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SPECIAL SMARANDACHE CURVES ACCORDING TO DARBOUX FRAME IN E^3

ÖZCAN BEKTAŞ AND SALİM YÜCE

Abstract. In this study, we specify some special Smarandache curves with reference to Darboux frame in Euclidean 3-space.We dispose certain particularizations and outcomes about Smarandache curves. However we defray an example about our study.

1. Introduction

There are many substantial outcomes and features of curves. Investigators chases labours about the curves. In the light of the existing studies, researchers always qualify new curves such as Special Smarandache curves are one of them. This curve is defined by Turgut and Yılmaz, [4]. Special Smarandache curves have been studied by some authors [1, 2, 4]. Turgut and Yilmaz have introduced a particular circumstance of such curves. They intitle it Smarandache $\mathbf{TB_2}$ curves in the space E_1^4. Ali has illustrated certain special Smarandache curves in the Euclidean space, [1]. Special Smarandache curves in such a manner that Smarandache curves $\mathbf{TN}_1$, $\mathbf{TN}_2$, $\mathbf{N}_1\mathbf{N}_2$ and $\mathbf{TN}_1\mathbf{N}_2$with respect to Bishop frame in Euclidean 3-space have been seeked for by Çetin et al [2]. Furthermore, they worked differential geometric properties of these special curves and they checked out first and second curvature (natural curvatures) of these curves. Also they get the centers of the curvature spheres and osculating spheres of Smarandache curves.

In this study, we investigate special Smarandache curves just as Smarandache **Tg**, **Tn**, **gn** and **Tgn** according to Darboux frame in Euclidean 3-space. Furthermore, we get some stamps of these special curves and we compute: the Frenet frame, the curvature and the torsion of the new curves.

2. Preliminaries

In this part, we give a knowledge about special Smarandache curves and Darboux frame. Let M be an oriented surface and let take notice of a curve $\alpha(s)$ on the surface M . Since the curve $\alpha(s)$ is also in space, there exists Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ at each points of the curve where $\mathbf{T}$ is unit tangent vector, $\mathbf{N}$ is principal normal vector and $\mathbf{B}$ is binormal vector, respectively. The Frenet equations are given by

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Corresponding author.

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$$\begin{cases} \mathbf{T}' = \kappa \mathbf{N} \\ \mathbf{N}' = -\kappa \mathbf{T} + \tau \mathbf{B} \\ \mathbf{B}' = -\tau \mathbf{N} \end{cases}$$

where κ and τ are curvature and torsion of the curve $\alpha(s)$, respectively. Here and in the following, we use " ' " to denote the derivative with respect to the arc length parameter of a curve. Since the curve $\alpha(s)$ lies on the surface M there exists another frame of the curve $\alpha(s)$ which is named Darboux frame and denoted by $\{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$. In this frame $\mathbf{T}$ is the unit tangent of the curve, $\mathbf{n}$ is the unit normal of the surface M and $\mathbf{g}$ is a unit vector given by $\mathbf{g} = \mathbf{n} \times \mathbf{T}$. Since the unit tangent $\mathbf{T}$ is common in both Frenet frame and Darboux frame, the vectors $\mathbf{N}, \mathbf{B}, \mathbf{g}, \mathbf{n}$ lie on the same plane. So that the relations between these frames can be given as follows

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{g} \\ \mathbf{n} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix} \cdot \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix} \tag{2.1}$$

where φ is the angle between the vectors $\mathbf{g}$ and $\mathbf{N}$, [5]. The derivative formulae of the Darboux frame is

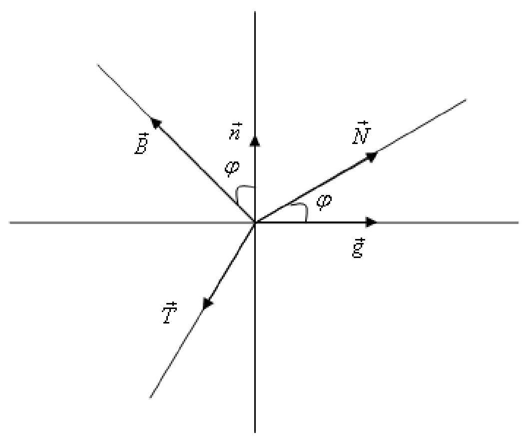

Figure 1.The Frenet Apparatus and Darboux Frame

and

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{g}' \\ \mathbf{n}' \end{bmatrix} = \begin{bmatrix} 0 & k_g & k_n \\ -k_g & 0 & \tau_g \\ -k_n & -\tau_g & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{T} \\ \mathbf{g} \\ \mathbf{n} \end{bmatrix} \tag{2.2}$$

where , k_g is the geodesic curvature, k_n is the normal curvature and τ_g is the geodesic torsion of $\alpha(s)$.

In the differential geometry of surfaces, for a curve $\alpha(s)$ lying on a surface M the followings are well-known

i) $\alpha(s)$ is a geodesic curve $\Leftrightarrow k_g = 0$,

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ii) $\alpha(s)$ is an asymptotic line $\Leftrightarrow k_n = 0$,
iii) $\alpha(s)$ is a principal line $\Leftrightarrow \tau_g = 0$, [3,5].

Let $\alpha = \alpha(s)$ and $\beta = \beta(s^*)$ be a unit speed regular curves in E^3 and $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be moving Serret-Frenet frame of $\alpha(s)$. Smarandache **TN** curves are introduced by

$$\beta(s^*) = \frac{1}{\sqrt{2}}(\mathbf{T}+\mathbf{N}),$$

Smarandache **NB** curves are described by

$$\beta(s^*) = \frac{1}{\sqrt{2}}(\mathbf{N}+\mathbf{B}),$$

Smarandache **TNB** curves are identified by

$$\beta(s^*) = \frac{1}{\sqrt{3}}(\mathbf{T}+\mathbf{N}+\mathbf{B}).$$

3. Smarandache Curves According to Darboux Frame

In this section, we investigate special Smarandache curves according to Darboux frame in E^3. Let $\alpha = \alpha(s)$ and $\beta = \beta(s^*)$ be a unit speed regular curves in E^3 and designed by $\{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$ and $\{\mathbf{T}^*, \mathbf{g}^*, \mathbf{n}^*\}$ be Darboux frame of these curves, respectively.

3.1. Tg$-$ Smarandache Curves.

Definition 1. *Let M be an oriented surface in E^3 and let consider the arc - length parameter curve $\alpha = \alpha(s)$ lying fully on M. Denote the Darboux frame of $\alpha(s)$* $\{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$.

Tg - Smarandache curve can be defined by

$$\beta(s^*) = \frac{1}{\sqrt{2}}(\mathbf{T}+\mathbf{g}). \tag{3.1}$$

Now, we can investigate Darboux invariants of **Tg**$-$ Smarandache curve according to $\alpha = \alpha(s)$. Differentiating (3.1) with respect to s, we get

$$\beta' = \frac{d\beta}{ds^*}\frac{ds^*}{ds} = \frac{-1}{\sqrt{2}}(k_g\mathbf{T} - k_g\mathbf{g} - (k_n+\tau_g)\mathbf{n}), \tag{3.2}$$

and

$$\mathbf{T}^*\frac{ds^*}{ds} = \frac{-1}{\sqrt{2}}(k_g\mathbf{T} - k_g\mathbf{g} - (k_n+\tau_g)\mathbf{n})$$

where

$$\frac{ds^*}{ds} = \sqrt{\frac{2k_g^2 + (k_n+\tau_g)^2}{2}}. \tag{3.3}$$

For the tangent vector of curve β, we can easily show that,

$$\mathbf{T}^* = \frac{-1}{\sqrt{2k_g^2 + (k_n+\tau_g)^2}}(k_g\mathbf{T} - k_g\mathbf{g} - (k_n+\tau_g)\mathbf{n}). \tag{3.4}$$

Differentiating (3.4) with respect to s, we obtain

$$\frac{d\mathbf{T}^*}{ds^*}\frac{ds^*}{ds}=\frac{1}{\left(2k_g^2+(k_n+\tau_g)^2\right)^{\frac{3}{2}}}\left(\Gamma_1\mathbf{T}+\Gamma_2\mathbf{g}+\Gamma_3\mathbf{n}\right) \tag{3.5}$$

where

$$\begin{cases} \Gamma_1=(k_n+\tau_g)\left\{k_g\left(k_n^{'}+\tau_g^{'}-k_nk_g-\tau_gk_g\right)-k_g^{'}(k_n+\tau_g)-k_n\left[2k_g^2+(k_n+\tau_g)^2\right]\right\}-2k_g^4 \\ \Gamma_2=(k_n+\tau_g)\left\{-k_g\left(k_n^{'}+\tau_g^{'}+k_nk_g+\tau_gk_g\right)+k_g^{'}(k_n+\tau_g)-\tau_g\left[2k_g^2+(k_n+\tau_g)^2\right]\right\}-2k_g^4 \\ \Gamma_3=k_g(k_n+\tau_g)\left[-2k_g^{'}-k_n+\tau_g(k_n+\tau_g)\right]+2k_g^2\left(k_g\tau_g+k_n^{'}+\tau_g^{'}\right). \end{cases}$$

Substituting (3.3) in (3.5), we get

$$\frac{d\mathbf{T}^*}{ds^*}=\frac{\sqrt{2}}{\left(2k_g^2+(k_n+\tau_g)^2\right)^2}\left(\Gamma_1\mathbf{T}+\Gamma_2\mathbf{g}+\Gamma_3\mathbf{n}\right).$$

Then, the curvature and principal normal vector field of curve β are respectively,

$$\kappa^*=\left\|\frac{d\mathbf{T}^*}{ds^*}\right\|=\frac{\sqrt{2\left(\Gamma_1^2+\Gamma_2^2+\Gamma_3^2\right)}}{\left(2k_g^2+(k_n+\tau_g)^2\right)^2}$$

and

$$\mathbf{N}^*=\frac{1}{\sqrt{\Gamma_1^2+\Gamma_2^2+\Gamma_3^2}}\left(\Gamma_1\mathbf{T}+\Gamma_2\mathbf{g}+\Gamma_3\mathbf{n}\right).$$

On the other hand, we express

$$\mathbf{B}^*=\mathbf{T}^*\times\mathbf{N}^*=\frac{1}{\sqrt{2k_g^2+(k_n+\tau_g)^2}\sqrt{\Gamma_1^2+\Gamma_2^2+\Gamma_3^2}}\begin{vmatrix}\mathbf{T} & \mathbf{g} & \mathbf{n}\\ -k_g & k_g & k_n+\tau_g\\ \Gamma_1 & \Gamma_2 & \Gamma_3\end{vmatrix}.$$

So, the binormal vector of curve β is

$$\mathbf{B}^*=\frac{1}{\sqrt{2k_g^2+(k_n+\tau_g)^2}\sqrt{\Gamma_1^2+\Gamma_2^2+\Gamma_3^2}}\left(\mu_1\mathbf{T}+\mu_2\mathbf{g}+\mu_3\mathbf{n}\right)$$

where

$$\begin{cases}\mu_1=k_g\Gamma_3-(k_n+\tau_g)\Gamma_2\\ \mu_2=(k_n+\tau_g)\Gamma_1+k_g\Gamma_3\\ \mu_3=-k_g\Gamma_2-k_g\Gamma_1.\end{cases}$$

We differentiate (3.2) with respect to s in order to calculate the torsion

$$\beta^{''}=\frac{-1}{\sqrt{2}}\left\{\left(k_g^{'}+k_g^2+k_n(k_n+\tau_g)\right)\mathbf{T}+\left(-k_g^{'}+k_g^2+\tau_g(k_n+\tau_g)\right)\mathbf{g}+\left(k_nk_g-k_g\tau_g-k_n^{'}-\tau_g^{'}\right)\mathbf{n}\right\},$$

and similarly

$$\beta^{'''}=\frac{-1}{\sqrt{2}}\left(\eta_1\mathbf{T}+\eta_2\mathbf{g}+\eta_3\mathbf{n}\right),$$

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where

$$\begin{cases} \eta_1 = k_g^{''} + 2k_n\left(k_n^{'} + \tau_g^{'}\right) + k_g\left(3k_g^{'} - \tau_g^2 - k_n^2 - k_g^2\right) + k_n^{'}\left(k_n + \tau_g\right) \\ \eta_2 = -k_g^{''} + 2\tau_g\left(k_n^{'} + \tau_g^{'}\right) + k_g\left(3k_g^{'} + \tau_g^2 + k_n^2 + k_g^2\right) + \tau_g^{'}\left(k_n + \tau_g\right) \\ \eta_3 = -k_g^{''} - \tau_g^{''} + k_g\left(k_n^{'} - \tau_g^{'}\right) + \left(k_n + \tau_g\right)\left(\tau_g^2 + k_n^2 + k_g^2\right) + 2k_g^{'}\left(k_n - \tau_g\right). \end{cases}$$

The torsion of curve β is

$$\tau^* = -\frac{1}{\sqrt{2}} \frac{\left(\eta_1 + \eta_2\right)\left[k_g\left(k_n^{'} + \tau_g^{'}\right) - k_g^{'}\left(k_n + \tau_g\right)\right] + \left(2k_g^2 + \left(k_n + \tau_g\right)^2\right)\left[\tau_g\eta_1 - k_n\eta_2 + k_g\eta_3\right]}{\begin{array}{c} 2\left(k_g^{'2} + k_g^4\right) + \left(k_n + \tau_g\right)\left[\left(k_n + \tau_g\right)\left(k_n^2 + \tau_g^2\right) + 2k_n\left(k_g^{'} + k_g^2\right) + 2\tau_g\left(k_g^{'} - k_g^2\right)\right] + \\ \left(k_n^{'} + \tau_g^{'}\right)\left[k_n^{'} + \tau_g^{'} - 2k_g\left(k_n - \tau_g\right)\right] + k_g^2\left(k_n - \tau_g\right)^2 \end{array}}.$$

Corollary 1. *Consider that $\alpha(s)$ is a geodesic curve, then the following equation holds,*

i) $\kappa^* = \frac{\sqrt{2\left(k_n^2 + \tau_g^2\right)}}{\left(k_n + \tau_g\right)}$,

ii) $\tau^* = \frac{-1}{\sqrt{2}} \frac{\left(k_n + \tau_g\right)^2 + \left(k_n + \tau_g\right)\left(k_n^{'}\ \tau_g - k_n\tau_g^{'}\right)}{\left(k_n + \tau_g\right)^3\left(k_n^2 + \tau_g^2\right) + \left(k_n^{'} + \tau_g^{'}\right)^2}$.

3.2. **Tn $-$ Smarandache Curves.**

Definition 2. *Let M be an oriented surface in E^3 and let consider the arc - length parameter curve $\alpha = \alpha(s)$ lying fully on M. Denote the Darboux frame of $\alpha(s)$* $\{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$.

Tn - Smarandache curve can be defined by

$$\beta(s^*) = \frac{1}{\sqrt{2}}\left(\mathbf{T} + \mathbf{n}\right). \tag{3.6}$$

Now, we can investigate Darboux invariants of **Tn** $-$ Smarandache curve according to $\alpha = \alpha(s)$. Differentiating (3.6) with respect to s, we get

$$\beta^{'} = \frac{d\beta}{ds^*}\frac{ds^*}{ds} = \frac{-1}{\sqrt{2}}\left(k_n\mathbf{T} + \left(\tau_g - k_g\right)\mathbf{g} - k_n\mathbf{n}\right), \tag{3.7}$$

and

$$\mathbf{T}^*\frac{ds^*}{ds} = \frac{-1}{\sqrt{2}}\left(k_n\mathbf{T} + \left(\tau_g - k_g\right)\mathbf{g} - k_n\mathbf{n}\right)$$

where

$$\frac{ds^*}{ds} = \sqrt{\frac{2k_n^2 + \left(\tau_g - k_g\right)^2}{2}}. \tag{3.8}$$

The tangent vector of curve β can be written as follow,

$$\mathbf{T}^* = \frac{-1}{\sqrt{2k_n^2 + \left(\tau_g - k_g\right)^2}}\left(k_n\mathbf{T} + \left(\tau_g - k_g\right)\mathbf{g} - k_n\mathbf{n}\right). \tag{3.9}$$

Differentiating (3.9) with respect to s, we obtain

$$\frac{d\mathbf{T}^*}{ds^*}\frac{ds^*}{ds} = \frac{-1}{\left(2k_n^2 + \left(\tau_g - k_g\right)^2\right)^{\frac{3}{2}}}\left(\gamma_1\mathbf{T} + \gamma_2\mathbf{g} + \gamma_3\mathbf{n}\right) \tag{3.10}$$

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where

$$\begin{cases} \gamma_1 = (\tau_g - k_g)\left\{k_n\left(-k_g^{'} + \tau_g^{'} + k_n k_g - \tau_g k_n\right) - k_n^{'}(\tau_g - k_g) + k_g\left[2k_n^2 + \tau_g - k_g\right]\right\} - 2k_n^4 \\ \gamma_2 = k_n(\tau_g - k_g)\left[2k_n^{'} + \tau_g\left(k_g^2 - \tau_g^2\right)\right] - 2k_n^2\left(k_n k_g + \tau_g^{'} - k_g^{'} - k_n\tau_g\right) \\ \gamma_3 = (\tau_g - k_g)\left\{-k_n\left(-k_g^{'} + \tau_g^{'} - k_n k_g + \tau_g k_n\right) + k_n^{'}(\tau_g - k_g) - \tau_g\left[2k_n^2 + \tau_g - k_g\right]\right\} - 2k_n^4. \end{cases}$$

Substituting (3.8) in (3.10), we get

$$\frac{d\mathbf{T}^*}{ds^*} = \frac{\sqrt{2}}{\left(2k_n^2 + (\tau_g - k_g)^2\right)^2}\left(\gamma_1\mathbf{T} + \gamma_2\mathbf{g} + \gamma_3\mathbf{n}\right).$$

Then, the curvature and principal normal vector field of curve β are respectively,

$$\kappa^* = \left\|\frac{d\mathbf{T}^*}{ds^*}\right\| = \frac{\sqrt{2\ \left(\gamma_1^2 + \gamma_2^2 + \gamma_3^2\right)}}{\left(2k_n^2 + (\tau_g - k_g)^2\right)^2}$$

and

$$\mathbf{N}^* = \frac{1}{\sqrt{\gamma_1^2 + \gamma_2^2 + \gamma_3^2}}\left(\gamma_1\mathbf{T} + \gamma_2\mathbf{g} + \gamma_3\mathbf{n}\right).$$

On the other hand, we express

$$\mathbf{B}^* = \mathbf{T}^* \times \mathbf{N}^* = \frac{1}{\sqrt{2k_n^2 + (\tau_g - k_g)^2}\sqrt{\gamma_1^2 + \gamma_2^2 + \gamma_3^2}}\begin{vmatrix} \mathbf{T} & \mathbf{g} & \mathbf{n} \\ -k_n & k_g - \tau_g & k_n \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix}.$$

So, the binormal vector of curve β is

$$\mathbf{B}^* = \frac{1}{\sqrt{2k_n^2 + (\tau_g - k_g)^2}\sqrt{\gamma_1^2 + \gamma_2^2 + \gamma_3^2}}\left(\nu_1\mathbf{T} + \nu_2\mathbf{g} + \nu_3\mathbf{n}\right)$$

where

$$\begin{cases} \nu_1 = (k_g - \tau_g)\,\gamma_3 - k_n\gamma_2 \\ \nu_2 = k_n\gamma_1 + k_n\gamma_3 \\ \nu_3 = -k_n\gamma_2 + (\tau_g - k_g)\gamma_1. \end{cases}$$

We differentiate (3.7) with respect to s in order to calculate the torsion

$$\beta^{''} = \frac{-1}{\sqrt{2}}\left\{\left(k_g^{'} + k_n^2 - k_g(\tau_g - k_g)\right)\mathbf{T} + \left(k_n k_g + \left(\tau_g^{'} - k_g^{'}\right) + k_n\tau_g\right)\mathbf{g} + \left(k_n^2 + (\tau_g - k_g)\,\tau_g - k_n^{'}\right)\mathbf{n}\right\},$$

and similarly

$$\beta^{'''} = \frac{-1}{\sqrt{2}}\left(\omega_1\mathbf{T} + \omega_2\mathbf{g} + \omega_3\mathbf{n}\right),$$

where

$$\begin{cases} \omega_1 = k_n^{''} - 2k_g\left(\tau_g^{'} - k_g^{'}\right) + k_n\left(3k_n^{'} - \tau_g^2 - k_n^2 - k_g^2\right) + k_g^{'}(k_g - \tau_g) \\ \omega_2 = k_g^{''} - \tau_g^{''} + k_n\left(k_g^{'} + \tau_g^{'}\right) + (k_g - \tau_g)\left(\tau_g^2 + k_n^2 + k_g^2\right) + \tau_g\left(2k_n^{'} - k_n^2\right) + 2k_g k_n^{'} \\ \omega_3 = -k_n^{''} + 2\tau_g\left(\tau_g^{'} - k_g^{'}\right) + k_n\left(3k_n^{'} + \tau_g^2 + \tau_g(k_g + \tau_g)\right) + (k_g - \tau_g)\left(k_n k_g - \tau_g^{'}\right). \end{cases}$$

The torsion of curve β is

$$\tau^* = \frac{-1}{\sqrt{2}} \frac{\begin{array}{c}(\omega_1+\omega_3)\left[k_n\left(\tau_g^{'}-k_g^{'}\right)-k_n^{'}(\tau_g-k_g)+k_n^2(\tau_g+k_g)\right]+(\tau_g-k_g)^2(\tau_g\omega_1+k_g\omega_3)+\\ k_n(\tau_g-k_g)\left[k_n(\omega_1-\omega_3)+\omega_2(k_g-\tau_g)\right]-2k_n\left(k_n^{'}+k_n^2\right)\omega_2\end{array}}{\begin{array}{c}2\left(k_n^{'2}+k_n^4\right)+(\tau_g-k_g)\left[2\left(k_n^2-k_n^{'}\right)-2\left(k_n^2-k_n^{'}\right)k_g\right]+(\tau_g-k_g)^2\left(1+k_g^2\right)+\\ \left(\tau_g-k_g^{'}\right)\left[\left(\tau_g^{'}-k_g^{'}\right)+2(k_nk_g+k_n\tau_g)\right]+k_n^2(k_g+\tau_g)^2\end{array}}.$$

$$\overrightarrow{N}^* = \frac{1}{\sqrt{\psi}\pi}\left\{\left(\nu_1\cos\varphi^*-\sqrt{\psi}\sin\varphi^*\gamma_1\right)\overrightarrow{t}+\left(\nu_2\cos\varphi^*-\sqrt{\psi}\sin\varphi^*\gamma_2\right)\overrightarrow{g}+\left(\nu_3\cos\varphi^*-\sqrt{\psi}\sin\varphi^*\gamma_3\right)\overrightarrow{N}\right\}.$$

Corollary 2. *Consider that $\alpha(s)$ is an asymptotic line, then the following equation holds,*

i) $\kappa^* = \frac{\sqrt{2\left(k_g^2+\tau_g^2\right)}}{(\tau_g-k_g)^2}$,

ii) $\tau^* = \frac{-1}{\sqrt{2}}\frac{(k_g-\tau_g)\left(k_g^{'}\tau_g-k_g\tau_g^{'}\right)}{\left(1+k_g^2\right)\left(\tau_g-k_g^{'}\right)\left(\tau_g^{'}-k_g^{'}\right)(\tau_g-k_g)^{-2}}$.

3.3. gn− Smarandache Curves.

Definition 3. *Let M be an oriented surface in E^3 and let consider the arc - length parameter curve $\alpha=\alpha(s)$ lying fully on M. Denote the Darboux frame of $\alpha(s)$* $\{\mathbf{T},\mathbf{g},\mathbf{n}\}$.

gn - Smarandache curve can be defined by

$$\beta(s^*) = \frac{1}{\sqrt{2}}(\mathbf{g}+\mathbf{n}). \tag{3.11}$$

Now, we can investigate Darboux invariants of **gn**− Smarandache curve according to $\alpha=\alpha(s)$. Differentiating (3.11) with respect to s, we get

$$\beta^{'} = \frac{d\beta}{ds^*}\frac{ds^*}{ds} = \frac{-1}{\sqrt{2}}\left((k_n+k_g)\mathbf{T}+\tau_g\mathbf{g}-\tau_g\mathbf{n}\right), \tag{3.12}$$

and

$$\mathbf{T}^*\frac{ds^*}{ds} = \frac{-1}{\sqrt{2}}\left((k_n+k_g)\mathbf{T}+\tau_g\mathbf{g}-\tau_g\mathbf{n}\right)$$

where

$$\frac{ds^*}{ds} = \sqrt{\frac{2\tau_g^2+(k_n+k_g)^2}{2}}. \tag{3.13}$$

The tangent vector of curve β can be written as follow,

$$\mathbf{T}^* = \frac{-1}{\sqrt{2\tau_g^2+(k_n+k_g)^2}}\left((k_n+k_g)\mathbf{T}+\tau_g\mathbf{g}-\tau_g\mathbf{n}\right). \tag{3.14}$$

Differentiating (3.14) with respect to s, we obtain

$$\frac{d\mathbf{T}^*}{ds^*}\frac{ds^*}{ds} = \frac{-1}{\left(2\tau_g^2+(k_n+k_g)^2\right)^{\frac{3}{2}}}\left(\lambda_1\mathbf{T}+\lambda_2\mathbf{g}+\lambda_3\mathbf{n}\right) \tag{3.15}$$

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where

$$\begin{cases} \lambda_1 = 2\tau_g\tau_g^{'}(k_n+k_g) - 2\tau_g^2\left(k_n^{'}+k_g^{'}\right) + \tau_g(k_g-k_n)\left(2\tau_g^2+(k_n+k_g)^2\right) \\ \lambda_2 = -2\tau_g^4 + \tau_g(k_n+k_g)\left[\left(k_n^{'}+k_g^{'}\right) - 2\tau_g k_g\right] - (k_n+k_g)^2\left((k_n+k_g)k_g + \tau_g^{'}+\tau_g^2\right) \\ \lambda_3 = -2\tau_g^4+\tau_g(k_n+k_g)\left[\left(k_n^{'}+k_g^{'}\right) - 2\tau_g k_n\right] + (k_n+k_g)^2\left(-(k_n+k_g)k_n + \tau_g^{'}-\tau_g^2\right) \end{cases}$$

Substituting (3.13) in (3.15), we get

$$\frac{d\mathbf{T}^*}{ds^*} = \frac{\sqrt{2}}{\left(2\tau_g^2+(k_n+k_g)^2\right)^2}\left(\lambda_1\mathbf{T}+\lambda_2\mathbf{g}+\lambda_3\mathbf{n}\right).$$

Then, the curvature and principal normal vector field of curve β are respectively,

$$\kappa^* = \left\|\frac{d\mathbf{T}^*}{ds^*}\right\| = \frac{\sqrt{2\ \left(\lambda_1^2+\lambda_2^2+\lambda_3^2\right)}}{\left(2\tau_g^2+(k_n+k_g)^2\right)^2}$$

and

$$\mathbf{N}^* = \frac{1}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\left(\lambda_1\mathbf{T}+\lambda_2\mathbf{g}+\lambda_3\mathbf{n}\right).$$

On the other hand, we express

$$\mathbf{B}^* = \mathbf{T}^*\times\mathbf{N}^* = \frac{1}{\sqrt{2k_n^2+(\tau_g-k_g)^2}\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\begin{vmatrix} \mathbf{T} & \mathbf{g} & \mathbf{n} \\ -(k_n+k_g) & -\tau_g & \tau_g \\ \lambda_1 & \lambda_2 & \lambda_3 \end{vmatrix}.$$

So, the binormal vector of curve β is

$$\mathbf{B}^* = \frac{1}{\sqrt{2\tau_g^2+(k_n+k_g)^2}\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\left(\rho_1\mathbf{T}+\rho_2\mathbf{g}+\rho_3\mathbf{n}\right)$$

where

$$\begin{cases} \rho_1 = -\tau_g\lambda_3 - \tau_g\lambda_2 \\ \rho_2 = \tau_g\lambda_1 + (k_n+k_g)\lambda_3 \\ \rho_3 = -(k_n+k_g)\lambda_2+\tau_g\lambda_1. \end{cases}$$

We differentiate (3.7) with respect to s in order to calculate the torsion

$$\beta^{''} = \frac{-1}{\sqrt{2}}\left\{\left(k_g^{'}+k_n^{'}+\tau_g(k_n-k_g)\right)\mathbf{T}+\left(k_g(k_n+k_g)+\tau_g^{'}+\tau_g^2\right)\mathbf{g}+\left(k_n(k_n+k_g)-\tau_g^{'}+\tau_g^2\right)\mathbf{n}\right\},$$

and similarly

$$\beta^{'''} = \frac{-1}{\sqrt{2}}\left(\chi_1\mathbf{T}+\chi_2\mathbf{g}+\chi_3\mathbf{n}\right),$$

where

$$\begin{cases} \chi_1 = k_n^{''}+k_g^{''} - 2\dot{\tau}_g(k_n-k_g) + \tau_g\left(k_n^{'}-k_g^{'}\right) - (k_n+k_g)\left(\tau_g^2+k_n^2+k_g^2\right) \\ \chi_2 = \tau_g^{''} + 2k_g\left(k_n^{'}+k_g^{'}\right) + 3\tau_g\tau_g^{'} + (k_n+k_g)\left(k_g^{'}-\tau_g k_n\right) + k_g\tau_g(k_n-k_g) - \tau_g^3 \\ \chi_3 = \tau_g^{''} + 2k_n\left(k_n^{'}+k_g^{'}\right) + 3\tau_g\tau_g^{'} + (k_n+k_g)\left(k_g^{'}+\tau_g k_g\right) + k_n\tau_g(k_n-k_g) + \tau_g^3. \end{cases}$$

The torsion of curve β is

$$\tau^{*}=\frac{-1}{\sqrt{2}}\frac{\begin{array}{c}(k_n+k_g)^2\left[\left(\left(\tau_g^{'}+\tau_g\right)^2+k_g\left(k_n+k_g\right)\right)\chi_3-\left(\left(\tau_g^{'}-\tau_g\right)^2-k_n\left(k_n+k_g\right)\right)\chi_2+\right.\\ \left.\tau_g\left(k_n+k_g\right)\chi_1\right]-\tau_g\left(\chi_2+\chi_3\right)\left[\left(k_n^{'}+k_g^{'}\right)+\tau_g\left(k_n-k_g\right)\right]^2\end{array}}{\begin{array}{c}(k_n+k_g)^2\left(k_n^2+k_g^2\right)+2\left(k_n+k_g\right)\left[k_g\left(\tau_g^{'}+\tau_g^2\right)+2k_n\left(\tau_g^2-\tau_g^{'}\right)\right]+\\ 2{\tau_g^{'}}^2+2\tau_g^4+\left[\left(k_n^{'}+k_g^{'}\right)+\tau_g\left(k_n-k_g\right)\right]^2\end{array}}.$$

Corollary 3. *Consider that $\alpha(s)$ is a principal line, then the following equation holds,*

i) $\kappa^{*}=\frac{\sqrt{2\left(k_n^2+\tau_g^2\right)}}{(k_n+\tau_g)}$,

ii) $\tau^{*}=\frac{-1}{\sqrt{2}}\frac{(k_n+\tau_g)^2+\left(k_gk_n^{'}-k_nk_g^{'}\right)}{\left(k_n^2+k_g^2\right)+\left(k_n^{'}+\tau_g^{'}\right)^2(k_n+k_g)^{-2}}$.

3.4. **Tgn − Smarandache Curves.**

Definition 4. *Let M be an oriented surface in E^3 and let consider the arc - length parameter curve $\alpha=\alpha(s)$ lying fully on M. Denote the Darboux frame of $\alpha(s)$* $\{\mathbf{T},\mathbf{g},\mathbf{n}\}$.

Tgn- Smarandache curve can be defined by

$$\beta(s^{*})=\frac{1}{\sqrt{3}}(\mathbf{T}+\mathbf{g}+\mathbf{n}). \tag{3.16}$$

Now, we can investigate Darboux invariants of **Tgn** − Smarandache curve according to $\alpha=\alpha(s)$. Differentiating (3.16) with respect to s, we get

$$\beta^{'}=\frac{d\beta}{ds^{*}}\frac{ds^{*}}{ds}=\frac{-1}{\sqrt{3}}\left(\left(k_n+k_g\right)\mathbf{T}+\left(\tau_g-k_g\right)\mathbf{g}-(\tau_g+k_n)\mathbf{n}\right), \tag{3.17}$$

and

$$\mathbf{T}^{*}\frac{ds^{*}}{ds}=\frac{-1}{\sqrt{3}}\left(\left(k_n+k_g\right)\mathbf{T}+\left(\tau_g-k_g\right)\mathbf{g}-(\tau_g+k_n)\mathbf{n}\right)$$

where

$$\frac{ds^{*}}{ds}=\sqrt{\frac{\left(k_n+k_g\right)^2+\left(\tau_g-k_g\right)^2+\left(\tau_g+k_n\right)^2}{3}}. \tag{3.18}$$

The tangent vector of curve β can be written as follow,

$$\mathbf{T}^{*}=\frac{-1}{\sqrt{\left(k_n+k_g\right)^2+\left(\tau_g-k_g\right)^2+\left(\tau_g+k_n\right)^2}}\left(\left(k_n+k_g\right)\mathbf{T}+\left(\tau_g-k_g\right)\mathbf{g}-(\tau_g+k_n)\mathbf{n}\right). \tag{3.19}$$

Differentiating (3.19) with respect to s, we obtain

$$\frac{d\mathbf{T}^{*}}{ds^{*}}\frac{ds^{*}}{ds}=\frac{-1}{\left(\left(k_n+k_g\right)^2+\left(\tau_g-k_g\right)^2+\left(\tau_g+k_n\right)^2\right)^{\frac{3}{2}}}\left(\delta_1\mathbf{T}+\delta_2\mathbf{g}+\delta_3\mathbf{n}\right) \tag{3.20}$$

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where

$$\begin{cases} \delta_1 = (k_n+k_g)^2\left[k_g(\tau_g - k_g) - k_n(\tau_g + k_n)\right] + (k_n+k_g)\left[(\tau_g - k_g)\left(\tau_g' - k_g'\right) + (\tau_g + k_g)\left(\tau_g' + k_n'\right)\right] + \\ \left((\tau_g - k_g)^2 + (\tau_g + k_n)^2\right)\left[k_g(\tau_g - k_g) - \left(k_g' + k_n'\right) - k_n(\tau_g + k_n)\right] \\ \\ \delta_2 = (\tau_g - k_g)^2\left[-k_g(k_n+k_g) - \tau_g(\tau_g + k_n)\right] + (\tau_g - k_g)\left[(k_g+k_n)\left(k_g' + k_n'\right) + (\tau_g + k_n)\left(\tau_g' + k_n'\right)\right] + \\ \left((k_g+k_n)^2 + (\tau_g + k_n)^2\right)\left[-k_g(k_g+k_n) + \left(k_g' - \tau_g'\right) - \tau_g(\tau_g + k_n)\right] \\ \\ \delta_3 = (\tau_g+k_n)^2\left[\tau_g(k_g - \tau_g) - k_n(k_g+k_n)\right] + (\tau_g+k_n)\left[-(k_g+k_n)\left(k_g' + k_n'\right) - (\tau_g - k_g)\left(\tau_g' - k_g'\right)\right] + \\ \left((k_g+k_n)^2 + (\tau_g - k_g)^2\right)\left[\tau_g(k_g - \tau_g) + \left(\tau_g' + k_n'\right) - k_n(k_g + k_n)\right] \end{cases}$$

Substituting (3.18) in (3.20), we get

$$\frac{d\mathbf{T}^*}{ds^*} = \frac{\sqrt{3}}{\left((k_n+k_g)^2 + (\tau_g - k_g)^2 + (\tau_g + k_n)^2\right)^2}\left(\delta_1\mathbf{T} + \delta_2\mathbf{g} + \delta_3\mathbf{n}\right).$$

Then, the curvature and principal normal vector field of curve β are respectively,

$$\kappa^* = \left\|\frac{d\mathbf{T}^*}{ds^*}\right\| = \frac{\sqrt{3\ \left(\delta_1^2 + \delta_2^2 + \delta_3^2\right)}}{\left((k_n+k_g)^2 + (\tau_g - k_g)^2 + (\tau_g + k_n)^2\right)^2}$$

and

$$\mathbf{N}^* = \frac{1}{\sqrt{\left(\delta_1^2 + \delta_2^2 + \delta_3^2\right)}}\left(\delta_1\mathbf{T} + \delta_2\mathbf{g} + \delta_3\mathbf{n}\right).$$

On the other hand, we express

$$\mathbf{B}^* = \mathbf{T}^* \times \mathbf{N}^* = \frac{1}{\sqrt{(k_n+k_g)^2 + (\tau_g - k_g)^2 + (\tau_g + k_n)^2}\sqrt{\left(\delta_1^2 + \delta_2^2 + \delta_3^2\right)}}\begin{vmatrix} \mathbf{T} & \mathbf{g} & \mathbf{n} \\ -(k_n+k_g) & k_g - \tau_g & \tau_g + k_n \\ \delta_1 & \delta_2 & \delta_3 \end{vmatrix}.$$

So, the binormal vector of curve β is

$$\mathbf{B}^* = \frac{1}{\sqrt{(k_n+k_g)^2 + (\tau_g - k_g)^2 + (\tau_g + k_n)^2}\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\left(\sigma_1\mathbf{T} + \sigma_2\mathbf{g} + \sigma_3\mathbf{n}\right)$$

where

$$\begin{cases} \sigma_1 = (k_g - \tau_g)\,\delta_3 - (\tau_g + k_n)\,\delta_2 \\ \sigma_2 = (\tau_g + k_n)\,\delta_1 + (k_n + k_g)\delta_3 \\ \sigma_3 = -(k_n + k_g)\,\delta_2 - (k_g - \tau_g)\,\delta_1. \end{cases}$$

We differentiate (3.17) with respect to s in order to calculate the torsion

$$\beta'' = \frac{-1}{\sqrt{3}}\left\{\begin{array}{l} \left(k_g' + k_n' + k_g(k_g - \tau_g) + k_n(\tau_g + k_n)\right)\mathbf{T} + \left(k_g(k_n+k_g) + \left(\tau_g' - k_g'\right) + \tau_g(\tau_g + k_n)\right)\mathbf{g} + \\ \left(k_n(k_n+k_g) + \tau_g(\tau_g - k_g) - \left(\tau_g' + k_n'\right)\right)\mathbf{n} \end{array}\right\},$$

and similarly

$$\beta''' = \frac{-1}{\sqrt{3}}\left(\xi_1\mathbf{T} + \xi_2\mathbf{g} + \xi_3\mathbf{n}\right),$$

where

$$\begin{cases} \xi_1 = k_n'' + k_g'' - 2k_g'\left(k_g' - \tau_g'\right) - (k_n + k_g)\left(k_n^2 + k_g^2\right) + (k_g - \tau_g)\left(k_g' + k_n\tau_g\right) + \\ 2k_n\left(k_n' + \tau_g'\right) + (k_n + \tau_g)\left(k_n' - k_g\tau_g\right) \\ \\ \xi_2 = \tau_g'' - k_g'' + 2\tau_g\left(k_n' + \tau_g'\right) + 2k_g\left(k_n' + k_g'\right) + (k_n + k_g)\left(k_g' - k_n\tau_g\right) + \\ (\tau_g + k_n)\left(k_nk_g + \tau_g'\right) + (k_g - \tau_g)\left(k_g^2 + \tau_g^2\right) \\ \\ \xi_3 = -\left(\tau_g'' + k_n''\right) + 2k_n\left(k_n' + k_g'\right) + (\tau_g - k_g)\left(\tau_g' - k_nk_g\right) + (k_n + k_g)\left(k_n' + k_g\tau_g\right) + \\ 2\tau_g\left(\tau_g' - k_g'\right) + (k_n + \tau_g)\left(k_n^2 + \tau_g^2\right). \end{cases}$$

The torsion of curve β is

$$\tau^* = \frac{-1}{\sqrt{3}} \frac{\begin{array}{l} (k_n + k_g)(k_g - \tau_g)\left(\tau_g\xi_2 - k_n\xi_1\right) + (k_g - \tau_g)^2\left(\tau_g\xi_1 + k_g\xi_3\right) + \\ \left(k_n' + \tau_g'\right)\left[(k_g - \tau_g)\xi_1 + (k_n + k_g)\xi_2\right] - \\ \left(k_n' + k_g'\right)\left[\left(k_n' + \tau_g\right)\xi_1 + (k_g - \tau_g)\xi_3 - (k_n + \tau_g)\xi_2\right] + \\ (k_n + \tau_g)^2\left(\tau_g\xi_1 - k_n\xi_2\right) + \\ (k_n + \tau_g)(k_n + k_g)\left(\tau_g\xi_3 + k_g\xi_1\right) + (k_n + \tau_g)^2\left(k_g\xi_3 - k_n\xi_2\right) + \\ (k_g - \tau_g)(k_n + \tau_g)\left(k_n\xi_3 - k_g\xi_2\right) + (k_n + k_g)\left(\tau_g' - k_g'\right) \end{array}}{\begin{array}{l} (k_g - \tau_g)^2\left(k_g^2 + \tau_g^2\right) + 2\left(k_n' + k_g'\right)\left[k_g(k_g - \tau_g) + k_n(k_n + \tau_g)\right] + \\ (k_n + \tau_g)^2\left(k_n^2 + \tau_g^2\right) + 2(k_n + \tau_g)\left[\tau_g\left(\tau_g' - k_g'\right) + k_g^2(k_n + \tau_g)\right] + \\ (k_n + k_g)^2\left(k_n^2 + \tau_g^2\right) + 2(k_n + k_g)\left[k_g\left(\tau_g' - k_g'\right) - k_n\left(\tau_g' + k_g'\right)\right] + \\ k_n\tau_g(\tau_g - k_g) + \left(k_n' + k_g'\right)^2 + \left(k_n^2 + \tau_g^2\right) + \left(\tau_g' - k_g'\right)^2 + \left(\tau_g' - k_n'\right)^2 + \\ -2\tau_g(\tau_g - k_g)\left(\tau_g' + k_n'\right). \end{array}}.$$

Example 1. *Let us consider the following unit speed curve:*

$$\alpha(t) = \left(-\cos t, \frac{\sin^2 t}{2}, \frac{1}{4}\sin 2t + \frac{t}{2}\right)$$

We can show smarandache curves in the following figure.

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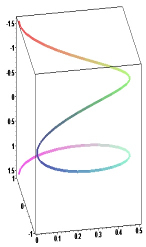
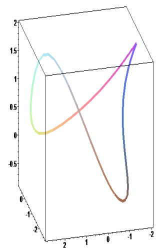
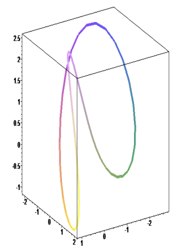


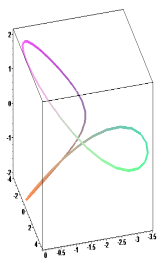
The curve $\alpha = \alpha(t)$



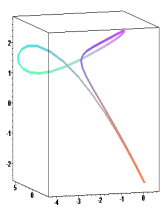
Tg-Smarandache curve



Tn-Smarandache curve



gn-Smarandache curve



Tgn-Smarandache curve

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YILDIZ TECHNICAL UNIVERSITY, FACULTY OF ARTS AND SCIENCES, DEPARTMENT OF MATHEMATICS, 34210, ESENLER, İSTANBUL, TURKEY
E-mail address: obektas@yildiz.edu.tr

Current address: YILDIZ TECHNICAL UNIVERSITY, FACULTY OF ARTS AND SCIENCES, DEPARTMENT OF MATHEMATICS, 34210, ESENLER, İSTANBUL, TURKEY
E-mail address: sayuce@yildiz.edu.tr