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NON-NULL CURVES OF TZITZEICA TYPE IN MINKOWSKI 3-SPACE

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Abstract. In this paper, we study non-null curves of Tzitzeica type in Minkowski 3-space $\mathbb{E}_1^3$. We find a simple link between Tzitzeica curves and Rectifying curves in $\mathbb{E}_1^3$. Next, we derive certain results for non-null general helices and pseudospherical curves to satisfy Tzitzeica condition in $\mathbb{E}_1^3$. Further, we interest Tzitzeica pseudospherical indicatrices of a spacelike curve in $\mathbb{E}_1^3$.

Keywords. Tzitzeica curve, Rectifying curve, General helix, Pseudosphere, Minkowski space.

AMS Subject Classification. 53B30, 53C50.

1. Introduction

Gheorghe Tzitzeica who is a Romanian mathematician (1873-1939) introduced a class of curves, nowadays called Tzitzeica curves and a class of surfaces of the Euclidean 3-space, called Tzitzeica surfaces. A Tzitzeica curve is a curve for which the ratio of its torsion and the square of the distance d_1 from the origin to the osculating plane at arbitrary point of the curve is constant, i.e.,

$$\frac{\tau}{d_1^2} = c_1, \tag{1.1}$$

where c_1 is nonzero constant. In [5], the connections between Tzitzeica curves and surfaces in Minkowski 3-space and the original ones from the Euclidian 3-space were given. The author, in [9], determined the elliptic and hyperbolic cylindrical curves satisfying Tzitzeica condition in Euclidian 3-space. Morever, the elliptic cylindrical curves verifying Tzitzeica condition were adapted to Minkowski 3-space in [14]. A necessary and sufficient condition was also found, in [3], for a space curve to be a Tzitzeica one.

On the other side, a Tzitzeica surface is a spatial surface for which the ratio of its

Gaussian curvature and the distance d_2 from the origin to the tangent plane at any arbitrary point of the surface is constant, namely; $K/d_2^4 = c_2$ for a constant c_2. This class of surface is of great interest, having important applications both in mathematics and in physics (see [19]). The relation between Tzitzeica curves and surfaces is the following: For a Tzitzeica surface with negative Gaussian curvature, the asimptotic lines are Tzitzeica curves [9]. It was given that a necessary and sufficient condition, in [19], for Cobb-Douglas production hypersurface to be a Tzitzeica hypersurface. In addition, a new Tzitzeica hypersurface was obtained in parametric, implicit and explicit forms in [8].

In this paper, we are interested in the curves of Tzitzeica type, more precisely we investigate the conditions for non-null general helices, pseudospherical curves and pseudospherical general helices to be of Tzitzeica type in Minkowski space $\mathbb{E}_1^3$. Next, we derive some characterizations about Tzitzeica tangent and binormal indicatrices of a spacelike curve in $\mathbb{E}_1^3$.

2. Preliminaries

The Minkowski 3-space $\mathbb{E}_1^3$ is the real vector space $\mathbb{R}^3$ provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}_1^3$. Recall that an arbitrary vector $v \in \mathbb{E}_1^3$ can be spacelike if $g(v,v) > 0$ or $v = 0$, timelike if $g(v,v) < 0$ and null (lightlike) if $g(v,v) = 0$ and $v \neq 0$ [15,17]. The norm of a vector v is given $\|v\| = \sqrt{|g(v,v)|}$ and two vectors v and w are said to be orthogonal, if $g(v,w) = 0$. An arbitrary curve $\alpha(s)$ in $\mathbb{E}_1^3$, can locally be spacelike, timelike or null (lightlike), if all its velocity vectors $\alpha'(s)$ are spacelike, timelike or null, respectively. A spacelike or timelike curve $\alpha(s)$ has unit speed, if $g(\alpha'(s), \alpha'(s)) = \pm 1$ [10,11,12] .

Now let $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ be two vectors in $\mathbb{E}_1^3$, then the Minkowski cross product $v \times_1 w$ is defined by the formula ([5])

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$$v \times_1 w = \begin{vmatrix} -\vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the moving Frenet frame along a curve α in $\mathbb{E}_1^3$, consisting of the tangent, principal normal and binormal vector field, respectively. If α is a non-null curve in $\mathbb{E}_1^3$, the Frenet equations are of the form ([1]):

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\varepsilon_1\varepsilon_2\kappa & 0 & \tau \\ 0 & \varepsilon_1\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}, \tag{2.1}$$

where the derivative with respect to the arc length s is denoted by a prime ($'$) and $\varepsilon_1 = g(\mathbf{T}, \mathbf{T}) = \pm 1$, $\varepsilon_2 = g(\mathbf{N}, \mathbf{N}) = \pm 1$, $g(\mathbf{B}, \mathbf{B}) = -\varepsilon_1\varepsilon_2$, respectively. For this moving Frenet frame, we write ([4])

$$\mathbf{T} \times_1 \mathbf{N} = \varepsilon_1\varepsilon_2\mathbf{B}, \quad \mathbf{N} \times_1 \mathbf{B} = -\varepsilon_1\mathbf{T}, \quad \mathbf{B} \times_1 \mathbf{T} = -\varepsilon_2\mathbf{N}. \tag{2.2}$$

We also recall from [12] that the pseudosphere of radius 1 and center at the origin is the hyperquadric in $\mathbb{E}_1^3$ defined by

$$\mathbb{S}_1^2(1) = \{v \in \mathbb{E}_1^3 \ : \ g(v, v) = 1\}, \tag{2.3}$$

the pseudohyperbolic space of radius 1 and center at the origin is the hyperquadric in $\mathbb{E}_1^3$ defined by

$$\mathbb{H}_0^2(1) = \{v \in \mathbb{E}_1^3 \ : \ g(v, v) = -1\},$$

and the pseudo-Riemannian lightlike cone (quadric cone) defined by

$$\mathbb{C} = \{v \in \mathbb{E}_1^3 \ : \ g(v, v) = 0\}.$$

3. The some curves satisfying Tzitzeica condition

3.1. The rectifying curves satisfying Tzitzeica condition. In three-dimensional Euclidean space $\mathbb{E}^3$, rectifying curves are introduced by B. Y. Chen in [6] as space curves whose position vector always lies in its rectifying plane of the curve. In this sense, the position vector, according to some chosen origin, of a rectifying curve α in $\mathbb{E}^3$

verifies the equation

$$\alpha(s) = \omega(s)\mathbf{T}(s) + \varpi(s)\mathbf{B}(s), \tag{3.1}$$

where ω and ϖ are some differentiable functions with respect to the arclength parameter s. The rectifying curves in a Euclidean space were studied in $[6], [7], [13]$.

We recall some known results on rectifying curves, in Minkowski 3-space, from $[11]$ for later use.

Theorem A. *Let $\alpha = \alpha(s)$ be a unit speed non-null rectifying curve in $\mathbb{E}_1^3$ with spacelike or timelike rectifying plane, the curvature $\kappa(s) > 0$ and $g(\mathbf{T},\mathbf{T}) = \varepsilon_1 = \pm 1$. Then the following statements hold:*

(i) *The distance function $\rho = \|\alpha\|$ satisfies $\rho^2 = |\varepsilon_1 s^2 + c_1 s + c_2|$, for some $c_1 \in \mathbb{R}$, $c_2 \in \mathbb{R}$.*

(ii) *The tangential component of the position vector of α is given by $g(\alpha,\mathbf{T}) = \varepsilon_1 s + c$, where $c \in \mathbb{R}$.*

(iii) *The normal component α^N of the position vector of the curve has a constant length and the distance function ρ is non-constant.*

(iv) *The torsion $\tau \neq 0$ and the binormal component of the position vector of the curve is constant, i.e. $g(\alpha,\mathbf{B})$ is constant.*

Conversely, if $\alpha(s)$ is a unit speed non-null curve in $\mathbb{E}_1^3$, with spacelike or timelike rectifying plane, the curvature $\kappa(s) > 0$, $g(\mathbf{T},\mathbf{T}) = \varepsilon_1 = \pm 1$ and one of the statements (i), (ii), (iii) and (iv) holds, then α is a rectifying curve.

Theorem B. *Let $\alpha = \alpha(s)$ be a unit speed non-null curve in $\mathbb{E}_1^3$, with a spacelike or a timelike rectifying plane and with the curvature $\kappa(s) > 0$. Then up to isometries of $\mathbb{E}_1^3$, the curve α is a rectifying if and only if there holds $\tau(s)/\kappa(s) = c_1 s + c_2$, where $c_1 \in R_0$, $c_2 \in R$.*

Now we give a very simple link between a rectifying curve and a Tzitzeica curve.

Proposition 1. *Let $\alpha : I \to \mathbb{E}_1^3$ be a non-null curve having constant torsion. Then the non-null curve α is of Tzitzeica type if and only if it is a rectifying curve.*

Proof. Let $\alpha : I \to \mathbb{E}_1^3$ be a non-null Tzitzeica curve with constant torsion. Then the distance $d(s)$ between the origin and its osculating plane at arbitrary point of the curve

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α is

$$d(s) = g(\mathbf{B}(s), \alpha(s)) = a_1, \tag{3.2}$$

for each $s \in I$ and nonzero constant a_1. Differentiating of (3.2) with respect to s, we conclude for each $s \in I$

$$g(\mathbf{N}(s), \alpha(s)) = 0$$

which implies the curve α is a rectifying curve.

Conversely, let us assume the curve α satisfies the following

$$\alpha(s) = \omega(s)\mathbf{T}(s) + \varpi(s)\mathbf{B}(s), \tag{3.3}$$

where $\mathbf{T}(s)$ and $\mathbf{B}(s)$ are the tangent and binormal vectors of α, respectively. From the statement (iv) of Theorem A and (3.3), the distance between the origin and the osculating plane at any point of the rectifying curve α is

$$d(s) = g(\mathbf{B}(s), \alpha(s)) = \varpi(s) = a_2, \tag{3.4}$$

for nonzero constant a_2. It follows from the hypothesis and (3.4) that every rectifying curve having constant torsion is a Tzitzeica curve.

3.2. The general helices satisfying Tzitzeica condition. A general helix in Euclidean space $\mathbb{E}^3$ is defined by the property that the tangent makes a constant angle with a constant direction. In $\mathbb{E}^3$, for general helices the Lancret Theorem is as following (see $[2]$ and $[16]$ for details)

Theorem C. (The Lancret theorem in Euclidean space). *A curve in $\mathbb{E}^3$ is a general helix if and only if there exists a constant b such that $\tau = b\kappa$.*

Now we present a condition for a general helix to be a Tzitzeica curve in Minkowski space $\mathbb{E}_1^3$.

Theorem 2. *Let $\alpha : I \to \mathbb{E}_1^3$ be a non-null general helix in $\mathbb{E}^3$. Then α is a Tzitzeica general helix if there exists a vector $\mathbf{X}(s) = 2b_1\varepsilon_1\mathbf{N}(s) - \left(\frac{\kappa'(s)}{\kappa^2(s)}\right)\mathbf{B}(s)$ in $\mathbb{E}_1^3$ such that*

$$g(\alpha(s), \mathbf{X}(s)) = 0$$

for each $s \in I$.

Proof. Since $\alpha : I \to \mathbb{E}_1^3$ is a general helix, we have $\tau = b_1\kappa$ for nonzero constant b_1. Now we can take

$$\frac{\tau(s)}{d^2(s)} = f(s), \tag{3.5}$$

where $f(s)$ is a differentiable function with respect to the arclength parameter s. From (3.5), we get

$$b_1 = \frac{f(s)}{\kappa} d^2(s)$$

and also, by using Frenet formulas (2.1),

$$\begin{aligned} 0 &= \left(\frac{f(s)}{\kappa}\right)' d^2(s) + 2b_1\varepsilon_1 f(s) g(\mathbf{B},\alpha) g(\mathbf{N},\alpha) \\ &= \left(\frac{\frac{df(s)}{ds}}{\kappa} g(\mathbf{B},\alpha) + f(s) g\left(2b_1\varepsilon_1\mathbf{N} - \left(\frac{\frac{d\kappa}{ds}}{\kappa^2}\right)\mathbf{B},\alpha\right)\right) g(\mathbf{B},\alpha). \end{aligned} \tag{3.6}$$

By hypothesis and (3.6), we obtain

$$\frac{df(s)}{ds} = 0,$$

which proves that α is a non-null Tzitzeica general helix.

Arbitrary curve in $\mathbb{E}_1^3$ is called *W-curve*, if all its curvature functions are constant $[10]$. All W-curves in the Minkowski 3-space $\mathbb{E}_1^3$ were completely classified and as example, the only planar spacelike W-curves are circles and hyperbolas (see $[18]$).

Thus we have a result as following.

Corollary 3. *There is no a non-null W- curve, in $\mathbb{E}_1^3$, satisfying Tzitzeica condition.*

Proof. From Theorem B and Proposition 1, the proof is obvious.

3.3. The pseudospherical curves satisfying Tzitzeica condition: Let $\alpha : I \to \mathbb{S}_1^2$ be a unit speed pseudospherical curve. In this subsection, we investigate the links between the pseudospherical curves and the Tzitzeica curves.

Theorem 4. *Let $\alpha : I \to \mathbb{E}_1^3$ be a non-null pseudospherical curve. Then the curve α is of Tzitzeica type provided there exists a nonconstant c_1 such that*

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$$\frac{\tau^3}{\left[\left(-\frac{\varepsilon_1}{\kappa}\right)'\right]^2}=c_1.$$

Proof. Let α be a unit speed pseudospherical curve. Without loss of generality, we take the $\mathbb{S}_1^2$ as a pseudosphere of radius 1 and center at the origin. Then we get

$$g(\alpha(s),\alpha(s))=1.$$

From this, by using Frenet formulas (2.1), we have

$$g(\mathbf{N}(s),\alpha(s))=-\frac{\varepsilon_1}{\kappa}$$

and

$$g(\mathbf{B}(s),\alpha(s))=\left(-\frac{\varepsilon_1}{\kappa}\right)'\frac{1}{\tau}. \tag{3.7}$$

Considering Tzitzeica condition and the hypothesis, we obtain

$$\begin{aligned}\frac{\tau}{d^2(s)}&=\frac{\tau}{\left[\left(-\frac{\varepsilon_1}{\kappa}\right)'\frac{1}{\tau}\right]^2}\\&=\frac{\tau^3}{\left[\left(-\frac{\varepsilon_1}{\kappa}\right)'\right]^2}\\&\Rightarrow\frac{\tau}{d^2(s)}=c_1,\end{aligned}$$

which implies the curve α is a Tzitzeica pseudospherical one.

Remark 5. According to $[16]$, we adapt spherical general helices to Minkowski 3-space, namely a pseudospherical general helix satisfy the following condition

$$\frac{\kappa'}{\kappa^2\sqrt{\kappa^2-1}}=\pm c_2,$$

for nonconstant c_2.

We have immediately the following result from the Theorem 4 and Remark 5,

Corollary 6. *Let* $\alpha : I\to\mathbb{E}_1^3$ *be a non-null pseudospherical general helix satisfying*

$$\frac{\kappa^3}{\kappa^2-1}=c_3,$$

where c_3 *is a nonconstant. Then the curve* α *is a Tzitzeica one.*

Next, we give some results for the pseudospherical indicatrices of a spacelike curve to satisfy Tzitzica condition.

Theorem 7. *Let* $\alpha = \alpha(s)$ *be a spacelike curve with timelike principal normal in* $\mathbb{E}_1^3$. *If* α *has the curvatures in the form*

$$\frac{\kappa(\tau/\kappa)'}{\tau^2} = const.,$$

then its tangent indicatrix is a Tzitzeica curve.

Proof. Let $\gamma = \gamma(s)$ be the tangent indicatrix of the spacelike curve α. Then, by Frenet formulas (2.1), we write

$$\begin{aligned} \frac{d\gamma}{ds} &= \kappa\mathbf{N}, \\ \frac{d^2\gamma}{ds^2} &= (\kappa^2)\mathbf{T} + (\kappa')\mathbf{N} + (\kappa\tau)\mathbf{B}, \\ \frac{d^3\gamma}{ds^3} &= (3\kappa\kappa')\mathbf{T} + (\kappa'' + \kappa^3 + \kappa\tau^2)\mathbf{N} \\ &\quad + (2\kappa'\tau + \kappa\tau')\mathbf{B}, \end{aligned}$$

also we have

$$\frac{d\gamma}{ds} \times_1 \frac{d^2\gamma}{ds^2} = (-\kappa^2\tau)\mathbf{T} + (\kappa^3)\mathbf{B},$$

and

$$g\left(\frac{d\gamma}{ds} \times_1 \frac{d^2\gamma}{ds^2}, \frac{d^3\gamma}{ds^3}\right) = \kappa^5\left(\frac{\tau}{\kappa}\right)'.$$

Denote by τ_γ and d_γ the torsion and distance from the origin to the osculating plane at arbitrary point of the curve γ, respectively. Then we derive

$$\frac{\tau_\gamma}{d_\gamma^2(s)} = \frac{g\left(\frac{d\gamma}{ds} \times_1 \frac{d^2\gamma}{ds^2}, \frac{d^3\gamma}{ds^3}\right)}{g\left(\gamma, \frac{d\gamma}{ds} \times_1 \frac{d^2\gamma}{ds^2}\right)^2} = \frac{\kappa\left(\frac{\tau}{\kappa}\right)'}{\tau^2},$$

which completes the proof.

We have the following result similar with previous theorem without proof.

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Theorem 8. *Let* $\alpha=\alpha(s)$ *be a spacelike curve with timelike principal normal in* $\mathbb{E}_1^3$. *If its curvatures satisfies following condition*

$$\frac{\tau(\tau/\kappa)'}{\kappa^2}=const.,$$

then the binormal indicatrix of α *is a Tzitzeica curve.*

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