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POLYNOMIAL ORTHOGONAL BASES FOR A CLASS OF CLOSED RINGS

ANGEL POPESCU

Abstract

Let (K, v) be a locally compact complete valued field relative to a Krull valuation v, let $\overline{K}$ be a fixed algebraic closure of K and let w be a pseudovaluation on $\overline{K}$ which extends the valuation v. Let $\widetilde{\overline{K}}$ be the completion of $(\overline{K}, w)$ and let x be a transcendental element of $\widetilde{\overline{K}}$, relative to K. Let E be the topological closure of $K[x]$ in $\widetilde{\overline{K}}$. In this note we construct a polynomial orthogonal basis for the K-Banach algebra E.

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Introduction

Let (K, v) be a rank 1 discrete locally compact valued field relative to a Krull valuation v (see[7]). It is very known [2] that in this case K is either a finite extension of $\mathbb{Q}_p$, the p-adic number field, or a finite extension of $\mathbb{F}_p((X))$ for a prime number p. In all these cases the valuation ring O_K is a compact subset of K.

Let $\overline{K}$ be a fixed algebraic closure of K and let w be a pseudovaluation of rank 1 on $\overline{K}$, which extends v. A pseudovaluation w of rank 1 on a field F is a mapping $w : F \to \mathbb{R} \cup \{\infty\}$ such that:

i) $w(a) = \infty$ if and only if $a = 0$

ii) $w(ab) \geq w(a) + w(b)$

iii) $w(a + b) \geq min\{w(a), w(b)\}$, for any a, b in F.

Let $\widetilde{\overline{K}}$ be the completion of $(\overline{K}, w)$. For instance, if $K = \mathbb{Q}_p$, and if w is the unique valuation on $\overline{K}$ which extends the standard p-adic valuation of $\mathbb{Q}$, then $\widetilde{\overline{K}}$ is $\mathbb{C}_p$, the p-adic complex number field. If $K = \mathbb{F}_p((X))$ and v is the X-adic valuation then, if we take for w the unique valuation on $\overline{\mathbb{F}_p((X))}$,

Department of Mathematics, Technical University of Civil Engineering of Bucharest, B=ul Lacul Tei 124, RO-72302, Bucharest 38, ROMANIA.
e-mail address: angel.popescu@gmail.com.

an algebraic closure of K, $\tilde{\overline{K}}$ is a field Ω_p, on which we know almost nothing. A description of $\overline{K}$ in this last case we can find in [5]. If we consider on $\mathbb{Q}$ the usual p-adic valuation and if we fix an extension u of it to $\overline{\mathbb{Q}}$, we can consider the (spectral) pseudovaluation $w(x) = min\{\ u(\sigma(x)) \mid \sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})\ \}$ and denote by $\tilde{\overline{\mathbb{Q}}}_p$ the completion ring of $\overline{\mathbb{Q}}$ relative to w (see[6]). It is clear that $K = \mathbb{Q}_p$ is a subfield in $\tilde{\overline{\mathbb{Q}}}_p$. We can consider E to be the topological closure in $\tilde{\overline{\mathbb{Q}}}_p$ of $K[x]$, where x is a transcendental element of $\tilde{\overline{\mathbb{Q}}}_p$, relative to K. So the main hypothesis above is the locally compact property of $\mathbb{Q}_p$ and not the exact way of construction of $\tilde{\overline{K}}$, starting from K. This means that instead of $\tilde{\overline{K}}$ we can consider any complete K-algebra relative to a rank 1 pseudovaluation which extends v.

In this note we extend the technique used in [1] in order to construct a polynomial orthonormal basis for a subring E of $\tilde{\overline{K}}$, of the form $E = \widetilde{K[x]}$, where this is the topological closure of $K[x]$ in $\tilde{\overline{K}}$ and x is in $\tilde{\overline{K}}$, transcendental over K. All of these remarks will be useful in a future study of Ω_p or for extending the results of [3] and [4].

The point in our study is to substitute the pseudovaluation w with the integral part of it, i.e. with $[w]$ defined by $[w](x) = [w(x)]$ for any x of $\tilde{\overline{K}}$. It is easy to see that $[w]$ is also a pseudovaluation, but its image is a discret subset of $\mathbb{R}$. For simplity, we put $[w] = z$.

1. Main results

Proposition 1.1. *With the above notations, the pseudovaluation z induces on $\tilde{\overline{K}}$ the same topology like w.*

Proof. Since $z(a) \le w(a) < z(a) + 1$, then $w(a) \to \infty$ if and only if $z(a) \to \infty$. □

Definition 1.1. *Let k be a positive integer and let $x \in \tilde{\overline{K}}$ be a fixed transcendental element over K. An admissible polynomial of degree k for x is a monic polynomial $P_k(X) \in K[X]$ of degree k such that*

$$z(P_k(x)) \ge z(Q_k(x))$$

for any other monic polynomial $Q_k(X) \in K[X]$ of degree k. A sequence of polynomials $\{f_k\}_{k\ge0}$ such that for any $k \ge 0$, f_k is admissible of degree k for x will be called an "admissible sequence of polynomials for x".

Proposition 1.2. *For any integer $k \ge 0$ and for any transcendental element $x \in \tilde{\overline{K}}$ over K there exists an admissible polynomial of degree k for x.*

Proof. Let $\gamma_k(x) = sup\{z(Q(x)) \mid Q(X) \in K[X]$, monic, $deg\ Q = k\}$. We shall prove that $\gamma_k(x) < \infty$. For this, let $\{Q_m\}_m$ be a sequence of monic polynomials of degree k such that $z(Q_m(x)) \to \gamma_k(x)$ as $m \to \infty$.

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Case 1. Suppose that the set of coefficients of all the polynomials Q_m is bounded in K. Since K is locally compact one can choose a convergent subsequence $\{Q_{m_j}\}_j$ to a polynomial $Q(X)$ which is monic of degree k and $z(Q(x)) = \gamma_k(x) < \infty$.

Case 2. Now assume that the set of coefficients of all polynomials Q_m is unbounded. Let O_K be the valuation ring of K. It is a compact subset of K (look at the structure of K!). Let π be a prime element of O_K ($v(\pi) = 1$) and let b_m be the smallest positive integer such that $f_m = \pi^{b_m} Q_m \in O_K[X]$. " Unbounded " means that at least for an infinite subset of $\{Q_m\}_m$, $b_m \to \infty$. Let us start with this subsequence from the beginning and denote it again by $\{Q_m\}_m$. Since O_K is a compact subset, let $\{f_{m_j}\}_j$ be a convergent (to $f \in O_K[X]$) subsequence of $\{f_m\}_m$. Since f_m is primitive for every m, one has that f is primitive. In particular $f \neq 0$. Now $z(f_m(x)) = b_m + z(Q_m(x)) \to \infty$ (since $z(Q_m(x)) \to \gamma_k(t) \neq -\infty$) which implies $z(f(x)) = \infty$, i.e. $f(x) = 0$ and x would be algebraic over K. Hence " case 2 " cannot appear.

Theorem 1.3. *Let $x \in \tilde{\tilde{\overline{K}}}$ be a transcendental element over K and let $\{f_k(X)\}_k$ be an admissible sequence of polynomials for x. Let $r_k = z(f_k(x))$ and $M_k(x) = \pi^{-r_k} f_k(x)$, where π is a prime element fixed in O_K, the valuation ring of (K, v). Then $\{M_k(x)\}_k$ is an integral (polynomial) basis of $E = \widetilde{K[x]}$ as a Banach space over K, relative to the pseudovaluation z. More precisely:*

i) For any $y \in E$ there exists a unique sequence $\{c_k\}_k$ in K such that $v(c_k) \to \infty$ and $y = \sum_k c_k M_k(x)$.

ii) $z(y) = min_k v(c_k)$

iii) Let $O_E = \{t \in E \mid z(t) \geq 0\}$. Then $y \in O_E$ if and only if $c_k \in O_K$ for any $k \geq 0$.

□

Proof. Since $\{M_k(x)\}_k$ are of degrees $0, 1, 2, ...$, any polynomial in x, $P(x) = \sum b_k M_k(x)$, $b_k \in K$ in a unique way, this sum being finite (use mathematical induction).

First of all we shall prove that $z(P(x)) = min_k\ z(b_k M_k(x))$.

Case1. Assume $min_k\ z(b_k M_k(x)) = z(b_d M_d(x))$, where $d = deg\ P(x)$. If $z(P(x)) > z(b_d M_d(x))$, then $v(\pi^{r_d} \frac{P(x)}{b_d}) > z(f_d(x))$. But this contradicts the choice of f_d, so the proof is done in the case 1.

Case 2. Assume now that the greatest index k_0 such that

$min_k z(b_k M_k(x)) = z(b_{k_0} M_{k_o}(x))$ is such that $k_0 < d$. Let $P(x) = b_d M_d(x) + P_1(x)$, where $deg\ P_1 < d$. Use now mathematical induction on the degree of $P(x)$. So that $z(P_1(x)) = z(b_{k_0} M_{k_0}(x))$. But, because of the choice of k_0, one has $z(P(x)) = z(P_1(x)) = min_k\ z(b_k M_k(x))$. Since the pseudovaluation z is discrete one has for ii) : $z(y) = z(\sum_{k=0}^{N} c_k M_k(x))$ for a sufficiently large N. Using the just proved formula above we get that $z(y) = min_{k=0,N}\ z(c_k M_k(x))$.

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But, from i) $z(c_kM_k(x)) = v(c_k) \to \infty$, so if we increase N above, $min_{k=0,N}$ $z(c_kM_k(x)) = min_k z(c_kM_k(x)) = min_k v(c_k)$, i.e. ii).

Let us prove i). Let $y \in E = \widetilde{K[x]}$ and let $\{P_m(x)\}_m$ a sequence of polynomials such that $P_m(x) \xrightarrow{z} y$. Let $k_m = deg\ P_m(x)$. Let us write $P_m(x) = \sum_{j=0}^{k_m} c_{m,j}M_j(x)$, in a unique way such that $z(P_m(x)) = min_j v(c_{m,j})$ (see the above discussion). As the sequence $\{P_m(x)\}_m$ is a Cauchy sequence relative to z, for each j, the sequence $\{c_{m,j}\}_m$ is a Cauchy sequence in v; since K is complete, let $c_j = \lim_{m\to\infty} c_{m,j}$. Let M be big enough, $M > 0$ and let fix m_0 such that $z(P_{m_0}(x) - y) > M$. For m large enough $z(P_m(x) - P_{m_0}(x)) > M$, i.e. $v(c_{m,j} - c_{m_0,j}) > M$ for all j. If $m \to \infty$ we get $v(c_j - c_{m_0,j}) > M$, or $v(c_j) > M$ for $j > k_{m_0}$. This means that $v(c_j) \to \infty$ as $j \to \infty$. Hence the element

$$\widetilde{y} = \sum_{m=0}^{\infty} c_m M_m(x) \in \widetilde{\widetilde{K}}$$

Since $P_m(x) \xrightarrow{z} \widetilde{y}$ $(c_{m,j} \to c_m)$ and $P_m(x) \to y$ one has that $y = \widetilde{y}$ and i) is proved (the uniqueness follows from ii)).

For iii), let us consider the folowing equivalences: $y \in O_E$ if and only if $z(y) \geq 0 \Leftrightarrow min_k\ v(c_k) \geq 0 \Leftrightarrow v(c_k) \geq 0$ for any $k \geq 0 \Leftrightarrow c_k \in O_K$ for all $k \geq 0$. □

Remark 1.1. *If $K = \mathbb{Q}_p, v = v_p$, the p-adic numbers with the standard valuation, if $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ in $\overline{\mathbb{Q}}_p$, a fixed algebraic closure of $\mathbb{Q}_p$ and if v is a valuation on $\overline{\mathbb{Q}}$ which extends v_p, then $w(\alpha) = min\{v(\sigma(\alpha)) \mid \sigma \in G = Gal(\overline{\mathbb{Q}}/\mathbb{Q})\}$ is a pseudovaluation on $\overline{\mathbb{Q}}$ which does not depend on v (see[3],[8]). Let $\widetilde{\overline{\mathbb{Q}}}_p$ be the completion of $\overline{\mathbb{Q}}$ relative to z and let L be a subfield of $\overline{\mathbb{Q}}$. Then $\widetilde{L}$, the topological closure of L in $\widetilde{\overline{\mathbb{Q}}}_p$, is of the form: $\widetilde{L} = \widetilde{\mathbb{Q}_p[x]}$ (see [3]). Theorem 3 says that in $\widetilde{L}$ we can construct a polynomial orthonormal basis $\{M_k(x)\}_k$. In particular any algebraic number y of L can be uniquely written as: $y = \sum_{k\geq 0} c_kM_k(x)$, with $c_k \in \mathbb{Q}_p$, which is a generalization of the primitive element theorem: $[L : \mathbb{Q}\ ] = n \Rightarrow L = \mathbb{Q}[\beta]$, $\beta \in L$ and for any $y \in L$ one has $y = c_0 + c_1\beta + ... + c_{n-1}\beta^{n-1}$.*

If $K = \mathbb{Q}_p$, $v = v_p$ and if v^* is the unique valuation of $\overline{\mathbb{Q}}_p$ which extends v, then an analogous result for $E = \widetilde{\mathbb{Q}_p[x]}$ takes place. Here $x \in \mathbb{C}_p$ is transcendental over $\mathbb{Q}_p$. Another important particular case is when one consider the Fontaine-Colmez ring, B^+_{dR}. See for instance [1] for such applications.

If $K = k((X))$, where k is a field of characteristic 0, we have a description of $\overline{K}$ (see [9] for instance). Let v be the unique valuation on $\overline{K}$ which extends the usual X-adic valuation of K. Let $w(x) = min\{v(\sigma(x)) \mid \sigma \in Gal(\overline{K}/K)\}$ be the corresponding pseudovaluation. Let $\widetilde{\overline{K}}$ be the completion *of* $\overline{K}$ *with respect to* w. *We do not know yet if any closed subring of* $\widetilde{\overline{K}}$ *has an orthonormal bases as above.*

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