ON A LAGUERRE'S THEOREM

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ABSTRACT. In this note we make some remarks on the classical Laguerre's theorem and extend it and some other old results of Walsh and Gauss-Lucas ([6], Th. XXVIII-XXX) to the so called trace series associated with transcendental elements of $\widetilde{\overline{\mathbb{Q}}}$, the completion of the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, with respect to the spectral norm: $\|x\| = max\left\{|\sigma(x)|, \sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})\right\}.$

1. INTRODUCTORY REMARKS ON LAGUERRE THEOREM

In [6], Theorems XXVIII-XXVIX, we find the following result (due to E. Laguerre).

Theorem 1.1. *(Laguerre) Let $f(z)$ be a polynomial in $\mathbb{C}[z]$, z_1, z_2, ..., z_n be its roots in $\mathbb{C}$ (not necessarily distinct) and $n = \deg f$ be its degree. Let w be a complex number such that $f(w) \neq 0$ and $w^* \in \mathbb{C}$ such that*

$$(1) \qquad \frac{f^{/}(w)}{f(w)} = \frac{n}{w - w^*}$$

Let (C) be an arbitrary circumference which contains w and w^*. Then, either all z_1, z_2, ..., z_n belong to (C) or, in each of the two connected components of $\mathbb{C}\backslash(C)$ we find at least one z_i, i.e. (C) separates the set of roots of $f(X)$. Moreover, if (H) is a circular region which contains all the roots z_1, z_2, ..., z_n, then w and w^* cannot be simultaneously outside (H).

Here by a circumference (C) we mean either a boundary of an usual ball $B(z_0, \rho) = \{y \in \mathbb{C}, |y - z_0| < \rho\}$ or a straight line, i.e. (C) is a circumference in the Riemann sphere $\mathbb{C} \cup \{\infty\}$. A circular region is a region (H) bounded by a circumference (C).

There exist many proofs and generalizations of this old and important theorem (see [1], [8], [6], [13], [7], [5]). Let us write formula (1) in another way:

$$(2) \qquad \sum_{j=1}^{n} \frac{1}{w - z_j} = \frac{n}{w - w^*}$$

and compute w^* as a function of z_j, $j = 1, 2, ..., n$ and w :

$$(3) \qquad w^* = \frac{\sum_{j=1}^{n} \frac{z_j}{w - z_j}}{\sum_{j=1}^{n} \frac{1}{w - z_j}} = \frac{w f'(w) - n f(w)}{f'(w)}.$$

If z_j has the algebraic multiplicity k_j and if we have k distinct roots of $f(z)$, say $z_1, ..., z_k$, then formula (3) becomes:

$$(4) \qquad w^* = \frac{\sum_{j=1}^{k} \frac{k_j z_j}{w - z_j}}{\sum_{j=1}^{k} \frac{k_j}{w - z_j}}.$$

We define now a new complex function $L : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ (L from Laguerre!) by the formula: $L(w) = w^*$ if w is not a root of the polynomial f and $L(z_j) = z_j$ if $j = 1, ..., n$. This function is a univalent one. But its inverse $L^{-1}(w^*) = w$ is a $(n-1)$-valued function because, given w^* in formula (1), w is one of the roots of a polynomial of degree $n - 1$. It is easy to rewrite the statement of Laguerre theorem 1.1 in language of the (Laguerre) function L. In fact this function was partially studied in [8] for instance.

We define now a new modified Laguerre function $\widetilde{L} : \mathbb{C} \to \mathbb{C}$ by $\widetilde{L}(z_j) = z_j$ if $j = 1, 2, ..., n$ and, for w with $f(w) \neq 0$, we define: $\widetilde{L}(w) = \widetilde{w}$, where

$$(5) \qquad \widetilde{L}(w) = \widetilde{w} = \frac{\sum_{j=1}^{n} \frac{z_j}{|w - z_j|}}{\sum_{j=1}^{n} \frac{1}{|w - z_j|}}$$

For this new function we prove the following modified Laguerre theorem.

Theorem 1.2. *Let $f(z) \in \mathbb{C}[z]$ be a polynomial with complex coefficients and let z_1, z_2, ..., z_n be its roots, where $n = \deg f$. Let w be any complex number and $\widetilde{w} = \widetilde{L}(w)$. Then, for any circular region (H) which contains all the roots z_1, z_2, ..., z_n, w and $\widetilde{w}$ cannot be simultaneously outside (H). Moreover, if $f(w) \neq 0$, then any circumference*

(C) *which contains w and $\widetilde{w}$ separates the set z_1, z_2, ..., z_n, i.e. they all cannot be in one of the connected components of $\mathbb{C} \smallsetminus (C)$.*

We shall prove this theorem by a series of auxiliary results.

Moreover, our method can be easily extended to a compact subset K of $\mathbb{C}$ instead of a finite set of points $\{z_1, ..., z_n\}$.

First of all we can reduce our elementary considerations to the case of a straight line "circumference", i.e. to a circumference (C_1) which has the infinite point ∞ on it. Indeed, let us take a point $w_1 \notin \{w, w^*\}$ on a circular circumference (C) and transfer everything through the rational linear-fractional transformation $z \to \frac{1}{z-w}$. Since this transformation caries circular regions into circular regions and caries them back, the statements of Theorem 1.2 are invariant to such transformations.

In what follows we identify the complex number field $\mathbb{C}$ with an usual plane $\mathcal{P}$ in which we fix a Cartesian coordinate system $\{O, \overrightarrow{i}, \overrightarrow{j}\}$. So, to any point $M(x, y)$ we associate the complex number $z = x + iy, i = \sqrt{-1}$, the affix of M and conversely, to $z = x + iy$, $x, y \in \mathbb{R}$, we associate the point $M(x, y)$. A *loaded point* in $\mathbb{C}$ (or in $\mathcal{P}$) is a pair $[z, m]$ (or $[M(x, y), m]$), where $z \in \mathbb{C}$ and $m \in (0, \infty)$. We shall identify a simple ("unloaded") point z of $\mathbb{C}$ with the loaded point $[z, 1]$. We can also say that the function $f : \mathbb{C} \to (0, \infty)$, $f(z) = m$ if $[z, m]$ are loaded points which cover all $\mathbb{C}$, is a *density function* on $\mathbb{C}$. In the following we think that any complex number z "has a density m in it". If $m = 1$ it is a simple point in $\mathbb{C}$. Two loaded points $[M_1(x_1, y_1), m_1], [M_2(x_2, y_2), m_2]$ are considered to be equal if $M_1 \equiv M_2$ and, in this case, $m_1 = m_2$ (since the density function is an usual univalent function). If a loaded point $[M(x, y), m]$ is considered k-times, $k = 1, 2, ...$ (it has multiplicity k), we identify it with $[M(x, y), km]$.

Like usual, if $S = \{[M_1(x_1, y_1), m_1], ..., [M_n(x_n, y_n), m_n]\}$ is a system of n loaded points in $\mathbb{C}$, the loaded point $[G(x_G, y_G), \sum_{j=1}^{n} m_j]$, where $x_G = \frac{\sum_{j=1}^{n} x_j m_j}{\sum_{j=1}^{n} m_j}$ and $y_G = \frac{\sum_{j=1}^{n} y_j m_j}{\sum_{j=1}^{n} m_j}$, is called the *centre of mass of S*. It is easy to prove the following elementary result:

Lemma 1.3. *Let $S = \{[M_1(x_1, y_1), m_1], [M_2(x_2, y_2), m_2]\}$ be a set of two distinct loaded points in $\mathbb{C}$. Then,*

a) the center of mass $[G(x_G, y_G), m_1 + m_2]$ is a point on the open segment (M_1, M_2).

b) if (d) is a straight line which passes through $G(x_G, y_G)$ and does not contain the entire segment $[M_1 M_2]$, then (d) separates the set of points $\{M_1, M_2\}$, i.e. M_1, M_2 cannot simultaneously be in one of the two (open) connected components of $\mathbb{C} \smallsetminus (d)$.

Lemma 1.4. *Let* $S = \{[M_1(x_1, y_1), m_1], ..., [M_n(x_n, y_n), m_n]\}$, $n > 1$, *be a system of n distinct loaded points in $\mathbb{C}$ with its center of mass $[G(x_G, y_G), \sum_{j=1}^{n} m_j]$ and let (d) be a straight line which contains the point $G(x_G, y_G)$ and does not contain all the points $M_j(x_j, y_j)$, $j = 1, 2, ..., n$. Then (d) separates the set S, i.e. S is not contained in any of the two connected components of $\mathbb{C} \smallsetminus (d)$. In particular, if $B(\alpha, r)$ is an open disc of radius r and center $\alpha \in \mathbb{C}$ which contains all the points $M_1, ..., M_n$, then the point $G(x_G, y_G)$ is contained in $B(\alpha, r)$. The same statement is true if we substitute the ball $B(\alpha, r)$ with a rectangle D which contains all the points $M_1, ..., M_n$.*

Proof. We use mathematical induction relative to n. For $n = 2$ the statement is true from Lemma 1.3. Let us assume that $n > 2$ and that the statement is true for any $k \leq n - 1$. We shall prove it for $k = n$. Let us take the system $S_1 = \{[M_1(x_1, y_1), m_1], [M_2(x_2, y_2), m_2]\}$ and let $[M_2^*(x^*, y^*), m_1 + m_2]$ be its centre of mass. It is not difficult to see that the centre of mass of S is equal to the centre of mass of the new system $S_2 = \{[M_2^*(x^*, y^*), m_1 + m_2], [M_3(x_3, y_3), m_3]..., [M_n(x_n, y_n), m_n]\}$ of $n - 1$ loaded points. Let now (d) be a straight line which contains this last centre of mass and does not contain all the points of S. If it contained all the points of S_2, then M_1 and M_2 would belong to one and the same connected component of $\mathbb{C} \smallsetminus (d)$ and the proof of the statement would be done. Since S_2 contains at most $n-1$ distinct points, we apply the induction hypothesis and find that not all the points of S_2 are in one of the two connected components of $\mathbb{C} \smallsetminus (d)$ which are also convex subsets of $\mathbb{C}$. If all the points of S were in one of these connected components, say C_1, then the centre of mass $[M_2^*(x^*, y^*), m_1 + m_2]$ would be there and not both M_1, M_2 would be in C_1 and the proof again would be done. Thus, the statement of the lemma is true for any $n = 2, 3, ...$. The other statements are now obvious. $\square$

Let now K be a compact subset of $\mathbb{C}$ with its boundary ∂K a continuous piecewise smooth curve, i.e. a curve which is smooth but a finite number of points of it, at which it is continuous. We suppose that $K \smallsetminus \partial K$ is an open nonempty subset of $\mathbb{C}$. Let $f : K \to \mathbb{R}_+ = (0, \infty)$ be a piecewise continuous (density) function (i.e. a continuous function except a set of area zero) defined on K with nonnegative real values. Then the pair $[K, f]$ is called a *loaded region* in $\mathbb{C}$ and its centre of mass $[G(x_G, y_G), mass[K, f]]$ is computed as follows:

$$(6) \qquad mass[K, f] = \iint_K f(x, y) dx dy,$$

$$x_G = \frac{\iint_K xf(x,y)dxdy}{mass[K,f]}, \quad y_G = \frac{\iint_K yf(x,y)dxdy}{mass[K,f]}$$

where the integral is the usual Riemann double integral. The following result is a natural "generalization" of Lemma 1.4.

Proposition 1.5. *Let $[K,f]$ be a loaded region as above, let*

$$[G(x_G,y_G), mass[K,f]]$$

be its centre of mass and let (d) a straight line which contains G. Then K cannot be contained in one of the connected components of $\mathbb{C} \smallsetminus (d)$. In particular, if $K \subset B(\alpha, r)$, then $[G(x_G,y_G), mass[K,f]] \in B(\alpha, r)$.

Proof. Let us assume that one connected component, say C_1, of $\mathbb{C} \smallsetminus (d)$, contains the whole K. We can easily embed K into the interior of a rectangular area $D \subset C_1$. So, G cannot belong to D. Now, it is easy to divide K into two compact subsets K_1 and K_2, $K_1 \cup K_2 = K$, $area(K_1 \cap K_2) = 0$, $area(K_1) \neq 0$ and $area(K_2) \neq 0$. Let $[G_1(x_1,y_1),$ $mass(K_1)]$ and $[G_2(x_2,y_2), mass(K_2)]$ be the centre of mass of K_1 and K_2 respectively. It is easy to see that G is also the centre of mass of the system $S = \{[G_1(x_1,y_1), mass(K_1)], [G_2(x_2,y_2), mass(K_2)]\}$ of the two loaded points. Thus G is on the segment (G_1, G_2) which is contained in the rectangle D, a contradiction! Hence K cannot be contained in C_1. $\qquad \square$

Now we can prove Theorem 1.2.

Proof. (for Theorem 1.2). In our case, $S = \{[z_1, \frac{1}{|w-z_1|}], ..., [z_n, \frac{1}{|w-z_n|}]\}$ is a system of loaded points and $\left[\widetilde{w}, \sum_{j=1}^{n} \frac{1}{|w-z_j|}\right]$ is its centre of mass. Then we simply apply Lemma 1.4 and the statements of Theorem 1.2 are proved. $\qquad \square$

Remark 1.6. Let K be a compact subset of $\mathbb{C}$ as in Proposition 1.5 and let w be a complex number which is outside K. For any $z \in K$ we define $f(z) = \frac{1}{|w-z|}$. In this way $[K,f]$ becomes a loaded compact subset of $\mathbb{C}$. Let $G(x_G,y_G)$ be its centre of mass ($w = u+iv$, $z = x+iy$):

$$mass[K,f] = \iint_K \frac{1}{|w-z|}dxdy,$$

$$x_G = \frac{\iint_K \frac{x}{|w-z|}dxdy}{mass[K,f]}, \quad y_G = \frac{\iint_K \frac{y}{|w-z|}dxdy}{mass[K,f]}.$$

Then, Proposition 1.5 can be applied and we find a natural generalization of Theorem 1.2 to the loaded compact $[K,f]$, where $f(z) = \frac{1}{|w-z|}$. This last result says that from the point of view of the Newtonian field

centered at w, we can substitute the compact subset K with the loaded point $[L(w), mass[K, f]]$. Moreover, this point cannot be outside any disc which contains K.

Remark 1.7. Let $\overline{\mathbb{Q}}$ be the algebraic closure of the rational number field $\mathbb{Q}$ in $\mathbb{C}$. Then, it is not difficult to see that the Laguerre function $L : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$,

$$(7) \qquad L(w) = w^* = \frac{wf'(w) - nf(w)}{f'(w)}$$

(see also formula (3)) satisfies the following properties:

1) $L(\overline{\mathbb{Q}}) = \overline{\mathbb{Q}}$ if $z_1, ..., z_n \in \overline{\mathbb{Q}}$. Moreover, $L^{-1}(w^*)$, $w^* \in \overline{\mathbb{Q}}$, is a subset of $\overline{\mathbb{Q}}$ which contains at most $n - 1$ distinct elements, i.e. the roots of the algebraic equation in w, $(w - w^*)f'(w) - nf(w) = 0$. Here $f(w) = (w - z_1)...(w - z_n)$. Moreover, for any $\mathbb{Q}$-automorphism $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, the absolute Galois group of $\mathbb{Q}$, and $w \in \overline{\mathbb{Q}}$, we have: $L(\sigma(w)) = \sigma(L(w))$, i.e L is a $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariant mapping, when it is restricted to $\overline{\mathbb{Q}}$.

2) $L(w_1) = \infty$ if and only if w_1 is a critical point of f, i.e. if and only if $f'(w_1) = 0$. In particular, L is a meromorphic function with the critical points of f as poles.

3) $L(w) = w$ if and only if $w \in \{z_1, ..., z_n\}$.

4) $L(\infty) = \frac{\sum_{i=1}^n z_i}{n}$.

2. A Laguerre type theorem for a pseudo-orbit $\mathcal{C}(\alpha)$ of a transcendental element α of $\widetilde{\overline{\mathbb{Q}}}$

Now, we intend to extend the result of Laguerre (Theorem 1.1) to the trace series-functions associated with an element $\alpha \in \widetilde{\overline{\mathbb{Q}}}$, the completion of $\overline{\mathbb{Q}}$ (the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$) relative to the spectral norm: $\|x\| = max\{|\sigma(x)|, \sigma \in G\}$, where $x \in \overline{\mathbb{Q}}$ and $G = Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ is the absolute Galois group of $\mathbb{Q}$ (see [9], [10], [11]). The notion of a trace function has deep roots in [2], [3] and connections with [12].

Let $\{\alpha_n\}_n$ be a Cauchy sequence in $\overline{\mathbb{Q}}$ with respect to spectral norm. Its class in $\widetilde{\overline{\mathbb{Q}}}$ will be denoted by $\alpha = \{\alpha_n\}_n$ or $\alpha \stackrel{\|.\|}{=} \lim_{n \to \infty} \alpha_n$. For any $\sigma \in G$ the sequence of complex numbers $\{\sigma(\alpha_n)\}_n$ is convergent in $\mathbb{C}$, say to $\alpha_{(\sigma)}$, the σ-component of α (see [9]). The function $\varphi_{(\alpha)} : G \to \mathbb{C}$, $\varphi_{(\alpha)}(\sigma) = \alpha_{(\sigma)}$ is a continuous function defined on G (with its Krull topology (see [4])) with values in $\mathbb{C}$ (with its usual topology) (see [10]). Therefore the range of $\varphi_{(\alpha)}$ is a compact set of $\mathbb{C}$, denoted by $\mathcal{C}(\alpha)$ and

called the *pseudo-orbit of* α. In [10] and [12] we studied these compact sets in connection with the arithmetic of $\overline{\mathbb{Q}}$.

Now we need the notion of an ε-neighborhood of a compact C of $\mathbb{C}$. Let $\varepsilon > 0$ and C be a compact subset of $\mathbb{C}$. For any $x \in C$ we write $B(x, \varepsilon)$ for the open ball with centre x and radius ε. $\mathcal{V}(C; \varepsilon) = \bigcup_{x \in C} B(x, \varepsilon)$ is said to be *the ε-neighborhood of C*. We say that a sequence of compact sets $\{C_n\}_n$ *is convergent to the compact C* if for any $\varepsilon > 0$ there exists a natural number N_ε such that $C_n \subset \mathcal{V}(C; \varepsilon)$ for any $n \geq N_\varepsilon$.

It is not difficult to see that if $\alpha_n \overset{\|\cdot\|}{\to} \alpha$ then $\mathcal{C}(\alpha_n)$ is convergent to $\mathcal{C}(\alpha)$.

Let $d_n = \deg \alpha_n$ (over $\mathbb{Q}$) and $f_n(X)$ be the monic minimal polynomial of α_n. Following [3] we denote by $Tr(\beta) = \frac{\beta_1 + \beta_2 + \cdots + \beta_{deg\beta}}{deg\beta}$, the *trace* of an element $\beta \in \overline{\mathbb{Q}}$, where $\beta = \beta_1, \beta_2, ..., \beta_{deg\beta}$ are all the conjugates of β (over $\mathbb{Q}$). If $f : G \to \mathbb{C}$ is a continuous function we denote by $\int_G f(\sigma)d\sigma$ the Haar measure of f $\left(\text{we assume that} \int_G d\sigma = 1\right)$. With these notations we obtain that $Tr(\beta) = \int_G \varphi_{(\beta)}(\sigma)d\sigma = \int_G \sigma(\beta)d\sigma$.

If $\{\alpha_n\}_n$ is a Cauchy sequence in $\overline{\mathbb{Q}}$, i.e. $\alpha_n \overset{\|\cdot\|}{\to} \alpha \in \widetilde{\overline{\mathbb{Q}}}$, then $\varphi_{(\alpha_n)}$ is uniformly convergent to $\varphi_{(\alpha)}$. Hence the sequence of rational numbers $\{Tr(\alpha_n)\}_n$ is convergent to the real number $\int_G \varphi_{(\alpha)}(\sigma)d\sigma$. We denote this last number by $Tr(\alpha)$. We analogously define $Tr(\alpha^k) = \lim_{n \to \infty} Tr(\alpha_n^k)$ for any $k = 0, 1, ...$

Let $x \notin \mathcal{C}(\alpha) = \{\alpha_{(\sigma)}, \sigma \in G\}$ and $\varepsilon > 0$ such that the ε-neighborhood $\mathcal{V}(\mathcal{C}(\alpha); \varepsilon)$ of $\mathcal{C}(\alpha)$ does not contain x.

Let $N_\varepsilon \in \mathbb{N}$ with $\mathcal{C}(\alpha_n) \subset \mathcal{V}(\mathcal{C}(\alpha); \varepsilon)$ for any $n \geq N_\varepsilon$. Let us fix such an $n \geq N_\varepsilon$ and consider the formula:

$$(8) \qquad \frac{f_n^/(x)}{f_n(x)} = \sum_{\sigma \in E_n} \frac{1}{x - \sigma(\alpha_n)}$$

where f_n is the monic minimal polynomial of α_n, $d_n = \deg f_n$ and E_n is a set of elements in G such that $\{\sigma(\alpha_n)\}_{\sigma \in E_n}$ is the set of all distinct conjugates of α_n in $\overline{\mathbb{Q}}$, over $\mathbb{Q}$. We can change $\{\alpha_n\}_n$ such that $\mathbb{Q}(\alpha_n) \subset \mathbb{Q}(\alpha_{n+1})$, $E_n \subset E_{n+1}$ for any $n = 1, 2, ...$ while $\alpha \overset{\|\cdot\|}{=} \lim_{n \to \infty} \alpha_n \in \widetilde{\overline{\mathbb{Q}}}$ remains unchanged (see [9] or [10]).

Let us consider $x \in \mathbb{C}$ with $|x| > \|\alpha\|$. Then $|x| > |\sigma(\alpha_n)|$ for all $\sigma \in E_n$ (n like above). Now we can write

$$\sum_{\sigma \in E_n} \frac{1}{x - \sigma(\alpha_n)} = \frac{1}{x} \sum_{\sigma \in E_n} \frac{1}{1 - \frac{\sigma(\alpha_n)}{x}} = \frac{1}{x} \sum_{\sigma \in E_n} \sum_{k=0}^{\infty} \frac{\sigma(\alpha_n^k)}{x^k}, \text{ or}$$

$$(9) \qquad \sum_{\sigma \in E_n} \frac{1}{x - \sigma(\alpha_n)} = \frac{d_n}{x} \sum_{k=0}^{\infty} Tr(\alpha_n^k) x^{-k}$$

Let us denote $\sum_{k=0}^{\infty} Tr(\alpha_n^k) x^{-k}$ by $T(\alpha_n; x)$ and call it the *trace series associated with* α_n. From (1) and (8) we obtain:

$$(10) \qquad \frac{x \cdot f_n^{/}(x)}{d_n \cdot f_n(x)} = T(\alpha_n; x) = \frac{x}{x - z_n}$$

where $z_n \in \mathbb{C}$ is uniquely defined by α_n and x (it is in fact $L(\alpha_n)$).

Since $Tr(\alpha_n^k) \to Tr(\alpha^k)$, the sequence of analytical functions $\{T(\alpha_n; x)\}_n$ is uniformly convergent to an analytical function say $T(\alpha; x)$ on $\mathbb{C} \backslash B[0, \|\alpha\|]$. In fact

$$(11) \qquad T(\alpha; x) = \sum_{k=0}^{\infty} Tr(\alpha^k) x^{-k}$$

We call the function $x \to T(\alpha; x)$ *the trace series associated with* $\alpha \in \overline{\widetilde{\mathbb{Q}}}$.

Moreover, $z_n = x - \frac{x}{T(\alpha_n; x)}$ is convergent to $z = x - \frac{x}{T(\alpha; x)}$ and (10) becomes

$$(12) \qquad T(\alpha; x) = \frac{x}{x - z}$$

Now we are able to give an extension of the Laguerre's theorem (Theorem 1.1).

Theorem 2.1. *Let* $\alpha \overset{\|.\|}{=} \lim_{n \to \infty} \alpha_n$, $\alpha_n \in \overline{\mathbb{Q}}$ *be an element in* $\overline{\widetilde{\mathbb{Q}}}$ *and* $T(\alpha; x)$ *be its trace series. Let* $x \in \mathbb{C}$ *such that* $|x| > \|\alpha\|$ *and* $z \in \mathbb{C}$ *which verifies (12). Let* (C) *be an arbitrary circumference which contains* x *and* z. *Then* $\mathcal{C}(\alpha)$ *is not contained in any of the two connected components of* $\mathbb{C} \backslash (C)$, *i.e.* (C) *separates* $\mathcal{C}(\alpha)$.

Proof. We assume that $\mathcal{C}(\alpha) \subset (C_1)$, one of the two connected components of $\mathbb{C} \backslash (C)$. Let $\mathcal{V}(\mathcal{C}(\alpha); \varepsilon)$ be such that $\mathcal{V}(\mathcal{C}(\alpha); \varepsilon) \subset (C_1)$ and $x \notin \mathcal{V}(\mathcal{C}(\alpha); \varepsilon)$. Let n be large enough such that $\mathcal{C}(\alpha_n) \subset \mathcal{V}(\mathcal{C}(\alpha); \varepsilon)$. Let (D_n) be a circumference which contains x and $z_n = x - \frac{x}{T(\alpha_n; x)}$

such that $(D_n) \cap \mathcal{V}(\mathcal{C}(\alpha); \varepsilon) = \emptyset$. It is easy to see that $\mathcal{C}(\alpha_n)$ is contained in one of the two connected components of $\mathbb{C} \backslash (D_n)$. Therefore we obtained a contradiction of the Laguerre's classical statement. $\quad \square$

In [6], we find other two classical results in polynomial geometry: Theorem XXIX (Walsh, [13]) and Theorem XXX (Gauss-Lucas).

We give in the following two theorems which are the analogous results of Walsh' and Gauss-Lucas' theorems.

A circular domain in $\mathbb{C}$ is one of the two connected components of $\mathbb{C} \backslash (C)$, where (C) is a circumference in $\mathbb{C} \cup \{\infty\}$.

Theorem 2.2. *Let* $\alpha \overset{\|.\|}{=} \underset{n \to \infty}{lim} \alpha_n$, $\alpha_n \in \overline{\mathbb{Q}}$ *be an element in* $\widetilde{\overline{\mathbb{Q}}}$ *and* $T(\alpha; x)$ *be its trace series. Let* (D) *be a circular domain in* $\mathbb{C}$ *which contains the ball* $B(0, \|\alpha\|)$. *Then, for any* $x \notin (D)$, *the complex number* $z = x - \frac{x}{T(\alpha; x)}$ *belongs to* (D).

Proof. We assume that $z \notin (D)$. Then there exists n large enough such that $z_n \notin (D)$, where $z_n = x - \frac{x}{T(\alpha_n; x)}$. From (10) we obtain that $\frac{f_n^/(x)}{f_n(x)} = \frac{d_n}{x - z_n}$. Applying now Walsh' Theorem (Theorem XXIX, [6]) to $f_n(X)$, the monic minimal polynomial of α_n, we obtain a contradiction. $\quad \square$

Theorem 2.3. *Let* $\alpha \overset{\|.\|}{=} \underset{n \to \infty}{lim} \alpha_n$, $\alpha_n \in \overline{\mathbb{Q}}$ *be an element in* $\widetilde{\overline{\mathbb{Q}}}$ *and* $T(\alpha; x)$ *be its trace series. Then* $T(\alpha; x)$ *has no zero in* $\mathbb{C} \backslash B(0, \|\alpha\|)$.

Proof. We directly apply (12) and Theorem 2.2. $\quad \square$

Remark 2.4. Let $x \notin \mathcal{C}(\alpha)$ and $x \notin \mathcal{V}(\mathcal{C}(\alpha); \varepsilon)$. The sequence

$$\left\{ \frac{x \cdot f_n^/(x)}{d_n \cdot f_n(x)} \right\}_{n \geq N_\varepsilon},$$

where $\mathcal{C}(\alpha_n) \subset \mathcal{V}(\mathcal{C}(\alpha); \varepsilon)$, is convergent. Moreover, the sequence of analytic functions $\left\{ \frac{Z \cdot f_n^/(Z)}{d_n \cdot f_n(Z)} \right\}$ is uniformly convergent to an analytic function $\widetilde{T}(\alpha; x)$ on every compact connected set contained in

$$H = \mathbb{C} \cup \{\infty\} \backslash (\mathcal{C}(\alpha) \cup (C_\alpha)),$$

where (C_α) is the boundary of the ball $B(0, \|\alpha\|)$. But $\widetilde{T}(\alpha; x)$ is uniquely determined only on every connected component of H. Moreover, all the zeros of $\widetilde{T}(\alpha; x)$ are in the convex hull of $\mathcal{C}(\alpha)$ (see [6]).

References

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