by inductive hypothesis for each $i > d - 1$, $U_i^{\mathfrak{m}}(M, 0 :_N x) = 0$. Hence, for each $i > d$, $U_i^{\mathfrak{m}}(M, N) \cong x U_i^{\mathfrak{m}}(M, N)$. Using Theorem 2.1 (i), we get

$$U_i^{\mathfrak{m}}(M, N) \cong \bigcap_{s>0} x^s U_i^{\mathfrak{m}}(M, N) \subseteq \bigcap_{s>0} \mathfrak{m}^s U_i^{\mathfrak{m}}(M, N) = 0.$$

This completes the inductive step.

(ii) Similar to proof (i). □

5 Generalized local homology modules and total complex

In this section we assume that $\mathfrak{m} = (a_1, \ldots, a_n)$; $\mathrm{K}_\bullet^t$ is the Koszul complex of R with respect to the sequence $a_1^t, \ldots, a_n^t$. Also, if M is finitely generated and $\mathrm{P}_\bullet$ is a projective resolution of the R-module M, let $\mathrm{C}_\bullet^t$ be the total complex associated to the double complex $\mathrm{K}_\bullet^t \otimes_R \mathrm{P}_\bullet$. Hence there are isomorphisms

$$H_{\mathfrak{m}}^i(M, N) \cong \varinjlim_t H^i\left(\mathrm{Hom}_R\left(\mathrm{C}_\bullet^t, N\right)\right)$$

for all $i \in \mathbb{N}_0$ (see [8, satz 1.1.6]).

Thus it seems natural to look for its dual version.

Theorem 5.1. *Let M be a finitely generated and N an Artinian R-modules. Then, for each $i \in \mathbb{N}_0$*

$$U_i^{\mathfrak{m}}(M, N) \cong \varprojlim_t H_i\left(\mathrm{C}_\bullet^t \otimes_R N\right).$$

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2012, VOLUME 2, p.62-72

Proof. Converting local homology into local cohomology, Theorem 2.1 (ii), we get easily

$$\begin{aligned} U_i^{\mathfrak{m}}(M,N) &\cong D\left(H^i_{\mathfrak{m}}(M,D(N))\right) \cong D\left(\varinjlim_t H^i\left(\mathrm{Hom}_R\left(\mathrm{C}^t_\bullet, D(N)\right)\right)\right) \\ &\cong \varprojlim_t D\left(H^i\left(\mathrm{Hom}_R\left(\mathrm{C}^t_\bullet, D(N)\right)\right)\right) \cong \varprojlim_t H_i\left(D\left(\mathrm{Hom}_R\left(\mathrm{C}^t_\bullet, D(N)\right)\right)\right) \\ &\cong \varprojlim_t H_i\left(\mathrm{C}^t_\bullet \otimes_R D(D(N))\right) \cong \varprojlim_t H_i\left(\mathrm{C}^t_\bullet \otimes_R N\right). \end{aligned}$$

□

References

[1] M. H. Bijan-zadeh, A common generalization of local cohomology theories, *Glasgow Math. J.* **21** (1980), 173–181.

[2] M. H. Bijan-zadeh, S. Rasoulyar, Torsion theory, co-cohen-macaulay and local homology, *Bull. Korean Math. Soc.* **39** (2002), No. 4, 577–587.

[3] N. T. Coung, T. T. Nam, The *I*-adic completion and local homology for Artinian modules, *Math. Proc. Cambridge Philos. Soc.* **131** (2001), 61–72.

[4] K. Divaani-aazar, R. Sazeedeh, M. Tousi, On vanishing of generalized local cohomology modules, *Algebra colloq.* **12** (2005), 213–219.

[5] E. E. Enochs, O. M. G. Jenda, *Relative homological algebra*, Walter de Gruyter, Berlin, New York, 2000.

[6] J. P. C. Greenlees, J. P. May, Derived functors of *I*-adic completion and local homolgy, *J. Algebra*, **142** (1992), 438–453.

[7] A. Grothendieck, Local cohomology, Lecture Notes in Mathematics, **41** Springer, Berlin, 1967.

[8] J. Herzog, *Komplexe, Auflösungen und Dualitat in der Lokalen Algebra*, Habilitationsschrift, Universität Regensburg, 1970.

[9] A. Ooishi, Matlis duality and the width of a module, *Hiroshima Math. J.*, **6** (3) (1976), 573–587.

[10] R. N. Roberts, Krull dimension for Artinian modules over quasi-local commutative rings, *Quart. J. Math. Oxford* **(3)**, 26 (1975), 269–273.

[11] J. J. Rotman, *An introduction to homological algebra*, Academic Press, 1979.

Mohammad H. Bijan-Zadeh
Faculty of Mathematics and Computer Engineering,
Tehran Teacher Training University,
Tehran,
Iran
E-mail: mh_bijan@pnu.ac.ir

Karim Moslehi
Department of Mathematics,
Faculty science Payame Noor University(PNU),
Tehran,
Iran
E-mail: karim.moslehi@gmail.com

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2012, VOLUME 2, p.62-72