ON THE EXPONENTIAL BONDAGE NUMBER OF A GRAPH

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Abstract. The domination number is an important vulnerability parameter that has become one of the most widely studied topics in graph theory. The bondage number is related to the domination number, and also the most often studied property of vulnerability of communication networks. Recently, Dankelmann et al. have defined the exponential domination number in [17]. We investigate a refinement that involves the exponential bondage number of this parameter. Let $G = (V(G), E(G))$ be a simple graph. The exponential bondage number, denoted by $b_{exp}(G)$, is defined by $b_{exp}(G) = min\{|B_e| : B_e \subseteq E(G), \gamma_e(G - B_e) > \gamma_e(G)\}$, where $\gamma_e(G)$ is the exponential domination number of G. In this paper, the above mentioned new parameter is defined and examined. Then upper bounds, lower bounds and exact formulas are obtained for any graph G. Finally, the exact values have been determined for some well-known graph families.

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1. Introduction

Graph theory has seen an explosive growth due to interaction with areas like computer science, operation research, etc. Especially, it has become one of the most powerful mathematical tools in the analysis and study of the architecture of a network. A network is described as an undirected and unweighted graph in which vertices represent the processing and edges represent the communication channel between them [8, 9, 13].

It is known that communication systems are often exposed to failures and attacks. So vulnerability of the network topology is a key aspect in the design of computer networks. The vulnerability value of a communication network shows the resistance of the network after the disruption of some centers or connection lines until a communication breakdown. In the literature, various measures have been defined to measure the robustness of network and a variety of graph theoretic parameters have been used to derive formulas to calculate network vulnerability. Graph vulnerability relates to the study of graph when some of its elements (vertices or edges) are removed. The measures of graph vulnerability are usually invariants that measure how a deletion of one or more network elements changes properties of the network. The best known measure of reliability of a graph is its connectivity. The vertex (edge) connectivity is defined to be the minimum number of vertices (edges) whose deletion results in a disconnected or trivial graph [8]. Then toughness [25], integrity [5], domination number [6, 19], bondage number [3, 4, 7, 10], reinforcement number [12] etc. have been proposed for measuring the vulnerability of networks. Recently, some average vulnerability parameters such as average lower independence number [2, 14, 24], average lower domination number [14, 23], average connectivity number [15], average lower bondage number [20] and average lower reinforcement number [21] have been defined.

Let $G = (V(G), E(G))$ be a simple undirected graph of order n. We begin by recalling some standard definitions that we need throughout this paper. For any vertex $v \in V(G)$, the *open neighborhood* of v is $N_G(v) = \{u \in V(G)|uv \in E(G)\}$ and *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. The degree of v in G denoted by $deg(v)$, is the size of its open neighborhood. A vertex v is said to be pendant if $deg(v) = 1$ [6, 19]. A vertex u is called support vertex if u is adjacent to a pendant vertex. The graph G is called r-regular graph if $deg(v) = r$ for every vertex $v \in V(G)$. The complement $\overline{G}$ of a graph G has $V(G)$ as its vertex sets, but two vertex are adjacent in $\overline{G}$ if only if they are not adjacent in G. The *distance* $d(u, v)$ between two vertices u and v in G is the length of a shortest path between them. The *diameter* of G, denoted by $diam(G)$ is the largest distance between two vertices in $V(G)$ [6, 19]. A cycle passing through all the vertices of a graph is called a *Hamiltonian cycle*. A graph containing a Hamiltonian cycle is called a *Hamiltonian graph*, and also if there exist a path P in the graph G such that $V(P) = V(G)$, then G is called *semihamiltonian* [19]. A *vertex-transitive graph* is a graph such that every pair of vertices is equivalent under some element of its automorphism group [26]. A set of pairwise independent edges of G is called a *matching* in G, while a matching of maximum cardinality is a maximum matching in G [19]. A *vertex cover* of a graph G can also more simply be thought of as a set S of vertices of G such that every edge of G has at least one of member of S as an endpoint. The vertex set of a graph is therefore always a vertex cover. The smallest possible vertex cover for a given graph G is known as a minimum vertex cover, and its size is called the *vertex cover number*, denoted by $\tau(G)$. An *edge cover* is a subset of edges defined similarly to the vertex cover, namely a collection of graph edges such that the union of edge endpoints corresponds to the entire vertex set of the graph. Therefore, only graphs with no isolated points have an edge cover. An edge cover having the smallest possible number of edges for a given graph is known as a *edge cover number*, denoted by $\tau'(G)$ [18]. The smallest integer not less than x is denoted by $\lceil x \rceil$. A set $S \subseteq V(G)$ is a *dominating set* if every vertex in $V(G) - S$ is adjacent to at least one vertex in S. The minimum cardinality taken over all dominating sets of G is called the *domination number* of G is denoted by $\gamma(G)$ [6, 19].

There are different application of domination problems. For instance, dominating sets in graphs are natural models for facility location problems in operations research [19] or domination number is the one of the most important vulnerability parameter for networks [19, 23]. When investigating the domination number of a given graph G, one may want to know the answer of the following question: How does the domination number increases in a graph G? or How many edges need to be added to decrease the domination number of the original graph? One of the vulnerability parameters known as *bondage number* in a graph G answers the former question. The bondage number $b(G)$ was introduced by Fink et al. [10] and is defined as follows:

$$b(G) = min\{|B| : B \subseteq E(G), \gamma(G - B) > \gamma(G)\}.$$

We call such an edge set B that $\gamma(G-B) > \gamma(G)$ the *bondage set* and the minimum one the *minimum bondage set*. If $E(G) = \emptyset$, then we say that $b(G) = \infty$.

The *reinforcement number* in a graph G answers the latter question. The reinforcement number $r(G)$ was introduced by Kok et al.[12] and is defined as follows:

$$r(G) = min\{|R| : R \subseteq E(\overline{G}), \gamma(G) > \gamma(G + R)\}.$$

We call such an edge set $R \subseteq E(\overline{G})$ a reinforcement set, if $\gamma(G) > \gamma(G + R)$.

In 2009, Dankelmann introduced the concept of *exponential domination*[17]. This new parameter is closely in relation with distance of each pair of vertices. The exponential domination number is the theoretical vulnerability parameters for a network that is represented by a graph. An exponential dominating set of graph G is subset $S \subseteq V(G)$ such that $\sum_{v \in S}(1/2)^{\overline{d}(u,v)-1} \geq 1, \forall v \in V(G)$, where $\overline{d}(u, v)$ is the length of a shortest path in $\langle V(G) - (S - \{u\})\rangle$ if such a path exist, and ∞ otherwise. The minimum exponential domination number, $\gamma_e(G)$ is the smallest cardinality of an exponential dominating set. We call such an edge set is a minimum exponential set or a γ_e-set.

Our aim in this paper is to define a new vulnerability parameter, so called exponential bondage number. In Section 2, some well-known basic results are given for exponential domination number and bondage number. In Section 3, we define a new parameter namely the exponential bondage number denoted by $b_{exp}(G)$. In Section 4, we determine upper bounds, lower bounds and exact solutions of the exponential bondage number for any graph G. Finally, the exponential bondage numbers of the popular well-known graphs are computed in Section 5.

2. Basic Results

In this section some well-known basic results are given with regard to exponential domination number and bondage number.

Theorem 2.1. [17] *The exponential domination number of*

(a) *the path graph P_n of order $n \geq 2$ is $\gamma_e(P_n) = \lceil \frac{n+1}{4} \rceil$.*

(b) *the cycle graph C_n of order $n \geq 4$ is* $\gamma_e(C_n) = \begin{cases} 2 & \text{, if } n = 4; \\ \lceil \frac{n}{4} \rceil & \text{, if } n \neq 4. \end{cases}$

Theorem 2.2. [17] *For every graph G, $\gamma_e(G) \leq \gamma(G)$, and also $\gamma_e(G) = 1$ if and only if $\gamma(G) = 1$.*

Theorem 2.3. [1] *Let G be any connected graph with n vertices and $\exists v \in V(G)$ such that $deg(v) = n-1$. Then, $\gamma_e(G) = 1$*

Theorem 2.4. [10] *If G is a connected graph of order $n \geq 2$, then $b(G) \leq n - \gamma(G) + 1$.*

Theorem 2.5. [10] *The bondage number of*

(a) *the path graph P_n of order $n \geq 2$ is* $b(P_n) = \begin{cases} 2 & \text{, if } n \equiv 1 (mod\ 3); \\ 1 & \text{, otherwise.} \end{cases}$

(b) *the cycle graph C_n of order $n \geq 3$ is* $b(C_n) = \begin{cases} 3 & \text{, if } n \equiv 1 (mod\ 3); \\ 2 & \text{, otherwise.} \end{cases}$

(c) *the complete graph K_n of order $n \geq 2$ is $b(K_n) = \lceil \frac{n}{2} \rceil$.*

(d) *the star graph S_n of order $n \geq 3$ is $b(S_n) = 1$.*

Theorem 2.6. [22] *If G is a nonempty graph with a unique minimum dominating set, then $b(G) = 1$.*

Theorem 2.7. [11] *Let G be a vertex-transitive graph. Then, $b(G) \geq \lceil \frac{n}{2\gamma(G)} \rceil$.*

3. The Exponential Bondage Number

In this section, we introduce a new graph theoretical parameter: the *exponential bondage number* and it is defined as:

$$b_{exp}(G) = min\{|B_e| : B_e \subseteq E(G), \gamma_e(G - B_e) > \gamma_e(G)\},$$

where $\gamma_e(G)$ is the exponential domination number of the graph G. We call such an edge set B_e that $\gamma_e(G - B_e) > \gamma_e(G)$ the *exponential bondage set* and the minimum one the *minimum exponential bondage set.*

If we think of a graph as a modeling of network, the exponential bondage number may be more sensitive than other measures of vulnerability such as connectivity, domination number, exponential domination number and bondage number for distinguish two graphs whose number the vertices and edges are the same. For example, consider two graphs G_1 and G_2 in Figure1, where $|V(G_1)| = |V(G_2)| = 7$ and $|E(G_1)| = |E(G_2)| = 8$. They have not only equal connectivity but also equal domination number, exponential domination number and bondage number such as $k(G_1) = k(G_2) = 1$, $\gamma(G_1) = \gamma(G_2) = 2$, $\gamma_e(G_1) = \gamma_e(G_2) = 2$ and $b(G_1) = b(G_2) = 1$. These values could be easily checked by readers. So, how can we distinguish between the graphs G_1 and G_2?

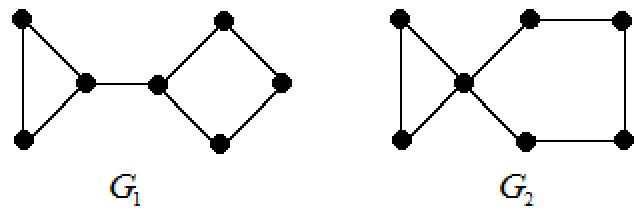

FIGURE 1. The graphs G_1 and G_2

When the exponential bondage numbers of these two graphs G_1 and G_2 are computed, $b_{exp}(G_1) = 1$ and $b_{exp}(G_2) = 2$ are obtained. The results could be checked by readers. Thus, the exponential bondage number may be used for distinguish between these two graphs G_1 and G_2.

4. Upper Bounds, Lower Bounds and Exact Formulas

Theorem 4.1. *If $\gamma(G) = \gamma_e(G)$, then $b_{exp}(G) \geq b(G)$.*

Proof. Let B_e be a minimum exponential bondage set of graph G. Then $\gamma(G) = \gamma_e(G) \leq \gamma_e(G - B_e) \leq \gamma(G - B_e)$ by Theorem 2.2. Thus, we have $b_{exp}(G) \geq b(G)$.
The proof is completed. □

Theorem 4.2. *If G is a connected graph, then $b_{exp}(G) \leq \delta(G)$.*

Proof. To exponentially dominate all the vertices as the minimum number, first of all the vertex u with maximum degree should be taken to γ_e-set. That is, the vertex u belongs to a minimum exponential dominating set. Let D be γ_e-set for G. We know that $u \in D$. Suppose that the graph G has a vertex v with minimum degree, $deg(v) = \delta(G)$, that does not belong to D. Let E_v denote the set of edges incident with v. The minimum exponential dominating set for $G - E_v$ is $D + \{v\}$. Hence $\gamma_e(G - E_v) > \gamma_e(G)$ is obtained. As a result, we have $b_{exp}(G) \leq |E_v| = deg(v) \leq \delta(G)$.
The proof is completed. □

Theorem 4.3. *If G is a vertex-transitive graph and $\gamma(G) = \gamma_e(G)$, then*

$$b_{exp}(G) \geq 2\lceil \frac{n}{2\gamma_e(G)} \rceil + \gamma_e(G) - (n + 1).$$

Proof. By Theorems 2.4 and 4.1, we have $b_{exp}(G) \geq 2b(G) + \gamma_e(G) - (n + 1)$. By using Theorem 2.7, it is trivial that

$$b_{exp}(G) \geq 2\lceil \frac{n}{2\gamma_e(G)} \rceil + \gamma_e(G) - (n + 1).$$

The proof is completed. □

Theorem 4.4. *Let G be a connected graph of order n. If G includes only one pendant vertex, then $b_{exp}(G) = 1$.*

Proof. Let v be the pendant vertex of G. The removal of an edge e which is incident to v from the graph G leaves a graph $G' \cong G - \{e\}$ consisting of two components. One of these is an isolated vertex and the other is connected graph H with $n - 1$ vertices. That is, $G' = H \cup \{v\}$. Let D be γ_e-set for G. Since the set D must include the support vertex u which is adjacent to the vertex v, it is easy to see that $\gamma_e(H) = \gamma_e(G)$. Hence the γ_e-set for G' is $D \cup \{v\}$. Consequently, since $\gamma_e(G') > \gamma_e(G)$ is obtained, the minimum exponential bondage set of G has an edge which is incident to vertex v. Thus, we have $b_{exp}(G) = 1$.
The proof is completed. □

5. The Exponential Bondage Number of Some Well-Known Graphs

In this section we calculate the exponential bondage number of some well known graphs such as the path graph P_n, the cycle graph C_n, the complete graph K_n, the star graph S_n and the wheel graph W_n. Furthermore, we give some corollaries for any graph G.

Theorem 5.1. *The exponential bondage number of the path graph P_n of order $(n \geq 2)$ is given by $b_{exp}(P_n) = 1$.*

Proof. While we are calculating the exponential bondage number of the path graph P_n, we have four cases according to the number of vertices of P_n.

Case 1. $n \equiv 0(mod\ 4)$.

We know that $\gamma_e(P_n) = \lceil \frac{n+1}{4} \rceil$ by Theorem 2.1. The removal of an edge from P_n leaves a graph H consisting of two paths P_{n_1} and P_{n_2}, where $n_1+n_2 = n$. Then, one of the following statements is satisfied.

- $n_1 \equiv 1(mod\ 4)$ and $n_2 \equiv 3(mod\ 4)$
- $n_1 \equiv n_2 \equiv 2(mod\ 4)$
- $n_1 \equiv n_2 \equiv 0(mod\ 4)$

In the first case,

$$\gamma_e(H) = \gamma_e(P_{n_1}) + \gamma_e(P_{n_2}) = \lceil \tfrac{n_1+1}{4} \rceil + \lceil \tfrac{n_2+1}{4} \rceil$$
$$= \tfrac{n_1+3}{4} + \tfrac{n_2+1}{4} = \tfrac{n+4}{4} = \lceil \tfrac{n+1}{4} \rceil = \gamma_e(P_n).$$

In the second case,

$$\gamma_e(H) = \gamma_e(P_{n_1}) + \gamma_e(P_{n_2}) = \lceil \tfrac{n_1+1}{4} \rceil + \lceil \tfrac{n_2+1}{4} \rceil$$
$$= \tfrac{n_1+2}{4} + \tfrac{n_2+2}{4} = \tfrac{n+4}{4} = \lceil \tfrac{n+1}{4} \rceil = \gamma_e(P_n).$$

In the last case,

$$\gamma_e(H) = \gamma_e(P_{n_1}) + \gamma_e(P_{n_2}) = \lceil \tfrac{n_1+1}{4} \rceil + \lceil \tfrac{n_2+1}{4} \rceil$$
$$= \tfrac{n_1+4}{4} + \tfrac{n_2+4}{4} = \tfrac{n+8}{4} = \tfrac{n+4}{4} + 1$$
$$= \lceil \tfrac{n+1}{4} \rceil + 1 = \gamma_e(P_n) + 1 > \gamma_e(P_n).$$

So, if the graph H is obtained by removing exactly one edge from P_n such that $n_1 \equiv n_2 \equiv 0(mod\ 4)$, then we get $b_{exp}(P_n) = 1$.

Case 2. $n \equiv 1(mod\ 4)$.

Let H be a graph obtained by the deletion of an edge of P_n. Then, we get either $n_1 \equiv 1(mod\ 4)$ and $n_2 \equiv 0(mod\ 4)$, or $n_1 \equiv 2(mod\ 4)$ and $n_2 \equiv 3(mod\ 4)$.

In the former case,

$$\gamma_e(H) = \gamma_e(P_{n_1}) + \gamma_e(P_{n_2}) = \lceil \tfrac{n_1+1}{4} \rceil + \lceil \tfrac{n_2+1}{4} \rceil$$
$$= \tfrac{n_1+3}{4} + \tfrac{n_2+4}{4} = \tfrac{n+7}{4} = \tfrac{n+3}{4} + 1$$
$$= \lceil \tfrac{n+1}{4} \rceil + 1 = \gamma_e(P_n) + 1 > \gamma_e(P_n).$$

In the latter case,

$$\gamma_e(H) = \gamma_e(P_{n_1}) + \gamma_e(P_{n_2}) = \lceil \tfrac{n_1+1}{4} \rceil + \lceil \tfrac{n_2+1}{4} \rceil$$

$= \frac{n_1+2}{4} + \frac{n_2+1}{4} = \frac{n+3}{4} = \lceil \frac{n+1}{4} \rceil = \gamma_e(P_n)$.

If the graph H is obtained by removing one edge from P_n consists of an isolated vertex and a path of order $n-1$, then we get $\gamma_e(H) > \gamma_e(P_n)$. Consequently, we have $b_{exp}(P_n) = 1$.

Case 3. $n \equiv 2(mod\ 4)$.

Let H be a graph obtained by removing an edge from P_n. Then, one of the following statements is satisfied.

- $n_1 \equiv n_2 \equiv 1(mod\ 4)$
- $n_1 \equiv n_2 \equiv 3(mod\ 4)$
- $n_1 \equiv 2(mod\ 4)$ and $n_2 \equiv 0(mod\ 4)$

In the first case,

$\gamma_e(H) = \gamma_e(P_{n_1}) + \gamma_e(P_{n_2}) = \lceil \frac{n_1+1}{4} \rceil + \lceil \frac{n_2+1}{4} \rceil$

$= \frac{n_1+3}{4} + \frac{n_2+3}{4} = \frac{n+6}{4} = \lceil \frac{n+2}{4} \rceil + 1 = \gamma_e(P_n) + 1 > \gamma_e(P_n)$.

In the second case, it is easy to see that $\gamma_e(H) = \gamma_e(P_n)$.

In the last case,

$\gamma_e(H) = \gamma_e(P_{n_1}) + \gamma_e(P_{n_2}) = \lceil \frac{n_1+1}{4} \rceil + \lceil \frac{n_2+1}{4} \rceil$

$= \frac{n_1+2}{4} + \frac{n_2+4}{4} = \frac{n+6}{4} = \frac{n+2}{4} + 1$

$= \lceil \frac{n+1}{4} \rceil + 1 = \gamma_e(P_n) + 1 > \gamma_e(P_n)$.

If an edge is removed from P_n such that either $n_1 \equiv n_2 \equiv 1(mod\ 4)$ or, $n_1 \equiv 2(mod\ 4)$ and $n_2 \equiv 0(mod\ 4)$, then $\gamma_e(H) > \gamma_e(P_n)$. Hence, we get $b_{exp}(P_n) = 1$.

Case 4. $n \equiv 3(mod\ 4)$.

Removing an edge from P_n, we have either $n_1 \equiv 1(mod\ 4)$ and $n_2 \equiv 2(mod\ 4)$, or $n_1 \equiv 0(mod\ 4)$ and $n_2 \equiv 3(mod\ 4)$.

In the former case,

$\gamma_e(H) = \gamma_e(P_{n_1}) + \gamma_e(P_{n_2}) = \lceil \frac{n_1+1}{4} \rceil + \lceil \frac{n_2+1}{4} \rceil$

$= \frac{n_1+3}{4} + \frac{n_2+2}{4} = \frac{n+5}{4} = \frac{n+1}{4} + 1$

$= \lceil \frac{n+1}{4} \rceil + 1 = \gamma_e(P_n) + 1 > \gamma_e(P_n)$.

In the latter case,

$\gamma_e(H) = \gamma_e(P_{n_1}) + \gamma_e(P_{n_2}) = \lceil \frac{n_1+1}{4} \rceil + \lceil \frac{n_2+1}{4} \rceil$

$= \frac{n_1+4}{4} + \frac{n_2+1}{4} = \frac{n+5}{4} = \lceil \frac{n+1}{4} \rceil + 1$

$= \lceil \frac{n+1}{4} \rceil + 1 = \gamma_e(P_n) + 1 > \gamma_e(P_n)$.

By the above both cases, we have $\gamma_e(H) > \gamma_e(P_n)$. Thus, $b_{exp}(P_n) = 1$ is obtained.

By Cases 1, 2, 3 and 4, the exponential bondage number of the path graph of order n is $b_{exp}(P_n) = 1$.

The proof is completed. □

Theorem 5.2. *The exponential bondage number of the cycle graph C_n of order $(n \geq 4)$ is given by*

$$b_{exp}(C_n) = \begin{cases} 1 & , \text{if } n \equiv 0(mod\ 4); \\ 2 & , \text{otherwise.} \end{cases}$$

Proof. We know that $\gamma_e(C_n) = \lceil \frac{n}{4} \rceil$ for $n \neq 4$ by Theorem 2.1. We have four cases depending on n.

Case 1. $n \equiv 0(mod\ 4)$.

The removal of one edge from the graph C_n leaves a graph H consisting of a path of order n. Thus,

$$\gamma_e(H) = \gamma_e(P_n)$$

$$= \tfrac{n+4}{4} = \tfrac{n}{4} + 1 = \gamma_e(C_n) + 1 > \gamma_e(C_n).$$

Since $\gamma_e(H) > \gamma_e(C_n)$ is obtained, we get $b_{exp}(C_n) = 1$.

Case 2. $n \equiv 1(mod\ 4)$.

If the graph H is obtained by removing an edge from C_n, then the remaining graph is the path graph with n vertices. Since $\gamma_e(H) = \gamma_e(P_n) = \frac{n+3}{4} = \gamma_e(C_n)$ for $n > 4$, we have $b_{exp}(C_n) \geq 2$.

If $n \equiv 1(mod\ 4)$, the removal of two edges from C_n, leaves a graph H consisting of two paths P_{n_1} and P_{n_2}, where $n_1 + n_2 = n$.

By using the Case 2 of the proof of Theorem 5.1, we obtain $n_1 \equiv 1(mod\ 4)$ and $n_2 \equiv 0(mod\ 4)$. Thus,

$$\gamma_e(H) = \gamma_e(P_{n_1}) + \gamma_e(P_{n_2}) = \tfrac{n_1+3}{4} + \tfrac{n_2+4}{4}$$

$$= \tfrac{n+7}{4} = \tfrac{n+3}{4} + 1 = \gamma_e(C_n) + 1 > \gamma_e(C_n).$$

Since $\gamma_e(H) > \gamma_e(C_n)$, we have $b_{exp}(C_n) = 2$.

Case 3. $n \equiv 2(mod\ 4)$.

This case is very similar to the Case 2. By using the Case 3 of the proof of Theorem 5.1, we obtain directly either $n_1 \equiv n_2 \equiv 1(mod\ 4)$ or, $n_1 \equiv 2(mod\ 4)$ and $n_2 \equiv 0(mod\ 4)$.
In the former case,

$$\gamma_e(H) = \gamma_e(P_{n_1}) + \gamma_e(P_{n_2}) = \tfrac{n_1+3}{4} + \tfrac{n_2+3}{4}$$

$$= \tfrac{n+6}{4} = \tfrac{n+2}{4} + 1 = \gamma_e(C_n) + 1 > \gamma_e(C_n).$$

In the latter case,

$$\gamma_e(H) = \gamma_e(P_{n_1}) + \gamma_e(P_{n_2}) = \tfrac{n_1+2}{4} + \tfrac{n_2+4}{4}$$

$$= \tfrac{n+6}{4} = \tfrac{n+2}{4} + 1 = \gamma_e(C_n) + 1 > \gamma_e(C_n).$$

Since $\gamma_e(H) > \gamma_e(C_n)$, we have $b_{exp}(C_n) = 2$.

Case 4. $n \equiv 3(mod\ 4)$.

This case is also very similar to the Case 2. By using the Case 4 of the proof of Theorem 5.1, it is easy to see that we have either $n_1 \equiv 1(mod\ 4)$ and $n_2 \equiv 2(mod\ 4)$ or, $n_1 \equiv 0(mod\ 4)$ and $n_2 \equiv 3(mod\ 4)$.
In the former case,

$$\gamma_e(H) = \gamma_e(P_{n_1}) + \gamma_e(P_{n_2}) = \tfrac{n_1+3}{4} + \tfrac{n_2+2}{4}$$

$$= \tfrac{n+5}{4} = \tfrac{n+1}{4} + 1 = \gamma_e(C_n) + 1 > \gamma_e(C_n).$$

In the latter case,

$$\gamma_e(H) = \gamma_e(P_{n_1}) + \gamma_e(P_{n_2}) = \frac{n_1+4}{4} + \frac{n_2+1}{4}$$

$$= \frac{n+5}{4} = \frac{n+1}{4} + 1 = \gamma_e(C_n) + 1 > \gamma_e(C_n).$$

Since $\gamma_e(H) > \gamma_e(C_n)$, we have $b_{exp}(C_n) = 2$.

Consequently, By Cases 1, 2, 3 and 4, the proof is completed. □

From Theorem 5.2, we know that the exponential bondage number of $n - cycle$ is $b_{exp}(C_n) = 1$ if $n \equiv 0(mod4)$ and $b_{exp}(C_n) = 2$ otherwise. So, we get following corollaries.

Corollary 5.3. *If G is hamiltonian with $n > 4$ vertices and $\gamma_e(G) = \lceil \frac{n}{4} \rceil$, then $b_{exp}(G) \geq 2$ and in addition $b_{exp}(G) \geq 1$ if $n \equiv 0(mod4)$.*

Corollary 5.4. *If G is semihamiltonian with $n > 4$ vertices and $\gamma_e(G) = \lceil \frac{n}{4} \rceil$ and $0 < n \equiv 0(mod4)$, then $b_{exp}(G) \geq 2$.*

Theorem 5.5. *The exponential bondage number of the star graph S_n of order $(n \geq 3)$ is $b_{exp}(S_n) = 1$.*

Proof. We know that $\gamma_e(S_n) = 1$ by Theorem 2.2. The removal of an edge from the graph S_n, leaves a graph H consisting of an isolated vertex and star graph order $n - 1$. Thus,

$$\gamma_e(H) = 1 + \gamma_e(S_{n-1}) = 2 > \gamma_e(S_n).$$

Since $\gamma_e(H) > \gamma_e(S_n)$ is obtained, the exponential bondage number of the star graph is $b_{exp}(S_n) = 1$.
The proof is completed. □

Theorem 5.6. *The exponential bondage number of the complete graph of order n is $b_{exp}(K_n) = \lceil \frac{n}{2} \rceil$.*

Proof. Clearly, $\gamma_e(K_n) = 1$ by Theorem 2.2. If any B_e-set F does not cover some vertex u, then $\{u\}$ is a γ_e-set of $K_n - F$, so $\gamma_e(K_n - F) = \gamma_e(K_n)$, a contradiction. Hence F must cover every vertex, which implies $|F| \geq \lceil \frac{n}{2} \rceil$. Now define a specific F as follows. If n is even, let F be a matching of size $\frac{n}{2}$. If n is odd, let F consists of a matching of size $\frac{n-1}{2}$ plus one vertex. So $|F| = \lceil \frac{n}{2} \rceil$. Let $H = K_n - F$. Then $\gamma_e(H) \geq 2$ since F covers every vertex. So, F is a B_e-set. Consequently, the exponential bondage number of complete graph is $b_{exp}(K_n) = \lceil \frac{n}{2} \rceil$.
The proof is completed. □

As an immediate corollaries to Theorem 5.6, we have the following.

Corollary 5.7. *If K_n is a complete graph of order n, then $b_{exp}(K_n) = \tau'(K_n)$.*

Corollary 5.8. *If G is a connected graph of order n, then $b_{exp}(G) \leq \lceil \frac{n}{2} \rceil$.*

Theorem 5.9. *The exponential bondage number of the wheel graph of order n is $b_{exp}(W_n) = 1$.*

Proof. Let c be the central vertex of W_n. Since $deg(c) = n - 1$, we have $\gamma_e(W_n) = 1$ by Theorem 2.2. The removal of any edge which is incident to the vertex c in the graph W_n leaves a graph H. In the graph H, clearly $deg(c) = n - 2$. Now we determine the exponential domination number of H. Let D be a γ_e-set of the graph H. If $D = \{c\}$, then D exponentially dominates $n-1$ vertices. Thus, there remains only one vertex does not exponentially dominated by D. Clearly, this vertex also must be in D. Then, we get $\gamma_e(H) = 2$.

Since $\gamma_e(H) > \gamma_e(W_n)$, the exponential bondage number of the wheel graph is $b_{exp}(W_n) = 1$.
The proof is completed. □

6. Conclusion

In this study, a new graph theoretical parameter namely the exponential bondage number has been presented for the network vulnerability. The stability of popular interconnection networks has been studied and their exponential bondage numbers have been computed. These networks have been modeled with the complete graphs, the path graphs, the cycle graphs, the star graphs and the wheel graphs. Then upper bounds, lower bounds and exact formulas of the exponential bondage number have been obtained for any given graph G. As a further study, exact formulas or bounds may be obtained for graph operations and trees.

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