ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2012, VOLUME 2, ISSUE 2, p.23-35

On Some New Class of Arithmetic Convolutions Involving Arbitrary Sets of Integers

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Abstract. In this paper we define a new type of arithmetic convolution called the S_B – product and denote it by $*_{S_B}$.Let $R_{S_B} = < C^N, +, *_{S_B} >$ be the set of all complex valued arithmetic functions with ordinary addition and with a S_B – product considered as multiplication. We give conditions on $*_{S_B}$ which are necessary and sufficient for R_{S_B} to be commutative, and associative. We also investigate some other algebraic properties of R_{S_B} such as the existence of identity, of zero divisors. We determine all invertible elements of R_{S_B} and we establish the conditions under which R_{S_B} is a local ring .We then give a definition for completely multiplicative B-product and study some of its properties. We then study some important relations between S_B – product, B – product and unitary convolution. We conclude our discussion by considering an example of S_B -product and investigate whether the corresponding S_B – product is commutative, associative, has an identity etc.

2000 Mathematics Subject Classification: 11A25

Keywords and phrases: Arithmetic convolution, characteristic function, completely multiplicative function, Narkiewicz's A-product, B-product, K_B – product, S-convolution, S_B –product.

1. Introduction

In a previous paper [2], the B – product is defined as follows. For every natural number n, let B_n be the set of some pairs of divisors of n. For arithmetical functions f and g, their B-product is given by

$$(f *_B g)(n) = \sum_{(u,v)\in B_n} f(u)\, g(v), \qquad \text{for} \quad n = 1,2,3,\ldots .$$

This B-product generalizes simultaneously the A-product of Narkiewicz [12] and the *lcm* product and it has a non-void intersection with the Ψ- product of Lehmer [10]. The τ - product of Scheid [13] is also a particular case of B-product. There are several classes of arithmetic convolutions which can be found in Apostol [1], Cohen [7], Davison [8] , McCarthy [11], Sivaramakrishnan [14], Vaidyanathaswamy [18], Subbarao [15] and more recently in the papers of Haukkanen [9], Tóth [16], [17] and Bhattacharjee [2]-[6]. In this paper, we define a new type of arithmetic convolution and we call it the S_B-product and denote it by $*_{S_B}$. We study in detail about the S_B-product in Section **2** of this paper. We then define completely multiplicative B-product and study its properties in Section **3**. In Section **4** we recall some identities mentioned in Tóth [16] and also study some important relations between S_B-product, B -product and unitary convolution. We conclude our discussion in Section **5** by considering an example of S_B -product and investigate whether the corresponding S_B-product is commutative, associative, has an identity, has an inverse etc.

2. S_B - product and its properties

Let **N** denote the set of natural numbers and let S be an arbitrary subset of **N**. For every natural number n we say that the pair of divisors (u, v) to be S_B divisors of n if $(u, v)\in B_n$ where B_n is a set of some pairs of divisors of n and gcd $(u, v) \in S$.

For arithmetical functions f and g, their S_B-product $f *_{S_B} g$ is given by

$$(f *_{S_B} g)(n) = \sum_{\substack{(u,v)\in B_n, \\ \gcd(u,v)\in S}} f(u)\, g(v), \qquad \text{for} \quad n = 1,2,3,\ldots$$

$$= \sum_{(u,v)\in B_n} \rho_S(<u,v>)\, f(u)\, g(v),$$

where ρ_S stands for the characteristic function of S and $< u, v>$ stands for gcd (u, v).

If $S = \mathbf{N}$ where **N** is the set of all natural numbers and

$$B_n = \{ (u, v): uv = n, \gcd (u, v) \in S \},$$

then S_B-product is the Dirichlet's convolution. Let S be an arbitrary subset of **N** and

$$B_n = \{(u, v): uv=n \text{ and } \gcd(u, v) \in S\},$$

then S_B-product is the S-convolution of Tóth [16]. If B_n are sets of pairs of divisors of n defining the B – product, let us consider the following set B'_n of pairs of divisor of n :

$$B'_n = \{(u, v) : (u, v) \in B_n, \gcd (u, v) \in S \}.$$

Then the S_B-product is the B – product defined by the sets B'_n. If $S=\{1\}$ and

$$B_n = \{(u, v): uv=n, \gcd(u, v) \in S\},$$

then S_B-product is the unitary convolution. For $K(u, v)= \rho_S< u, v>$, where $K(u, v)$ is a function of two variables u and v and range of $K \subseteq S \subseteq \mathbf{N}$, S_B-product is a special type of K_B -product of Bhattacharjee [4].

Let $R_{S_B} = < C^{\mathbf{N}}, + ,*_{S_B}>$ be the set of all complex valued arithmetic functions with the ordinary addition and with a S_B-product considered as multiplication.

For a natural number k we define the function e_k as follows:

$$e_k(n) = \begin{cases} 0, \text{ if } n \neq k \\ 1, \text{ if } n = k. \end{cases}$$

Thus $e_k(n) = \delta_{k,n}$ (the Kronecker delta).

The system $R_{S_B} = < C^{\mathbf{N}}, + , *_{S_B} >$ is a ring like structure which is neither commutative nor associative in general. We now discuss some properties of the S_B-product.

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Theorem 2.1. R_{S_B} *is commutative if and only if for every n,*

$$(u, v) \in B_n \iff (v, u) \in B_n .$$

Proof. Follows from the definition of S_B –product. □

Theorem 2.2. R_{S_B} *is associative if and only if for fixed n, d_1, d_2, d_3 the following equality holds.*

$$\sum_{\substack{r \\ (r, d_1) \in B_n \\ (d_2, d_3) \in B_r}} \rho_S(<r, d_1>)\, \rho_S(<d_2, d_3>) = \sum_{\substack{w \\ (d_2, w) \in B_n \\ (d_3, d_1) \in B_w}} \rho_S(<d_2, w>)\, \rho_S(<d_3, d_1>) .$$

Proof. $\Leftarrow$ For every arithmetic functions f, g, h we have

$$[(f *_{S_B} g) *_{S_B} h](n) = \sum_{x, t, u \in \mathbf{N}} f(t)g(u)h(x) \sum_{\substack{r \\ (r, x) \in B_n,\ (t, u) \in B_r}} \rho_S(<r, x>)\, \rho_S(<t, u>).$$

On the other hand

$$[f *_{S_B} (g *_{S_B} h)](n) = \sum_{x, t, u \in \mathbf{N}} f(t)g(u)h(x) \sum_{\substack{w \\ (t, w) \in B_n,\ (u, x) \in B_w}} \rho_S(<t, w>)\, \rho_S(<u, x>).$$

By the assumption in both expressions the inner sums are equal. Therefore the S_B - product is associative.

$\Rightarrow$ Conversely suppose that the S_B -product is associative and fix $n, d_1, d_2, d_3 \in \mathbf{N}$. From the first part of the proof we get

$$[(e_{d_2} *_{S_B} e_{d_3}) *_{S_B} e_{d_1}](n) = \sum_{\substack{r \\ (r, d_1) \in B_n,\ (d_2, d_3) \in B_r}} \rho_S(<r, d_1>)\, \rho_S(<d_2, d_3>).$$

Similarly

$$[e_{d_2} *_{S_B} (e_{d_3} *_{S_B} e_{d_1})](n) = \sum_{\substack{w \\ (d_2, w) \in B_n,\ (d_3, d_1) \in B_w}} \rho_S(<d_2, w>)\, \rho_S(<d_3, d_1>).$$

Therefore the sums obtained are equal and the result follows. □

Theorem 2.3. *A function e is a right identity in the system* R_{S_B} *if and only if for every k and n we have*

$$\sum_{(k, v) \in B_n} \rho_S(<k, v>)\, e(v) = e_k(n).$$

Proof. $\Rightarrow$ For every k and n we have

$$e_k(n) = (e_k *_{S_B} e)(n) = \sum_{\substack{u,v \\ (u,v)\in B_n}} \rho_S(<u,v>)\, e_k(u)e(v) = \sum_{\substack{v \\ (k,v)\in B_n}} \rho_S(<k,v>)\, e(v)$$

$\Leftarrow$ Conversely suppose for every f and n we have

$$(f *_{S_B} e)(n) = \sum_{(u,v)\in B_n} \rho_S(<u,v>)\, f(u)\, e(v)$$

$$= \sum_{u} f(u) \sum_{(u,v)\in B_n} \rho_S(<u,v>)\, e(v)$$

$$= \sum_{u} f(u)\, e_u(n),$$

$$= f(n).$$ □

A similar condition characterizes left identities. Hence we get

Theorem 2.4. *A function e is an identity in the system R_{S_B} if and only if for every k and n we have*

$$\sum_{\substack{v \\ (k,v)\in B_n}} e(v)\, \rho_S(<k,v>) = e_k(n) = \sum_{\substack{u \\ (u,k)\in B_n}} e(u)\, \rho_S(<u,k>)\,.$$

Corollary 2.5. *If the system R_{S_B} has an identity e, then for every n there exist u and v such that* $(u, n) \in B_n$, $(n, v) \in B_n$, $\gcd(u, n) \in S$ and $\gcd(n, v) \in S$.

Hence $B_1 = \{(1,1)\}$, $\rho_S(<1,1>) = 1$ and $e(1) = 1/\rho_S(<1,1>) = 1$.

Corollary 2.6. *The function e_1 is the identity of the system R_{S_B} if and only if for every k (>1) and n we have*:

$(k,1) \in B_n$ *and* $\rho_S(<k,1>) = 1 \Leftrightarrow k = n$ *and* $\rho_S(<k,1>) = \rho_S(<1,k>)$

$\Leftrightarrow (1,k) \in B_n$ *and* $\rho_S(<1,k>) = 1$.

Theorem 2.7 (i). *If R_{S_B} is commutative, associative, has a unique identity e and $f \in R_{S_B}$ satisfies*

$$\sum_{(u,n)\in B_n} \rho_S(<u,n>)\, f(u) \neq 0, \text{ for every } n, \tag{1}$$

then f has a right inverse. Such an inverse g can be defined inductively by the formulas:

$$g(1) = [\, f(1)\, \rho_S(<1,1>)\,]^{-1}, \tag{2}$$

$$g(n) = \Big[e(n) - \sum_{\substack{v \\ v<n}} g(v) \sum_{\substack{u \\ (u,v)\in B_n}} f(u)\, \rho_S\,(<u,v>)\Big] \Big[\Big(\sum_{\substack{u \\ (u,n)\in B_n}} f(u)\Big)\, \rho_S\,(<u,n>)\Big]^{-1}, \quad \textit{for} \quad n>1.$$

(ii) *Moreover if* $f \in R_{S_B}$ *has a right inverse, then* (1) *holds*.

Proof. **(i)** From (1) for n=1 and Corollary 2.5 it follows that $\rho_S\,(<1,1>)\, f(1) \neq 0$. Therefore the formulas (2) define a function g. The verification of the formula

$$(f *_{S_B} g)(n) = e(n), \quad \text{for every } n,$$

is straightforward.

(ii) Let g be a right inverse of f i.e let $f *_{S_B} g = e$. From the associativity of the system R_{S_B} it follows that

$$f *_{S_B} (g *_{S_B} e_n) = (f *_{S_B} g) *_{S_B} e_n = e *_{S_B} e_n = e_n, \text{ for every } n.$$

Evidently for $v< n$, we have

$$(g *_{S_B} e_n)(v) = 0.$$

Therefore

$$1 = e_n(n) = \sum_{(u,v)\in B_n} f(u)\,(g *_{S_B} e_n)(v)\, \rho_S\,(<u,v>)$$
$$= (g *_{S_B} e_n)(n) \sum_{(u,n)\in B_n} f(u)\, \rho_S\,(<u,n>),$$

consequently (1) holds. □

Similar results can be proved for left inverses. Hence we get the following theorem.

Theorem 2.8 (i). *If* R_{S_B} *is commutative, associative, has a unique identity e and* $f \in R_{S_B}$ *satisfies*

$$\sum_{\substack{u \\ (u,n)\in B_n}} \rho_S\,(<u,n>)\, f(u) \neq 0 \neq \sum_{\substack{v \\ (n,v)\in B_n}} \rho_S\,(<n,v>)\, f(v), \ \textit{for every } n, \qquad (3)$$

then f is invertible and its left inverse g is given by the formulas (2), *and the right inverses by similar ones.*

(ii) *Moreover if* $f \in R_{S_B}$ *is invertible, then* (3) *holds.*

Corollary 2.9. *If* R_{S_B} *has an identity and* $f \in R_{S_B}$ *satisfies* $f(n) > 0$*, for every n, then f is invertible.*

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Proof. From Corollary **2.5** it follows that, for every n, the set B_n is non-empty. Consequently the condition (3) is satisfied. □

Theorem 2.10. *If the system* R_{S_B} *is associative and has an identity e, then the following conditions are equivalent:*

(i) *For every n,* **(** $(t, n) \in B_n$, $\gcd(t, n) \in S$ *if and only if* $t = 1$, $\rho_S(<t, n>) = \rho_S(<1, n>) = 1$**)**

and

($(n, t) \in B_n$, $\gcd(n, t) \in S$ *if and only if* $t=1$, $\rho_S(<n, t>) = \rho_S(<n, 1>) = 1$**).**

(ii) *Every* $f \in R_{S_B}$ *satisfying* $f(1) \neq 0$ *is invertible.*

If moreover the system R_{S_B} *is commutative, then the above conditions are equivalent to.*

(iii) R_{S_B} *is a local ring.*

Proof . **(i) ⇒ (ii)** From the assumption we conclude that

$$\sum_{(u, n) \in B_n} \rho_S(<u, n>) f(u) = \rho_S(<1, n>) f(1) = f(1) \neq 0$$

and

$$\sum_{(n, v) \in B_n} \rho_S(<n, v>) f(v) = \rho_S(<n, 1>) f(1) = f(1) \neq 0.$$

Therefore from Theorem 2.8 it follows that f is invertible.

(ii) ⇒(i) Since e_1 is invertible, from (3) with $f=e_1$ we get $(1,n),(n,1) \in B_n$,

$$\rho_S(<1, n>) = \rho_S(<n, 1>) = 1.$$

For $t>1$, $(e_1 - e_t)(1) = e_1(1) - e_t(1) = 1$, hence $e_1 - e_t$ is invertible.

Moreover

$$\sum_{\substack{u \\ (u,n)\in B_n}} (e_1 - e_t)(u)\rho_S(<u, n>) = \begin{cases} 0, \text{if } (t,n) \in B_n \text{ and } \gcd(t,n) \in S \\ 1, \text{ otherwise.} \end{cases}$$

Therefore by Theorem 2.8 (ii) it follows that $(t, n) \notin B_n$ or $\gcd(t, n) \notin S$. Analogously we get $(n, t) \notin B_n$ or $\gcd(n, t) \notin S$.

(ii) ⇒(iii) The set I of elements $f \in R_{S_B}$ satisfying $f(1)=0$ is an ideal of R_{S_B}. It is the unique maximal ideal since every $f \notin I$ is invertible. Hence R_{S_B} is a local ring.

(iii) ⇒(ii) Suppose that $f(1) \neq 0$. Since $g = f(1)e - f$ satisfies $g(1)=0$, the element g is not invertible. In a local ring the sum of invertible and not invertible elements is invertible. Consequently the element $f = f(1)e + (f - f(1).e)$ is invertible. □

Theorem 2.11. *If for every* n ,$\{(u, v): uv=n\} \subset B_n \subset \{(u, v) : uv \mid n\}$ *and* gcd $(u, v) \in S$ *for every* $(u, v) \in B_n$ i.e $\rho_S(<u, v>)=1$, *for every* $(u, v) \in B_n$, *then in* R_{S_B} *there are no zero divisors.*

Proof. Let $f, g \in R_{S_B}$, $f \neq 0$, $g \neq 0$.Then there exist u, v such that $f(u) \neq 0$, and $f(k) = 0$, for $k<u$, $g(v) \neq 0$ and $g(l)=0$, for $l < v$. Then for $n = uv$ we get ,

$$(f *_{S_B} g)(n) = \sum_{\substack{k,l \\ kl=n}} \rho_S(<k, l>) f(k)g(l) + \sum_{\substack{k,l \\ k l \mid n,\ k l < n \\ (k,l) \in B_n}} \rho_S(<k, l>) f(k)g(l)$$

$$= f(u)\, g(v) \neq 0,$$

since only the summand corresponding to $k = u$, $l = v$ is different from zero. Therefore $f *_{S_B} g \neq 0$. □

Theorem 2.12. *If there exist* u, v *such that* $(u, v) \notin B_n$ *, for every* n, *then* $e_u *_{S_B} e_v = 0$

Proof. We have

$$(e_u *_{S_B} e_v)(n) = \sum_{(k,l) \in B_n} \rho_S(<k, l>)\, e_u(k)\, e_v(l) = 0,$$

since $(u, v) \notin B_n$. □

Theorem 2.13. *If for a fixed* $u \neq 1$ *and every* n *divisible by* u *we have* $(u, n) \in B_n$ *and* $\rho_S(<u, n>) = 1$, *then* e_u *is a left zero divisor in the system* R_{S_B}.

Proof. We are looking for a function $f \neq 0$ satisfying $e_u *_{S_B} f = 0$. For every n, we have,

$$\left(e_u * s_B f\right)(n) = \sum_{\substack{v \\ (u,v) \in B_n}} f(v)\rho_S(<u,v>) = \begin{cases} 0, & \text{if } u \nmid n \\ f(n) + \sum_{\substack{v<n \\ (u,v) \in B_n}} f(v)\rho_S(<u,v>), & \text{if } u \mid n. \end{cases}$$

We define f inductively:

Let $f(n) = 1$ if $u \nmid n$ and $f(n) = -\left[\sum_{\substack{v<n \\ (u,v) \in B_n}} f(v)\, \rho_S(<u, v>)\right]$, if $u \mid n$. Then we get $e_u *_{S_B} f = 0$.

□

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3. Completely Multiplicative *B*-product

We say that a *B*-product is completely multiplicative if for every pair (m,n) of natural numbers we have

$$B_{mn}=\{(r_1r_2, s_1s_2) : (r_1,s_1)\in B_m, (r_2,s_2)\in B_n\}.$$

This definition can also be formulated as follows. For every pair (m,n) of natural numbers we have

$$(r_1r_2, s_1s_2)\in B_{mn} \text{ if and only if } (r_1,s_1)\in B_m, (r_2,s_2)\in B_n \quad (4)$$

[Note: r_1, s_1 are divisors of m and r_2, s_2 are divisors of n respectively.]

Let us recall that an arithmetical function f is completely multiplicative if $f(n)\neq 0$ for at least one integer n and if $f(mn) = f(m)f(n)$ for every n. We now discuss some property of completely multiplicative *B*-product. More precisely the following theorem holds.

Theorem 3.1. *The B-product of completely multiplicative functions is a completely multiplicative function if the B-product is completely multiplicative.*

Proof. Let f, g be completely multiplicative functions and let (m, n) be a pair of natural numbers. Then,

$$(f *_B g)(mn) = \sum_{(r,s)\in B_{mn}} f(r)g(s)$$

$$= \sum_{(r_1r_2, s_1s_2)\in B_{mn}} f(r_1r_2)g(s_1s_2) \quad [\text{where } r= r_1r_2,\ s= s_1s_2 \text{ and } r_1|m,\ s_1|m,\ r_2|n, s_2|n\ ]$$

$$= \sum_{(r_1,s_1)\in B_m}\ \sum_{(r_2,s_2)\in B_n} f(r_1)f(r_2)\, g(s_1)g(s_2) \quad [\text{since } f, g \text{ and the } B\text{-product are completely multiplicative}]$$

$$= \sum_{(r_1,s_1)\in B_m} f(r_1)\, g(s_1) \sum_{(r_2,s_2)\in B_n} f(r_2)\, g(s_2) = (f *_B g)(m)\,(f *_B g)(n).$$

Hence follows the theorem. □

4. Identities

For an arbitrary $S\subseteq \mathbf{N}$, let μ_S be the Möbius function of S defined by

$$\sum_{d|n} \mu_S(d) = \rho_S(n),\ n\in \mathbf{N}, \quad (5)$$

see Cohen[7], Tóth [16]. Therefore ,by Möbius inversion formula

$$\mu_S(n) = \sum_{d|n} \rho_S(d)\, \mu(n/d),\ n\in \mathbf{N}, \qquad (6)$$

where $\mu = \mu_{\{1\}}$ is the ordinary Möbius function.

Theorem 4.1. *If* $S \subseteq \mathbf{N}$ *and f and g are completely multiplicative functions and also the B-product is completely multiplicative, then, for every* $n\in \mathbf{N}$,

(i)
$$(f *_{S_B} g)(n) = \sum_{(j,j)\in B_j} \mu_S(j)\, f(j)\, g(j)\, (f *_B g)(n/j),$$

where $*_B$ *is the B-product of Bhattacharjee* [2].

(ii)
$$(f *_{S_B} g)(n) = \sum_{\substack{a\in S\\ (a,a)\in B_a}} \rho_S(a)\, f(a)\, g(a) \sum_{(i,j)\in B_{n/a},\ \gcd(i,j)=1} f(i)\, g(j)$$

$$= \sum \rho_S(a) f(a) g(a) (f \mathrm{X}\, g)(n/a),$$

if $B_{n/a}$ *consists of all pairs of divisors of n/a and* $\mathrm{X} \equiv *_{\{1\}}$ *is the unitary convolution.*

Proof. **(i)** Using identity (5) we have, for every $n\in\mathbf{N}$,

$$(f *_{S_B} g)(n) = \sum_{(u,v)\in B_n} \rho_S(<u,v>)\, f(u) g(v)$$

$$= \sum_{(u,v)\in B_n} \Big[\sum_{j|\gcd(u,v),\ \gcd(u,v)\in S} \mu_S(j)\Big] f(u)\, g(v)$$

Hence with $u=ja$, $v=jb$

$$(f *_{S_B} g)(n) = \sum_{(ja,jb)\in B_n} \Big[\sum_{j|\gcd(ja,jb)} \mu_S(j)\Big] f(ja)\, g(jb)$$

$$= \sum_{(ja,jb)\in B_n} \sum_{j|\gcd(ja,jb)} \mu_S(j)\, f(j)\, f(a) g(j) g(b)$$ [since f and g are completely multiplicative]

$$= \sum_{(j,j)\in B_j} \mu_S(j)) f(j) g(j) \sum_{(a,b)\in B_{n/j}} f(a)\, g(b)$$ [since the B- product is completely multiplicative]

$$= \sum_{(j,j)\in B_j} \mu_S(j) f(j)\, g(j)\, (f *_B g)(n/j).$$

(ii) Furthermore we have,

$$(f *_{S_B} g)(n) = \sum_{(u,v)\in B_n} \rho_S(<u,v>)\, f(u)\, g(v)$$

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$$= \sum_{a \in S} \rho_S(a) \sum_{\substack{(u,v) \in B_n \\ \gcd(u,v)=a}} f(u)g(v)$$

$$= \sum_{a} \rho_S(a) \sum_{\substack{(u,v) \in B_n \\ \gcd(u/a,\, v/a)=1}} f(u)g(v).$$

If $u=ai$, $v=aj$, we get

$$(f *_{S_B} g)(n) = \sum_{\substack{a \in S \\ (a,a) \in B_a}} \rho_S(a) \sum_{\substack{(i,j) \in B_{n/a} \\ \gcd(i,j)=1}} f(a)\, f(i)\, g(a) g(j)$$

[since f, g and the B-product are completely multiplicative]

$$= \sum_{\substack{a \in S \\ (a,a) \in B_a}} \rho_S(a)\, f(a)\, g(a) \sum_{\substack{(i,j) \in B_{n/a} \\ \gcd(i,j)=1}} f(i)\, g(j)$$

$$= \sum \rho_S(a)\, f(a)\, g(a)\, (f \text{ X } g)(n/a),$$

if $ij = n/a$ i.e if $B_{n/a}$ is the set of all pairs of divisors of n/a. Hence we have the theorem. □

5. Example

We conclude our discussion by considering an example of S_B - product and investigate whether the corresponding S_B - product is commutative, associative, has an identity, has inverses as well as zero divisors.

Example 5.1. Let $B_n = \{(1,1),(1,n),(n,1)\}$, $S=\{1\}$.

Solution.

(i) Commutativity: Follows clearly from Theorem 2.1.

(ii) Associativity: For associativity we have,

$$\sum_{\substack{r \\ (r,d_1) \in B_n \\ (d_2,d_3) \in B_r}} \rho_S(<r,d_1>)\rho_S(<d_2,d_3>) = \begin{cases} 2 & \text{for } d_1 = d_2 = d_3 = 1 \neq n \\ 1 & \text{for } n = 1 = d_1 = d_2 = d_3 \\ & \text{or } n \neq 1, d_i = n, d_j = d_k = 1, \{i,j,k\} = \{1,2,3\} \\ 0, & \text{otherwise.} \end{cases}$$

Since the result does not depend on the ordering of d_1, d_2, d_3 we obtain the associativity from Theorem 2.2.

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(iii) Existence of identity: By Corollary 2.6, we have

$$\sum_{\substack{v \\ (k,v)\in B_n}} \rho_S(<k,v>)e_1(v) = \rho_S(k,1) = \begin{cases} 1, & \text{if and only if } k=n \\ 0, & \text{otherwise} \end{cases}$$

$$= e_k(n).$$

Hence e_1 is a right identity of R_{S_B} and similarly e_1 is a left identity of R_{S_B}. Therefore e_1 is the identity of R_{S_B}.

(iv) Existence of inverse: We have

$$\sum_{\substack{u \\ (u,n)\in B_n}} \rho_S(<u,n>)f(u) = \begin{cases} f(1), & \text{if } n=1 \\ f(1), & \text{if } n>1. \end{cases}$$

Therefore from Theorem 2.7 (i) it follows that $f\in R_{S_B}$ is invertible if and only if $f(1) \neq 0$, for $n \geq 1$.

(v) Existence of zero divisors:

Here $f *_{S_B} g = 0$ if and only if $f = 0$ or $g = 0$ or $f(1) = g(1) = 0$

Proof. We have

$(f *_{S_B} g)(1) = \rho_S(<1,1>)f(1)g(1) = f(1)g(1),$

$(f *_{S_B} g)(n) = \rho_S(<1,1>)f(1)g(1) + \rho_S(<1,n>)f(1)g(n) + \rho_S(<n,1>)f(n)g(1)$ for $n>1$

$= f(1)g(1) + f(1)g(n) + f(n)g(1).$

$\Leftarrow$ Clear.

$\Rightarrow$ Assume $f *_{S_B} g = 0$, $f \neq 0$, $g \neq 0$ and $f(1) \neq 0$(say). Hence $g(1)=0$, $g(n)=0$ for $n>1$, a contradiction. Thus $f \in R_{S_B}$ is a zero divisor if and only if $f(1) \neq 0$.

Acknowledgements

The first author would like to thank Prof. dr. hab. Jerzy Browkin, Institute of Mathematics, Polish Academy of Sciences, Warsaw, Poland for his valuable suggestions.

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