ON A COHERENT MULTIPLICATION OF DISTRIBUTIONS

SOFIANE BOURADA, BEKKAI MESSIRDI AND LEKHMISSI BELAIB

Abstract. The goal of this paper is to investigate the multiplication of distributions by fractional calculus. We introduce in a fairly consistent way the multiplication of distributions by regularization using model delta-sequences. Our method combine the generalized Kaminski product and Fractional integration and differentiation, it serves as motivation for the development of new applications.

Mathematics Subject Classification (2010): 46F10, 46F30.
Key words: Distribution, Regularization, Fractional calculus, Model delta-sequence, Product of distributions.

Article history:
Received 01 October 2018
Accepted 27 March 2019

1. Introduction

The question of fractional derivatives was discussed in 1695 by G.W. Leibniz. More than 300 years later, several mathematical difficulties related to this derivative have been solved. Numerous mathematicians have studied this question, in particular L. Euler in 1730, N.H. Abel in 1823, J. Liouville in 1832 and B. Riemann in 1847 and others. Fractional Order Calculus (FOC) dates back to the birth of the theory of differential calculus or integer order calculus. However, FOC only began to be applied in the last two decades as a result of advances in the area of chaos, which revealed subtle relationships with the FOC concepts. Recent progress in the area of fractional derivatives and integrals implies a promising potential for future developments and application of the theory in various scientific areas. Some basic concepts of FOC and several applications in applied sciences and engineering have been presented. Recent work on the fractional derivative techniques of Li and Clarkson [4] and Odibat and Momani [6] can also be mentioned. Different approaches have been used to generalize the notion of derivation to non-integer orders. Indeed, the limit definition of a derivative of a function has been generalized by the formula of Grunwald-Letnikov; left and right Riemann–Liouville fractional integrals, left and right Caputo fractional derivatives are also used via Euler's gamma function. These approaches are particularly useful in numerical analysis. Fractional calculus is a classical area with many good books available. We refer the reader to Malinowska and Torres [5] (2012) and Podlubny [7] (1999). These different definitions have for a long time seemed not to always give the same results. This apparent inconsistency has been dissipated in the context of theory of distributions introduced by L. Schwartz in [8].

The notion of distributions makes it possible in particular to differentiate a function even if it is not continuous, which consequently allows the inclusion of discontinuous functions as solutions of differential equations. Practically, all the known operations on the functions were extended to the distributions, nevertheless L. Schwartz realized through a counter-example the impossibility to define a consistent multiplication of distributions which will be able to preserve the minimum of algebraic properties, such as the associativity, and which coincides with Schwartz's product of a distribution by a C^∞-function (see [9]). This constitutes a major disadvantage for this theory, namely its non-extension to non-linear problems.

This article shows the interest and the importance of FOC to introduce in a fairly consistent way the multiplication of distributions. It is described how multiplication can be correctly defined by the regularization techniques and fractional calculus principle : differentiation and integration of non-integer order. The C^∞-smooth approximations (process of smoothly regularizing a given distribution) are obtained by convolution with adequate mollifiers via the so-called delta-sequences or generalized sequences of functions convergent towards the Dirac distribution δ.

We strongly hope that our approach serves as motivation for the development of new applications. Our contribution provided a sufficient criterion to define multiplication of distributions by regularization and using a direct integral calculus. The method that we use here, although it uses Kaminski's product [3], is more general and easier for the introduction of a coherent multiplication operation in the class of distributions.

The paper is organized as follows. In sections 2 and 3, we give some definitions and preliminary results in which our investigation will be done, especially we review basic mathematical results about the distribution x_+^λ, $\lambda \in \mathbb{C}$, such as its analytic continuation and regularization. In section 4, we briefly introduce the necessary concepts and definitions of fractional calculus on distributions on $\mathbb{R}$. In section 5, we present the main result of this work, where sufficient conditions are obtained under which an explicit product of distributions is well defined.

2. Analytic continuation of distribution x_+^λ

Let λ be a complex number and let x_+^λ be the function defined on $\mathbb{R}$ by:

$$x_+^\lambda = \begin{cases} x^\lambda & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}.$$

If $\operatorname{Re}\lambda > -1$, then the function x_+^λ is locally integrable, so it defines a distribution:

$$< x_+^\lambda, \varphi > \; = \int_0^{+\infty} x^\lambda \varphi(x) dx \quad ; \quad \varphi \in C_0^\infty(\mathbb{R}). \tag{2.1}$$

$C_0^\infty(\mathbb{R})$ is the space of test functions on $\mathbb{R}$. For φ arbitrarily fixed in $C_0^\infty(\mathbb{R})$, it's clear that the function $\xi(\lambda) = < x_+^\lambda, \varphi >$ is analytic for $\operatorname{Re}\lambda > -1$, since for all $n \in \mathbb{N}^*$ the integral

$$\frac{\partial^n \xi}{\partial \lambda^n} = \int_0^{+\infty} x^\lambda (\ln x)^n \varphi(x) dx$$

is convergent. Note also that the second member of (2.1) can be expressed as follows:

$$\int_0^1 x^\lambda [\varphi(x) - \varphi(0)] dx \; + \int_1^{+\infty} x^\lambda \varphi(x) dx + \frac{\varphi(0)}{\lambda + 1}. \tag{2.2}$$

As $\varphi(x) - \varphi(0) = x\psi(x)$ with ψ bounded, then the first member of (2.2) is defined for $\operatorname{Re}\lambda > -2$, the second member exists for all values of λ and the third for $\lambda \neq -1$. Hence the possibility to extend analytically the distribution x_+^λ and the function $\xi(\lambda)$ for $\operatorname{Re}\lambda > -2$ and $\lambda \neq -1$. If we repeat this process, we can define x_+^λ for $\operatorname{Re}\lambda > -n-1$, $\lambda \neq -1, ..., -n$:

$$< \ x_+^\lambda, \varphi >= \int_0^1 x^\lambda \left[\varphi(x) - \varphi(0) - \frac{\varphi'(0)}{1!}x - .. - \frac{\varphi^{(n-1)}(0)}{(n-1)!}x^{n-1}\right] dx$$

$$+\int_1^{+\infty} x^\lambda \varphi(x)dx + \sum_{k=1}^{n} \frac{\varphi^{(k-1)}(0)}{(\lambda+k)\ (k-1)!};\ \varphi \in C_0^\infty(\mathbb{R}),\ n \in \mathbb{N}^*. \tag{2.3}$$

Remark 2.1. *When φ and all its derivatives vanish at 0, the distribution x_+^λ is regular equal to the monomial x^λ.*

Therefore, the restriction of the distribution defined by (2.3) to an interval of $]0, +\infty[$ is the function $x^\lambda \in C^\infty(]0, +\infty[)$. The formula (2.3) shows that the function $\xi(\lambda)$ has simple poles at $\lambda = -k$, $k \in \mathbb{N}^*$. The residue of $\xi(\lambda)$ at $\lambda = -k$ is $\frac{\varphi^{(k-1)}(0)}{(k-1)!}$, that of the distribution x_+^λ is then equal to $\frac{(-1)^{k-1}}{(k-1)!}\delta^{(k-1)}$, where δ is the Dirac distribution on $\mathbb{R}$ concentrated at 0.

Since the distribution x_+^λ, $\operatorname{Re}\lambda > -1$, is tempered we can thus make it acting on the Schwartz space $\mathcal{S}(\mathbb{R})$ of all functions whose derivatives are rapidly decreasing on $\mathbb{R}$. On the other hand, the integral (gamma function):

$$\Gamma(\lambda) = \int_0^{+\infty} x^{\lambda-1}e^{-x}dx$$

converges for $\operatorname{Re}\lambda > -1$, $\Gamma(\lambda)$ can be considered as the distribution $x_+^{\lambda-1}$ acting on the function $\varphi_0 \in \mathcal{S}(\mathbb{R})$ equal e^{-x} on $]0, +\infty[$.

Using formula (2.3), we get an explicit expression of $\Gamma(\lambda)$ for $\operatorname{Re}\lambda > -n-1, \lambda \neq -1, ., -n$, $n \in \mathbb{N}^*$:

$$\Gamma(\lambda) = \int_0^1 x^{\lambda-1}\left[e^{-x} - \sum_{k=0}^{n}\frac{(-1)^k}{k!}x^k\right]dx + \int_1^{+\infty} x^{\lambda-1}e^{-x}dx + \sum_{k=0}^{n}\frac{(-1)^k}{(\lambda+k)\ k!}. \tag{2.4}$$

3. Regularization of x_+^λ with respect to λ

x_+^λ has simple poles at $\lambda = -k$, $k \in \mathbb{N}^*$, it is then natural to try to eliminate these singularities for example by dividing it by a function having the same poles. Since $\frac{(-1)^{k-1}}{(k-1)!}\delta^{(k-1)}$ is the residue of x_+^λ at $\lambda = -k$, $k \in \mathbb{N}^*$, this function must be chosen such that the residues of $\xi(\lambda)$ are all non-zero, so the function and all its derivatives must be non-zero at $x = 0$. It is natural to take $\varphi_0(x) = e^{-x}$ on $]0, +\infty[$ and thus:

$$< x_+^\lambda, \varphi_0 >= \int_0^{+\infty} x^\lambda e^{-x}dx = \Gamma(\lambda+1).$$

Therefore, we can consider the distribution $\frac{x_+^\lambda}{\Gamma(\lambda+1)}$, it is in particular equal to the ratio of the residues of x_+^λ and $\Gamma(\lambda+1)$ at singularities $\lambda = -k$, $k \in \mathbb{N}^*$:

$$\frac{x_+^\lambda}{\Gamma(\lambda+1)}|_{\lambda=-k} = \delta^{(k-1)}. \tag{3.1}$$

4. Fractional integration and differentiation

The Cauchy formula for repeated integration, allows one to compress n antidifferentiations of a function $g(y)$ into a single integral:

$$\begin{aligned} g_n(x) &= \int_0^x \int_0^{y_{n-1}} ... \int_0^{y_1} g(y)dydy_1...dy_{n-1} \\ &= \frac{1}{(n-1)!}\int_0^x g(y)(x-y)^{n-1}dy. \end{aligned}$$

This formula can be expressed in term of convolution as follows:

$$g_n(x) = g(x) * \frac{x^{n-1}}{\Gamma(n)}$$

where g and x^{n-1} both vanish if $x < 0$, $n \in \mathbb{N}^*$.

It seems natural to generalize this formula to the case of any complex exponent λ and a distribution g concentrated on the half axis $x \geq 0$, by introducing the notion of differentiation of order λ of g as a convolution in the distributional sense in the space D'_+ of distributions on $\mathbb{R}$ with support in $[0, +\infty[$.

(4.1)
$$g_\lambda = g * \Phi_\lambda \text{ with } \Phi_\lambda = \frac{x_+^{\lambda-1}}{\Gamma(\lambda)}.$$

We note that the convolution on the right-hand side is well defined, as supports of g and Φ_λ are bounded on the same side.

By virtue of (3.1), we have:

$$\Phi_{-k} = \delta^{(k)}\ ,\ k \in \mathbb{N}$$

and thus:

$$\begin{aligned} g_0 &= g * \Phi_0 = g * \delta = g, \\ g_{-1} &= g * \Phi_{-1} = g * \delta' = g'. \end{aligned}$$

So we will denote for $\lambda \in \mathbb{C}$:

$$g_{-\lambda} = \frac{d^\lambda g}{dx^\lambda}.$$

as the fractional derivative of the distribution $g(x)$ with order λ for $\operatorname{Re}\lambda \geq 0$. Similarly, $\frac{d^\lambda g}{dx^\lambda}$ is interpreted as the fractional integral if $\operatorname{Re}\lambda < 0$. Note that $\Phi_\lambda \in D'_+$ and $g_\lambda \in D'_+$ if $g \in D'_+$, on other hand g_λ is C^∞-function if g is C^∞-function.

Example 4.1. *As an example of finding a fractional derivative of a distribution, we let $g \in D'_+$ be given by:*

$$g(x) = \begin{cases} 1 & \text{if } x \text{ is irrational and positive,} \\ 0 & \text{otherwise.} \end{cases}$$

Then the ordinary derivative of $g(x)$ does not exist. However, the distributional derivative of $g(x)$ does exist, and $g'(x) = \delta(x)$:

$$< g', \varphi >= - < g, \varphi' >= -\int_0^{+\infty} \varphi'(x)dx = \varphi(0) =< \delta, \varphi >$$

as the measure of rational numbers is zero. Therefore,

$$\frac{d^{1,5}g}{dx^{1,5}} = \frac{d^{0,5}\delta}{dx^{0,5}} = \frac{x_+^{-1,5}}{\Gamma(-0,5)} = -\frac{1}{2\sqrt{\pi}} x_+^{-1,5}.$$

Distributions Φ_λ have the following fundamental properties:

Proposition 4.2. *For all $\lambda, \mu \in \mathbb{C}$ and $k \in \mathbb{Z}$:*

$$\Phi_\lambda * \Phi_\mu = \Phi_{\lambda+\mu} \tag{4.2}$$

$$\frac{d^\lambda H(x)}{dx^\lambda} = \Phi_{1-\lambda} \tag{4.3}$$

$$\frac{d^\lambda \delta^{(k)}}{dx^\lambda} = \Phi_{-k-\lambda}. \tag{4.4}$$

where H is the Heaviside distribution on $\mathbb{R}$.

Proof. Formula (4.2) is a direct consequence of equality:

$$\frac{x_+^{\lambda-1}}{\Gamma(\lambda)} * \frac{x_+^{\mu-1}}{\Gamma(\mu)} = \frac{x_+^{\lambda+\mu-1}}{\Gamma(\lambda+\mu)}$$

with $\operatorname{Re}\lambda > 0$ and $\operatorname{Re}\mu > 0$, which remains, by uniqueness of analytic continuation, valid for the other values of λ and μ.

Taking $-\lambda$ instead of λ in (4.1), we have:

$$\frac{d^\lambda}{dx^\lambda}\left(\frac{x_+^{\mu-1}}{\Gamma(\mu)}\right) = \frac{x_+^{\mu-\lambda-1}}{\Gamma(\mu-\lambda)}.$$

For $\mu = 1$ we find (4.3) and $\mu = -k$ gives (4.4). □

5. Explicit multiplication in D'_+

In this section we give a sufficient criterion for the existence of the product in the convolution algebra D'_+. We introduce the multiplication on D'_+ by convolution and regularization of distributions also using the product of Kaminski (cf. [3]). In the following let D_+ be the space of C^∞-functions on $\mathbb{R}$ with compact support contained in $[0, +\infty[$.

Definition 5.1. *Let $\rho \in D_+$ such that:*

$$\rho \geq 0 \text{ and } \int_0^{+\infty} \rho(x)dx = 1.$$

The model delta-sequence $\{\rho_n\}_n \subset D_+$ is defined to be a sequence of testing functions:

$$\rho_n(x) = \beta_n \rho(\beta_n x) \quad (x \in \mathbb{R},\ n \in \mathbb{N}) \tag{5.1}$$

where $\beta_n \in \mathbb{R}$, $\beta_n \longrightarrow \infty$.

The choice of the number sequence (β_n) in (5.1) influences the speed of convergence of the sequence $\{\rho_n\}$ to the Dirac measure δ.

In fact, the condition $\rho \geq 0$ makes the regularizations ρ_n more specific than those defined by Kaminski in [3], therefore the product that we propose will be more general than that of Kaminski. Let S and T be distributions on $\mathbb{R}$. In order to introduce the product ST by regularization using model delta-sequences $\{\rho_n\}$ and $\{\eta_n\}$, Kaminski considered the following definitions:

$$\begin{aligned}
[S][T] &= \lim_{n\to\infty} (S * \rho_n)(T * \eta_n), \\
[S]T &= \lim_{n\to\infty} (S * \rho_n)T, \\
S[T] &= \lim_{n\to\infty} S(T * \rho_n), \\
[ST] &= \lim_{n\to\infty} (S * \rho_n)(T * \rho_n).
\end{aligned}$$

For each of the definitions of the product above it is required that the limit in the second member exists and does not depend on the choice of model delta-sequences $\{\rho_n\}$ and $\{\eta_n\}$.

The first three products of Kaminski are equivalent, while the fourth product called generalized Kaminski product is strictly more general (see [2]). Therefore, it is enough to give a criterion allowing to calculate explicitly the generalized Kaminski product. So we need the following lemma:

Lemma 5.2. *Let $(T_n)_n, T \in D'_+$. Then for any $p \in \mathbb{N}^*$, we have:*

$$T_n \underset{n\to\infty}{\longrightarrow} T \text{ in } D'_+ \Longleftrightarrow T_n * \frac{x_+^{p-1}}{(p-1)!} \underset{n\to\infty}{\longrightarrow} T * \frac{x_+^{p-1}}{(p-1)!} \text{ in } D'_+.$$

Proof. Implication from left to right is obtained by continuity of the convolution in D'_+. The converse implication results from the fact that $x_+^0 = H(x)$ and

$$T_n * \frac{x_+^{p-1}}{(p-1)!} * \delta^{(p-1)} = T_n * H(x).$$

Thus,

$$[T_n * \frac{x_+^{p-1}}{(p-1)!}] * \delta^{(p)} = T_n \underset{n\to\infty}{\longrightarrow} [T * \frac{x_+^{p-1}}{(p-1)!}] * \delta^{(p)} = T.$$

□

We establish now the following main result:

Theorem 5.3. *Let $S, T \in D'_+$, $(\rho_n)_n$ be a model delta-sequence and $p \in \mathbb{N}^*$. We use the following notations:*

$$\begin{aligned}
S_n &= S * \rho_n \\
T_n &= T * \rho_n \\
\Phi_n^j(y) &= \int_0^y t^j S_n(t) T_n(t) dt \ ; \ j = 0, 1, ..., p-1.
\end{aligned}$$

If Φ_n^j, $n \in \mathbb{N}$, $j = 0, 1, ..., p-1$, satisfy:

$$\left|\Phi_n^j(y)\right| \leq M_j(y) \in L^1_{loc}([0, +\infty[)$$

and

$$\lim_{n\to\infty} \Phi_n^j(y) = \Phi_j(y) \ , \ \textit{for all } y \in [0, +\infty[$$

independently on the choice of model delta-sequence $\{\rho_n\}$ *, then the product* $[ST]$ *exists and*

$$[ST] = \sum_{j=0}^{p-1} \frac{(-1)^j}{j!}\ \Phi_j(x)\ \frac{x_+^{p-1-j}}{(p-1-j)!} * \delta^{(p)}.$$

Proof. From Lemma 5.2, $S_nT_n \to h$ in D'_+ if and only if $S_nT_n * \frac{x_+^{p-1}}{(p-1)!} \underset{n\to\infty}{\longrightarrow} h * \frac{x_+^{p-1}}{(p-1)!}$ in D'_+. Let $\varphi \in D_+$. As $S_nT_n \in C^\infty([0,+\infty[)$, we infer that:

$$(5.2)\qquad \begin{aligned} < \ S_nT_n * \frac{x_+^{p-1}}{(p-1)!}, \varphi > &=< S_nT_n, < \frac{x_+^{p-1}}{(p-1)!}, \varphi(x+t) >> \\ &= \int_0^{+\infty} S_n(t)T_n(t) < \frac{x_+^{p-1}}{(p-1)!}, \varphi(x+t) > dt. \end{aligned}$$

Using the the change of variables $y = x + t$ and binomial expansion, we obtain:

$$(5.3)\qquad < \frac{x_+^{p-1}}{(p-1)!}, \varphi(x+t) >= \sum_{j=0}^{p-1} \left[\int_t^{+\infty} \frac{y^{p-1-j}}{(p-1-j)!}\varphi(y)dy\right] \frac{(-1)^j}{j!} t_+^j.$$

By virtue of (5.2) and using Fubini's theorem we also obtain:

$$(5.4)\qquad < S_nT_n * \frac{x_+^{p-1}}{(p-1)!}, \varphi >= \sum_{j=0}^{p-1} \frac{(-1)^j}{j!} \int_0^{+\infty} \Phi_n^j(y)\ \frac{y^{p-1-j}}{(p-1-j)!}\varphi(y)dy.$$

Using the hypotheses on $\left(\Phi_n^j\right)_{0\le j\le p-1}$ and Lebesgue's dominated convergence theorem, we have:

$$h * \frac{x_+^{p-1}}{(p-1)!}(\xi) = \sum_{j=0}^{p-1} \frac{(-1)^j}{j!}\ \Phi_j(y)\ \frac{y_+^{p-1-j}}{(p-1-j)!}$$

Finally,

$$h = [ST] = \sum_{j=0}^{p-1} \frac{(-1)^j}{j!}\ \left[\Phi_j(y)\ \frac{y_+^{p-1-j}}{(p-1-j)!}\right] * \delta^{(p)}.$$

□

Under the same conditions of Theorem 5.3, we have the following two corollaries:

Corollary 5.4. *If* $p = 1$ *and* $j = 0$,

$$(5.5)\qquad [ST] = \Phi'(y) + \Phi(0)\delta$$

where $\Phi(y) = \Phi_0(y) = \lim\limits_{n\to\infty} \int_0^y S_n(t)T_n(t)dt.$

Corollary 5.5. *If* $p = 1$, $j = 0$ *and* $\Phi_n^0(y) > 0$ *such that* $\lim\limits_{n\to\infty} \Phi_n^0(y) = +\infty$ *for all* $y > 0$, *then the generalized Kaminski product* $[ST]$ *does not exist.*

Proof. Let $\varphi_0 \in D_+$ with support contained in $[0, a]$, $a > 0$ and $\varphi_0 \ge 0$. Using (5.4) and the mean value theorem, we have:

$$< S_nT_n * H, \varphi >= \int_0^{+\infty} \Phi_n^0(y)\varphi_0(y)dy = \Phi_n^0(a\theta) \int_0^a \varphi_0(y)dy\ ;\ 0 < \theta < 1.$$

Since $\lim\limits_{n\to\infty} \Phi_n^0(y) = +\infty$, we deduce that:

$$\lim_{n\to\infty} < S_nT_n * H, \varphi_0 >= +\infty.$$

The result follows from Lemma 5.2. □

In the following examples we will use Corollary 5.4.

Example 5.6. *1) Distribution δH.*
Let $(\rho_n)_n$ be a model delta-sequence, then:

$$\begin{aligned} H_n(x) &= \int_0^x \rho_n(x-t)dt = \int_0^x \rho_n(z)dz \\ \Phi_n(y) &= \int_0^y \rho_n(t)H_n(t)dt = \int_0^y \rho_n(t)\left[\int_0^t \rho_n(z)dz\right]dt \\ &= \int_0^y H_n(t)dH_n(t)dt = \frac{1}{2}H_n^2(y). \end{aligned}$$

So,

$$\lim_{n\to\infty} \Phi_n(y) = \frac{1}{2}.$$

We also have from (5.5) :

$$\delta H = \frac{1}{2}\delta.$$

2) Distribution H^2.
This leads to compute $\Phi_n(y)$ by integration by parts:

$$\Phi_n(y) = \int_0^y H_n^2(t)dt = yH_n^2(y) - 2\int_0^y tH_n(t)dH_n(t),$$

we infer by mean value theorem that:

$$\begin{aligned} \int_0^y tH_n(t)dH_n(t) &= H_n(\theta_1 y)\int_0^y tdH_n(t) \\ &= H_n(\theta_1 y)\int_0^y t\rho_n(t)dt \ ; \ 0 < \theta_1 < 1. \end{aligned}$$

On the other hand,

$$\begin{aligned} \int_0^y t\rho_n(t)dt &= yH_n(y) - \int_0^y H_n(t)dt \\ &= yH_n(y) - H_n(\theta_2 y)y \\ &= y[H_n(y) - H_n(\theta_2 y)] \ ; \ 0 < \theta_2 < 1. \end{aligned}$$

Thus,

$$\lim_{n\to\infty} \int_0^y t\rho_n(t)dt = 0.$$

Consequently, $\lim_{n\to\infty} \Phi_n(y) = y$. By virtue of (5.5), *we then find $H^2 = 1H + 0\delta = H$, as well as $H^n = H$, for all $n \in \mathbb{N}^*$.*

Remark 5.7. *It follows immediately from Corollary* 5.5, *that δ^2 does not exist for the product described above. Indeed,*

$$\begin{aligned} \lim_{n\to\infty} \Phi_n(y) &= \lim_{n\to\infty} \int_0^y \rho_n^2(t)dt \\ &= \lim_{n\to\infty} \beta_n^2 \int_0^y \rho^2(\beta_n t)dt = +\infty \end{aligned}$$

since $\lim_{n\to\infty} \int_0^{\beta_n y} \rho^2(x)dx$ is bounded when $\rho \in D_+$.

In particular, Colombeau algebra remains a way to remedy such a situation in the class of generalized functions [1].

References

[1] J.F. Colombeau, *Elementary introduction to new generalized functions*, North Holland, 1985.

[2] J. Jelinek, *A contribution to the equivalence results for the product of distributions*, Comment. Math. Univ. Carolinae, 35(2) (1994), 263-266.

[3] A. Kaminski, *Convolution, Product and Fourier transformation of distributions*, Studia Math., 74 (1982), 83-96.

[4] C. Li and K. Clarkson, *Babenko's Approach to Abel's Integral Equations*, Mathematics MDPI, 6(3) (2018), 32, doi:10.3390/math6030032.

[5] A. B. Malinowska and D.F.M. Torres, *Introduction to the fractional calculus of variations*, Imperial College Press, London, 2012.

[6] Z. Odibat and S. Momani, *Numerical methods for nonlinear partial differential equations of fractional order*, Appl. Math. Modelling, 32(1) (2008), 28-39.

[7] I. Podlubny, *Fractional differential equations, Mathematics in Science and Engineering*, Academic Press, San Diego, 1999.

[8] L. Schwartz, *Théorie des distributions*, Hermann, Paris, 1966.

[9] L. Schwartz, *Sur l'impossibilité de la multiplication des distributions*, C.R. Acad. Sci. Paris, 239 (1954), 847–848.

Department of Mathematics, Faculy of Exact and Applicable Sciences. University of Oran1 Ahmed Ben Bella, Algeria.

E-mail address: bouradasofiane@yahoo.fr

Hight School of electrical engineering and energetics-Oran, Algeria., Laboratory of Fundamental and Applicable Mathematics of Oran (LMFAO).

E-mail address: bmessirdi@yahoo.fr

Department of Mathematics, Faculy of Exact and Applicable Sciences. University of Oran1 Ahmed Ben Bella, Algeria.

E-mail address: belaib_lekhmissi@yahoo.fr