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QUOTIENT PSEUDO d-ALGEBRAS

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Abstract. In this study, we introduce the notion of pseudo d^*-algebras and investigate relations between pseudo d-algebras, pseudo BCK-algebras and pseudo d^*-algebras. Furthermore, we describe the notion of a quotient pseudo d-algebra and pseudo d-homomorphism to prove a fundamental theorem of pseudo d-homomorphism as a consequence.

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1. Introduction

Y. Imai and K.Iséki introduced BCK-algebras and BCI-algebras which are two classes of abstract algebras in [1],[2]. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. G.Georgescu and A.lorgulesce introduced the idea of pseudo BCK-algebras as a generalization of the notion of BCK-algebras in [3], and Jun characterised the pseudo BCK-algebras in [4]. J. Neggers and H.S. Kim introduced the notion of d-algebras which is a generalization of BCK-algebras, and they investigated relations between d-algebras and BCK-algebras in [5]. J.Neggers, Y.B. Jun and H.S. Kim introduced pseudo d-algebras as a generalization of d-algebras in [6].

In this paper, we introduce the idea of pseudo d^*-algebras, and then we investigate some relations between pseudo BCK-algebras, pseudo d^*algebras and pseudo d-algebras. Furthermore, we introduce a compatible d^*-system and by favour of it we construct the quotient pseudo d-algebras. Lastly, we introduce pseudo d-homomorphism and we give that fundamental theorems of pseudo d-homomorphism.

2. Preliminaries

Definition 2.1. *A d-algebra* [4] *is a non-empty set* $\mathbb{X}$ *with a constant* 0 *and a binary operetion "$\star$" satisfying the following axioms:*

(I) $x \star x = 0$,
(II) $0 \star x = 0$,
(III) $x \star y = 0$ *and* $y \star x = 0$ *imply* $x = y$, *for all* $x, y \in \mathbb{X}$.

Definition 2.2. *A BCK-algebra* [4] *is a d-algebra* $(\mathbb{X}; \star, 0)$ *satisfying the following additional axioms:*

(IV) $\big((x \star y) \star (x \star z)\big) \star (z \star y) = 0$,
(V) $\big(x \star (x \star y)\big) \star y = 0$ *for all* $x, y, z \in \mathbb{X}$.

Definition 2.3. *A pseudo d-algebra* [6] *is non-empty set* $\mathbb{X}$ *with a constant* 0 *and two binary operations "$\star$" and "$\lhd$" satisfying the following axioms:*

(PI) $x \star x = 0 = x \lhd x$,
(PII) $0 \star x = 0 = 0 \star y$,
(PIII) $x \star y = 0$ *and* $y \lhd x = 0$ *imply* $x = y$ *for all* $x, y \in \mathbb{X}$.

Definition 2.4. *A pseudo BCK-algebra* [3] *is a structure* $(\mathbb{X}; \leqslant, \star, \lhd, 0)$, *where "*$\leqslant$*" is a relation on set* $\mathbb{X}$, *"*$\star$*" and "*$\lhd$*" are two binary operations on* $\mathbb{X}$ *and "*0*" is a element of* $\mathbb{X}$, *satisfying the following axioms:*

(pBCK-I) $(x \star y) \lhd (x \star z) \leqslant z \star y,\ (x \lhd y) \star (x \lhd z) \leqslant z \lhd y$
(pBCK-II) $x \star (x \lhd y) \leqslant y,\ x \lhd (x \star y) \leqslant y$
(pBCK-III) $x \leqslant x$
(pBCK-IV) $0 \leqslant x$
(pBCK-V) $x \leqslant y$ *ve* $y \leqslant x$ *ise* $x = y$
(pBCK-VI) $x \leqslant y \Longleftrightarrow x \star y = 0 \Longleftrightarrow x \lhd y = 0$

Proposition 2.5. *Let* $(\mathbb{X}; \leqslant, \star, \lhd, 0)$ *is pseudo BCK-algebra. Then satisfying following axioms;* (*respectively* [7],[8])

(a_1) $x \star y \leqslant x$ *and* $x \lhd y \leqslant x$,
(a_2) $x \leqslant y$ *and* $y \leqslant z$ *imply* $x \leqslant z$.

Theorem 2.6. $(\mathbb{X}, \leqslant, \star, \lhd, 0)$ *is a pseudo BCK-algebra* [9] *iff axioms (pBCK-I), (pBCK-V), (pBCK-VI) and* $(p1) : x \star (0 \lhd y) = x = x \lhd (0 \star y)$ *hold.*

3. Pseudo d-Algebras

Definition 3.1. *Let* $(\mathbb{X}; \star, \lhd, 0)$ *be pseudo d-algebra. If* $x \star y = 0$ *iff* $x \lhd y = 0$, *then* $\mathbb{X}$ *is said to be strong pseudo d-algebra*

Example 3.2. $\mathbb{X} := \{0, a, b, c\}$ *be a set with following Cayley tables:*

$\star$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	a	0

$\lhd$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	c	0

Then $(\mathbb{X}; \star, \lhd, 0)$ *is a strong pseudo d-algebra.*

Definition 3.3. *A pseudo d-algebra* $(\mathbb{X}; \star, \lhd, 0)$ *is said to be pseudo d-transitive, if* $x \star z = 0$ *and* $z \star y = 0$ *imply* $x \star y = 0$ *and also,* $x \lhd z = 0$ *and* $z \lhd y = 0$ *imply* $x \lhd y = 0$ *for all* $x, y, z \in \mathbb{X}$.

Example 3.4. *Let* $\mathbb{X} := \{0, a, b, c\}$ *be a set with following Cayley tables:*

$\star$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	0	0	b
c	c	c	c	0

$\lhd$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	0
c	c	c	0	0

Then $(\mathbb{X}; \star, \lhd, 0)$ *is pseudo d-algebra and it is pseudo d-transitive.*

Obviously, a pseudo BCK-algebra is a pseudo d-algebra and by Proposition 2.5 (a_2) it is transitive.

Definition 3.5. *A pseudo d-algebra* $(\mathbb{X}; \star, \lhd, 0)$ *is called a pseudo d*-algebra if* $(x \star y) \lhd x = 0$ *and* $(x \lhd y) \star x = 0$, *for all* $x, y \in \mathbb{X}$.

In example 3.4, $(\mathbb{X}; \star, \lhd, 0)$ is a pseudo d^*-algebra. Clearly, by Proposition 2.5 (a_1), a pseudo BCK-algebra is a pseudo d^*-algebra.

Definition 3.6. *Let* $(\mathbb{X}, \star, \lhd, 0)$ *be a d-algebra and* $x \in \mathbb{X}$. *Define*

$$x \star \mathbb{X} := \{x \star a \mid a \in \mathbb{X}\} \quad and \quad x \lhd \mathbb{X} := \{x \lhd a \mid a \in \mathbb{X}\}.$$

$\mathbb{X}$ *is said to be edge pseudo d-algebra if* $x \star \mathbb{X} = x \lhd \mathbb{X} = \{x, 0\}$ *for any* x *in* $\mathbb{X}$.

In example 3.4, $(\mathbb{X}; \star, \lhd, 0)$ is an edge pseudo d-algebra.

Lemma 3.7. *Let* $\mathbb{X}$ *be an edge pseudo d-algebra. Then* $x \star 0 = x = x \lhd 0$ *for any* x *in* $\mathbb{X}$.

Proof. Since $(\mathbb{X}; \star, \lhd, 0)$ is an edge pseudo d-algebra, $x \star 0 = x$ or $x \star 0 = 0$ and $x \lhd 0 = x$ or $x \lhd 0 = 0$ for all x in $\mathbb{X}$. If $x \star 0 = x = x \lhd 0$, then proof is completed. Assume that $x \neq 0$. Then, for the rest of conditions, we have $x = 0$, a condition. □

Theorem 3.8. *Let* $(\mathbb{X}; \star, \lhd, 0)$ *be an arbitrary pseudo d-algebra which is not edge. Define two binary operation* $\oplus_{1,2} : \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{X}$ *by*

$$x \oplus_1 y := \begin{cases} x, & x \star y \neq 0 \\ 0, & otherwise \end{cases} \qquad x \oplus_2 y := \begin{cases} x, & x \lhd y \neq 0 \\ 0, & otherwise. \end{cases}$$

Then $(\mathbb{X}; \oplus_1, \oplus_2, 0)$ *is a edge pseudo d-algebra and it is called the extended edge pseudo d-algebra.*

Proof. We can see easily that $(\mathbb{X}; \oplus_1, \oplus_2, 0)$ is a pseudo d-algebra. Assume that $x \oplus_1 \mathbb{X} = \{0\}$. Then $x \star y = 0$ for all $y \in \mathbb{X}$. Particularly, $x \star 0 = 0 = 0 \star x$, so that $x = 0$. Therefore, if $x \neq 0$, then $x \oplus_1 \mathbb{X} = \{0, x\}$. In a similar way, we can see that $x \oplus_2 \mathbb{X} = \{0, x\}$. □

Proposition 3.9. *A pseudo d-algebra* $(\mathbb{X}; \star, \lhd, 0)$ *is pseudo d-transitive if and only if its extended edge pseudo d-algebra* $(\mathbb{X}; \oplus_1, \oplus_2, 0)$ *is pseudo d-transitive.*

Proof. Let $(\mathbb{X}; \star, \lhd, 0)$ be pseudo d-transitive. If $x \oplus_1 z = 0$ and $z \oplus_2 y = 0$ then $x \star z = 0 = z \star x$, so that $x \star y = 0$, i.e. $x \oplus_1 y = 0$. Similarlay, $x \oplus_2 z = 0 = z \oplus_2 y = 0$ imply $x \oplus_2 y = 0$. Conversely, let $(\mathbb{X}; \oplus_1, \oplus_2, 0)$ be pseudo d-transitive. If $x \star z = 0$ and $z \star y = 0$ then $x \oplus_1 z = 0$ and $z \oplus_2 y = 0$, so that $x \oplus_1 y = 0$, i.e. $x \star y = 0$. Similarly, $x \lhd z = 0 = z \lhd y$ imply $x \lhd z = 0$. □

Lemma 3.10. *Let* $(\mathbb{X}; \star, \lhd, 0)$ *be an edge pseudo d-algebra. Then the condition (p1) holds.*

Proof. From (PIII) and Lemma 3.7, we have $x \star (0 \lhd y) = x \star 0 = x$ and $x \lhd (0 \star y) = x \lhd 0 = x$, for all $x, y \in \mathbb{X}$. □

Theorem 3.11. *Let "$\leqslant$" be a relation on* $\mathbb{X}$ *defined as follows*

$$x \leq y :\Longleftrightarrow x \star y = 0 \Longleftrightarrow x \lhd y = 0$$

and $(\mathbb{X}, \star, \lhd, 0)$ *be a transitive strong edge pseudo d-algebra. Then* $(\mathbb{X}; \star, \lhd, 0)$ *is a pseudo BCK-algebra.*

Proof. In order to $(\mathbb{X}; \star, \lhd, 0)$ is a pseudo BCK-algebra by Theorem 2.6, it is enough to show that conditons (pBCK-I), (pBCK-V) and (pBCK-IV) hold. Suppose that $\big((x \star y) \lhd (x \star z)\big) \star (z \star y) \neq 0$ for some $x, y, z \in \mathbb{X}$. Since $(x \star y) \lhd (x \star z) \in (x \star y) \lhd \mathbb{X} = \{x \star y, 0\}$, we have that

$$(x \star y) \lhd (x \star z) = x \star y. \tag{3.1}$$

If $x \star y = 0$, then $0 \neq \big((x \star y) \lhd (x \star z)\big) \star (z \star y) = \big(0 \lhd (x \star z)\big) \star (z \star y) = 0 \star (z \star y) = 0$, and this is a contradiction. Since $\mathbb{X}$ is an edge pseudo d-algebra and $x \star y \neq 0$,

$$x \star y = x. \tag{3.2}$$

Thus

$$\begin{aligned} x &= x \star y && \text{[by (3.2)]} \\ &= (x \star y) \lhd (x \star x) && \text{[by (3.1)]} \\ &= x \lhd (x \star x) && \text{[by (3.2)]} \end{aligned}$$

that is to say

$$x = x \lhd (x \star z). \tag{3.3}$$

If $x \star z \neq 0$, then $x \star z = x$, whence $\mathbb{X}$ is an edge pseudo d-algebra. Then

$$\begin{aligned} x &= x \lhd (x \star z) && \text{[by (3.3)]} \\ &= x \lhd x && [\, x \star z = x] \\ &= 0. && \text{[by (PI)]} \end{aligned}$$

Hence

$$\begin{aligned} 0 &\neq \big((x \star y) \lhd (x \star z)\big) \star (z \star y) \\ &= (x \lhd x) \star (x \star y) && [\, x \star z = x \text{ and by (3.2)}] \\ &= 0 \star (x \star y) \\ &= 0, \end{aligned}$$

and it is a contradiction. Thus we have that

$$x \star z = 0 \tag{3.4}$$

If $x \star y = 0$, then

$$\begin{aligned} 0 &\neq \big((x \star y) \lhd (x \star z)\big) \star (z \star y) \\ &= \big((x \star y) \lhd 0\big) \star z && [\, z \star y = z \text{ and (3.4)}] \\ &= (x \star z) \star z && \text{[by Lemma 3.7]} \\ &= x \star z && \text{[by (3.2)]} \\ &= 0, && \text{[by (3.4)]} \end{aligned}$$

a contradiction. Thus we obtain $x \star x = 0$ ve $x \star y = 0$, Since $\mathbb{X}$ is transitive, $x \star y = 0$ and hence

$$0 \neq \big(\underbrace{(x \star y)}_{=0} \lhd \underbrace{(x \star z)}_{=0}\big) \star \underbrace{(z \star y)}_{=0} = 0,$$

a contradiction. Similarly, can be seen that $\big((x \lhd y) \star (x \lhd z)\big) \star (z \lhd y) = 0$. This shows that (pBCK-I) is provided. By (PIII) and definition of $\leqslant$, we can see easily that (pBCK-V) are (pBCK-VI) provided. □

Converse of Theorem 3.11 need not be true.

Example 3.12. *Let $\mathbb{X} := \{a, b, c\}$ be a set with following Cayley tables:*

$\star$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
c	c	b	b	0

$\lhd$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
c	c	c	a	0

Then $(\mathbb{X}; \leqslant, \star, \lhd, 0)$ is a pseudo BCK-algebra. Also, $(\mathbb{X}; \leqslant, \star, \lhd, 0)$ is a transitive strong pseduo d-algebra. However it is not an edge pesudo d-algebra, since $c \star a = b \notin \{0, c\}$.

4. Direct Sum, Direct Product

Let $\{(\mathbb{X}_i;\star_i,\lhd_i,0_i)\}_{i\in I}$ be a non-empty family of d-algebras and

$$\prod_{i\in I}\mathbb{X}_i=\{(x_i)_{i\in I}\ \mid\ x_i\in\mathbb{X}_i\}.$$

Also, let the sequences $(0_i)_{i\in I}$ be represented by 0. If we define two binary operation by

$$(x_i)_{i\in I}\star(y_i)_{i\in I}:=(x_i\star_i y_i)_{i\in I}\quad\text{ve}\quad(x_i)_{i\in I}\lhd(y_i)_{i\in I}:=(x_i\lhd_i y_i)_{i\in I}$$

then $\left(\prod_{i\in I}\mathbb{X}_i;\star,\lhd,0\right)$ is a pseudo d-algebra, called *the direct product of the pseudo d-algebras* $\{(\mathbb{X}_i;\star_i,\lhd_i,0_i)\}_{i\in I}$.

Similarly, let consider the set

$$\bigoplus_{i\in I}\mathbb{X}_i=\{(x_i)_{i\in I}\ \mid\ x_i=0_i\in\mathbb{X}_i,\ \text{ except for a finite number of } i\}.$$

The set is closed under "$\star$", "$\lhd$" and $\bigoplus_{i\in I}\mathbb{X}_i\subset\prod_{i\in I}\mathbb{X}_i$, whence $\left(\bigoplus_{i\in I}\mathbb{X}_i;\star,\lhd,0\right)$ is a pseudo d-algebra, called *the direct sum of the pseudo d-algebras* $\{(\mathbb{X}_i;\star_i,\lhd_i,0_i)\}_{i\in I}$.

Let $(\mathbb{X};\star,\lhd,0)$ and $(\mathbb{Y};\star,\lhd,0)$ be pseudo d-algebras. Define two binary operations $\circledast_1$ and $\circledast_2$ on $\mathbb{X}\times\mathbb{Y}$ as follows:

$$(x,y)\circledast_1(a,b)=\begin{cases}(0,0), & x\star a=0=y\star b\\ (x,y), & \text{otherwise}\end{cases}\qquad (x,y)\circledast_2(a,b)=\begin{cases}(0,0), & x\lhd a=0=y\lhd b\\ (x,y), & \text{otherwise.}\end{cases}$$

We can easily see that $(\mathbb{X}\times\mathbb{Y};\circledast_1,\circledast_2,0_{\mathbb{X}\times\mathbb{Y}})$ is an edge pseudo d-algebras, denoted by $\mathbb{X}\circledast_{(1,2)}\mathbb{Y}$, and called *the edge product of pseudo d-algebras* $(\mathbb{X};\star,\lhd,0)$ and $(\mathbb{Y};\star,\lhd,0)$.

Even though $(\mathbb{X};\star,\lhd,0)$ and $(\mathbb{Y};\star,\lhd,0)$ are edge pseudo d-algebras, their direct sum $\mathbb{X}\oplus\mathbb{Y}$ need not have the edge property. For example, let $x\in\mathbb{X},y\in\mathbb{Y}$ and let $x\star a=x$ and $y\star b=0$ for some $a\in\mathbb{X}$ and $b\in\mathbb{Y}$. Then $(x\star y)\star(a\star b)=(x\star a,y\star b)=(x,0)\notin\{(x,y),(0,0)\}$ when $y\neq 0$.

5. Quotient Pseudo d-Algebras

Definition 5.1. *Let $(\mathbb{X};\star,\lhd,0)$ be a pseudo d-algebra and $\emptyset\neq I\subseteq\mathbb{X}$. I is said to be d*-system, if it satisfies the following axioms;*

(s_0) $0\in I$
(s_1) $x\star y\in I$ *and* $y\in I$ *imply* $x\in I$,
(s_2) $x\in I$ *and* $y\in\mathbb{X}$ *imply* $x\star y\in I$,
(s_3) $x\star y\in I$ *and* $y\star z\in I$ *imply* $x\star z\in I$,
(s_4) $x\star y\in I$ *and* $y\star x\in I$ *imply* $(x\star z)\star(y\star z)\in I$ *and* $(z\star x)\star(z\star y)\in I$

Also, d*-system can be described by the operation $\lhd$ instead of $\star$ which is used in axioms $(s_1),(s_2),(s_3)$ and (s_4).

Example 5.2. *Let $\mathbb{X}:=\{0,a,b,c\}$ be a set with following Cayley tables:*

$\star$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	a
c	c	c	a	0

$\lhd$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	0
c	c	c	0	0

Then $(\mathbb{X};\star,\lhd,0)$ is a pseudo d-algebra and the set $I:=\{0,a\}\subseteq\mathbb{X}$ is a d-system.*

Definition 5.3. *Let I be a d^*-system of a pseudo d-algebra $(\mathbb{X};\star,\lhd,0)$. If $x\star y\in I$ iff $x\lhd y\in I$, then I is called compatible.*

In Example 5.2, the set $I := \{0,a\}$ is a compatible d^*-system.

Let I be a compatible d^*-system of a pseudo d-algebra $(\mathbb{X};\star,\lhd,0)$. For any $x,y\in\mathbb{X}$, we define

$$x\theta y \Longleftrightarrow x\star y\in I \text{ and } y\star x\in I.$$

For any $x\in\mathbb{X}$, $x\star x=0\in I$ whence $x\theta x$, i.e. θ is reflexive. If $x\theta y, y\theta z$ then $x\star y, y\star x\in I$ and $y\star z, z\star y\in I$. By (s_3), $x\star z, z\star x\in I$ and hence $x\theta z$. This shows that θ is transitive. The symmetry of θ is trivial. By (s_3) and (s_4) we can easily see that θ is a congruence relation on $\mathbb{X}$.

We denote the congruence class containing x by $[x]_\theta$, i.e. $[x]_\theta=\{y\in\mathbb{X}\mid x\theta y\}$. We can see that $x\theta y$ iff $[x]_\theta=[y]_\theta$. Denote the set of all equivalence classes of $\mathbb{X}$ by $\mathbb{X}/I$, i.e. $\mathbb{X}/I=\{[x]_\theta\mid x\in\mathbb{X}\}$.

Lemma 5.4. *Let I be a pseudo d^*-system of a pseudo d-algebra $(\mathbb{X};\star,\lhd,0)$. Then $I=[0]_\theta$.*

Proof.

$$\begin{aligned}
[0]_\theta &= \{x\in\mathbb{X}\mid x\theta 0\}\\
&= \{x\in\mathbb{X}\mid x\star 0, 0\star x\in I\}\\
&= \{x\in\mathbb{X}\mid x\star 0\in I\} && \big(0\in I\big)\\
&= I && \big((s_1)\big)
\end{aligned}$$

□

Theorem 5.5. *Let $(\mathbb{X};\star,\lhd,0)$ be a pseudo d-algebra and I be a compatible d^*-system of $\mathbb{X}$. We define two operations by*

$$[x]_\theta\star[y]_\theta := [x\star y]_\theta \quad \textit{and} \quad [x]_\theta\lhd[y]_\theta := [x\lhd y]_\theta.$$

Then $(\mathbb{X}/I;\star,\lhd,0)$ be a pseudo d-algebra, called the quotient pseudo d-algebra.

Proof. Assume that $\big([x]_\theta,[y]_\theta\big)=\big([x']_\theta,[y']_\theta\big)$. It follows that $[x]_\theta=[x']_\theta$ and $[y]_\theta=[y']_\theta$ and hence $x\theta x'$ and $y\theta y'$. Since θ is a congruence relation on $\mathbb{X}$, $(x\star y)\theta(x'\star y')$, i.e. $[x\star y]_\theta=[x'\star y']_\theta$. In addition, since I is a compatible d^*-system, $(x\lhd y)\theta(x'\lhd y')$, i.e. $[x\lhd y]_\theta=[x'\lhd y']_\theta$ whence $x\theta x'$ and $y\theta y'$. This means that $[x]_\theta\star[y]_\theta := [x\star y]_\theta$ and $[x]_\theta\lhd[y]_\theta := [x\lhd y]_\theta$ are well-defined.

Let $[x]_\theta\star[y]_\theta=[0]_\theta=[y]_\theta\lhd[x]_\theta$ for $[x]_\theta,[y]_\theta$ in $\mathbb{X}/I$. Then $[x\star y]_\theta=I=[y\lhd x]_\theta$ and hence $x\star y\in I$ and $y\lhd x\in I$. Besides, since I is compatible, $y\star x\in I$. Thus, $x\theta y$ and $[x]_\theta=[y]_\theta$. The rest is trivial. □

6. Pseudo d-Homomorphism

Let $(\mathbb{X};\star,\lhd,0_\mathbb{X})$ and $(\mathbb{Y};\star,\lhd,0_\mathbb{Y})$ be pseudo d-algebras. If

$$f(x\star y)=f(x)\star f(y) \quad\text{and}\quad f(x\lhd y)=f(x)\lhd f(y),$$

for any $x,y\in\mathbb{X}$ then $f:\mathbb{X}\longrightarrow\mathbb{Y}$ is called a pseudo d-homomorphism. Note that $f(0_X)=0_Y$. Indeed, $f(0_\mathbb{X})=f(x\star x)=f(x\lhd x)=f(x)\star f(x)=f(x)\lhd f(x)=0_\mathbb{Y}$.

Given $\mathbb{X}\oplus\mathbb{Y}$ and $\mathbb{X}\circledast_{(1,2)}\mathbb{Y}$, there are inclusion mappings i_X and i_Y, and projections π_X and π_Y. The inclusion mapping i_X and projection π_X relative to $\mathbb{X}\oplus\mathbb{Y}$ are pseudo d-homomorphism: For $(x,y),(a,b)\in\mathbb{X}\oplus\mathbb{Y}$,

$$\pi_X\big((x,y)\star(a,b)\big)=\pi_X(x\star a, y\star b)=(x\star a)=\pi_X(x,y)\star\pi_X(a,b)$$
$$\pi_X\big((x,y)\lhd(a,b)\big)=\pi_X(x\lhd a, y\lhd b)=(x\lhd a)=\pi_X(x,y)\lhd\pi_X(a,b).$$

For $x,a\in\mathbb{X}$,

$$i_X(x\star a)=(x\star a,0)=(x,0)\star(a,0)=i_X(x)\star i_X(a)$$

$$i_X(x \lhd a) = (x \lhd a, 0) = (x, 0) \lhd (a, 0) = i_X(x) \lhd i_X(a)$$

Similarly, we can see easily that π_Y and i_Y relative to $\mathbb{X} \oplus \mathbb{Y}$ are pseudo d-homomorphism.

Proposition 6.1. *Let $(\mathbb{X}; \star, \lhd, 0)$ and $(\mathbb{Y}; \star, \lhd, 0)$ be pseudo d-algebras. Then $\mathbb{X}$ (or $\mathbb{Y}$, respectively) is a pseudo d-algebra iff inclusion mapping i_X (or i_Y, respectively) is a pseudo d-homomorphism.*

Proof. Assume that $i_X : \mathbb{X} \longrightarrow \mathbb{X} \circledast \mathbb{Y}$ is a pseudo d-homomorphism. Then $(x \star a, 0) = i_X(x) \circledast_1 i_X(a) = (x, 0) \circledast_1 (a, 0) = (0, 0)$ or $(x, 0)$ and hence $x \star a = 0$ or $x \star a = x$, for any $a \in \mathbb{X}$. As a result, $x \star \mathbb{X} = \{0, x\}$ for all x in $\mathbb{X}$. Similarly, $x \lhd \mathbb{X} = \{0, x\}$. Conversely, suppose $\mathbb{X}$ is a edge pseudo d-algebra. Consider the $i_X : \mathbb{X} \longrightarrow \mathbb{X} \circledast \mathbb{Y}$. Then $i_X(x \star a) = (x \star a, 0) = (x, 0)$ or $(0, 0)$ whence $x \star a \in \{0, x\}$, and $i_X(x) \circledast_1 i_X(a) = (x, 0) \circledast_1 (a, 0) = (x, 0)$ or $(0, 0)$ from definition of $\circledast_1$. In a similar way, we can see easily that $i_X(x \lhd a) = i_X(x) \circledast_2 i_X(a)$. Thus, i_X is a pseudo d-homomorphism. □

Proposition 6.2. *Let $f : \mathbb{X} \longrightarrow \mathbb{Y}$ be a pseudo d-homomorphism from a $(\mathbb{X}; \star, \lhd, 0)$ into a transitive pseudo d-algebra $(\mathbb{Y}; \star, \lhd, 0)$. Then $Kerf$ is a d*-system of $\mathbb{X}$.*

Proof. It can be easily seen that $(d_0), (d_1)$ and (d_2) are provided. If $x \star y, y \star z \in Kerf$, then $f(x) \star f(y) = 0_{\mathbb{Y}} = f(y) \star f(z)$. Since $\mathbb{Y}$ is pseudo d-transitive, we have $f(x) \star f(z) = 0$, i.e. $x \star z \in Kerf$, and hence (d_3) is provided. If $x \star y, y \star x \in Kerf$, then $f(x) \star f(y) = 0_{\mathbb{Y}} = f(y) \star f(x)$ and hence we obtain $f(x) = f(z)$ by (III). It follows that

$$\begin{aligned} f\big((x \star z) \star (y \star z)\big) &= f(x \star z) \star f(y \star z) \\ &= \big(f(x) \star f(z)\big) \star \big(f(y) \star f(z)\big) \\ &= \big(f(y) \star f(z)\big) \star \big(f(y) \star f(z)\big) && \big(f(x) = f(y)\big) \\ &= 0_{\mathbb{Y}} && \big((\text{I})\big) \end{aligned}$$

and hence $(x \star z) \star (y \star z) \in Kerf$. Similarly, $(z \star x) \star (z \star y) \in Kerf$, which proves (d_4). □

Proposition 6.3. *Let I be a d*-system of the pseudo d-algebra $\mathbb{X}$. Then the mapping $\pi : \mathbb{X} \longrightarrow \mathbb{X}/I$ defined by $\pi(x) = [x]_\theta$ is a pseudo d-homomorphism of $\mathbb{X}$ onto the quotient pseudo d-algebra $\mathbb{X}/I$ and the kernel of π is precisely the set I.*

Proof. We know that $[x \star y]_\theta = [x]_\theta \star [y]_\theta$ and $[x \lhd y]_\theta = [x]_\theta \lhd [y]_\theta$, Then π is a pseudo d-homomorphism. Also, by Lemma 5.4

$$\begin{aligned} Ker\pi &= \{x \in \mathbb{X} \mid \pi(x) = [0]_\theta\} \\ &= \{x \in \mathbb{X} \mid [x]_\theta = [0]_\theta\} \\ &= \{x \in \mathbb{X} \mid x\theta 0\} \\ &= [0]_\theta \\ &= I. \end{aligned}$$

□

Theorem 6.4. *Let $f : \mathbb{X} \longrightarrow \mathbb{Y}$ be a pseudo d-homomorphism from a pseudo d-algebra $\mathbb{X}$ onto a transitive pseudo d-algebra $\mathbb{Y}$. If $Kerf$ is compatible then $\mathbb{X}/Kerf \cong \mathbb{Y}$.*

Proof. Suppose $\phi : \mathbb{X}/Kerf \longrightarrow \mathbb{Y}$ such that $\phi([x]_\theta) = f(x)$. If $[x]_\theta = [y]_\theta$ then $x\theta y$, i.e. $x \star y, y \star x \in Kerf$. Since $Kerf$ is compatible, also we say that $x \lhd y, y \lhd x \in Kerf$. It follows that $f(x \star y) = f(x) \star f(y) = 0_{\mathbb{Y}}$ and $f(y \lhd x) = f(y) \lhd f(x) = 0_{\mathbb{Y}}$, and hence by (PIII) we obtain $f(x) = f(y)$, i.e. $\phi\big([x]_\theta\big) = \phi\big([y]_\theta\big)$, which proves ϕ is well-defined. Since f is onto, there is an x in $\mathbb{X}$ such that $f(x) = y$, for any $y \in \mathbb{Y}$. Thus $\phi\big([x]_\theta\big) = f(x) = y$, which shows that ϕ is onto. If $[x]_\theta \neq [y]_\theta \in \mathbb{X}/Kerf$ then $x \star y \notin Kerf$ or $y \star x \notin Kerf$. Without of generality we may suppose $x \star y \notin Kerf$. Then $f(x \star y) = f(x) \star f(y) \neq 0_{\mathbb{Y}}$ and hence $f(x) \neq f(y)$. This shows that ϕ is one-one. Since $\phi\big([x]_\theta \star [y]_\theta\big) = \phi\big([x \star y]_\theta\big) = f(x \star y) = f(x) \star f(y) = \phi\big([x]_\theta\big) \star \phi\big([y]_\theta\big)$ and similarly, $\phi\big([x]_\theta \lhd [y]_\theta\big) = \phi\big([x]_\theta\big) \lhd \phi\big([y]_\theta\big)$, ϕ is pseudo d-homomorphism. □

References

[1] K.Iséki, *On BCI-algebras*, Math. Seminar Notes 8 (1980) 125-130.
[2] K.Iséki, S.Tanaka, *An introduction to theory of BCK-algebras*, Math. Japonica 23 (1978) 1-26.
[3] G.Georgescu, A.lorgulescu, *Pseudo BCK-algebras: An extension of BCK-algebras*, Proceeding of DMTCS'01: Combinatorics, Computability and Logic (2001) 94-114.
[4] Y.B. Jun, *Characterization of pseudo BCK-algebras*, Scientiae Mathematicae Japonice 57 (2003) 265-270.
[5] J.Neggers, H.S. Kim, *On d-algebras*, Math. Slovaca 49 (1999) 19-26.
[6] Y.B. Jun, H.S. Kim and J. Neggers, *Pseudo d-algebras*, Information Sciences 179 (2009) 1751-1759
[7] G.Dymek, 2011, *Normal pseudo BCK-algebras*, Commentationes Mathematicae, Vol. 51 No. 1 99-107.
[8] L.C. Ciungu, *States on pseudo-BCK algebras*, Mathematical Reports 10(60) 1, (2008) 17-36.
[9] A. Walendziac, 2011, *On axiom systems of pseudo-BCK-algebras*, Bulletin of the Malaysian Mathematical Science Society, (2)(34)(2), (2011) 287-293.

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